

ON A MODIFIED HYERS-ULAM STABILITY OF HOMOGENEOUS EQUATION

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ABSTRACT. In this paper, a generalized Hyers-Ulam stability of the homogeneous equation shall be proved, i.e., if a mapping f satisfies the functional inequality $\|f(yx) - y^k f(x)\| \leq \varphi(x, y)$ under suitable conditions, there exists a unique mapping T satisfying $T(yx) = y^k T(x)$ and $\|T(x) - f(x)\| \leq \Phi(x)$

KEY WORDS AND PHRASES: Functional equation, homogeneous equation, stability.

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1. INTRODUCTION

It is well-known that if a real-valued mapping f defined on non-negative real numbers is a solution of the homogeneous equation, i.e. if f satisfies

$$f(yx) = y^k f(x), \quad (1.1)$$

where k is a given real number, then $f(x) = cx^k$ for some $c \in \mathbb{R}$.

In this note, we shall investigate a generalized Hyers-Ulam stability of the homogeneous equation (1.1) with extended domain and range by using ideas from the paper of Gávruta [1].

Let $(X, +, \cdot)$ be a field and $(X, +, \|\cdot\|)$ a real Banach space. In addition, we assume $\|xy\| = \|x\| \|y\|$ for all $x, y \in X$. For convenience, we write x^2, x^3, \dots instead of $x \cdot x, (x \cdot x) \cdot x, \dots$. If there is no confusion we use 0 and 1 to denote the 'zero-element' and the unity (the neutral element with respect to ' \cdot ') in X , respectively. By x^{-1} we denote the multiplicatively inverse element of x . Suppose k is a natural number. Let $\varphi : X \times X \rightarrow [0, \infty)$ be a mapping such that

$$\Phi_z(x) = \sum_{j=0}^{\infty} \|z\|^{-(j+1)k} \varphi(z^j x, z) < \infty \quad (1.2)$$

or

$$\tilde{\Phi}_z(x) = \sum_{j=0}^{\infty} \|z\|^{jk} \varphi(z^{-(j+1)} x, z) < \infty \quad (1.3)$$

for some $z \in X$ with $\|z\| > 1$ and all $x \in X$. Moreover, we assume that

$$\begin{cases} \Phi_z(\omega^n x) = o(\|\omega\|^{nk}) & (\text{if } \Phi_z(x) < \infty) \\ \tilde{\Phi}_z(\omega^n x) = o(\|\omega\|^{nk}) & (\text{if } \tilde{\Phi}_z(x) < \infty) \end{cases}, \tag{1.4}$$

as $n \rightarrow \infty$, for some $\omega \in X$ and all $x \in X$. Let a mapping $f : X \rightarrow X$ satisfy

$$\|f(yx) - y^k f(x)\| \leq \varphi(x, y) \tag{1.5}$$

and

$$\begin{cases} \varphi(z^n x, y) = o(\|f(z^n x)\|) \text{ as } n \rightarrow \infty & (\text{if } \Phi_z(x) < \infty) \\ \varphi(z^{-n} x, y) = o(\|f(z^{-n} x)\|) \text{ as } n \rightarrow \infty & (\text{if } \tilde{\Phi}_z(x) < \infty) \end{cases}, \tag{1.6}$$

for all x and $y \neq 0$ in X . If (1.3) holds true then we further assume $f(0) = 0$. Our main result is the following theorem.

THEOREM. There exists a unique mapping $T : X \rightarrow X$ satisfying (1.1) and

$$\|T(x) - f(x)\| \leq \begin{cases} \Phi_z(x) & (\text{if } \Phi_z(x) < \infty) \\ \tilde{\Phi}_z(x) & (\text{if } \tilde{\Phi}_z(x) < \infty) \end{cases}, \tag{1.7}$$

for all $x \in X$

2. PROOF OF THEOREM

'We use induction on n to prove

$$\|y^{-nk} f(y^n x) - f(x)\| \leq \sum_{j=0}^{n-1} \|y\|^{-(j+1)k} \varphi(y^j x, y) \tag{2.1}$$

for any $n \in \mathbb{N}$. By (1.5), it is clear for $n = 1$. If we assume that (2.1) is true for n , we get for $n + 1$

$$\begin{aligned} \|y^{-(n+1)k} f(y^{n+1} x) - f(x)\| &\leq \|y\|^{-(n+1)k} \|f(y^n x) - y^k f(y^n x)\| + \|y^{-nk} f(y^n x) - f(x)\| \\ &\leq \|y\|^{-(n+1)k} \varphi(y^n x, y) + \sum_{j=0}^{n-1} \|y\|^{-(j+1)k} \varphi(y^j x, y) \\ &= \sum_{j=0}^n \|y\|^{-(j+1)k} \varphi(y^j x, y) \end{aligned}$$

by using (1.5) and (2.1).

(a) First, we assume that $\Phi_z(x) < \infty$ for some $z \in X$ with $\|z\| > 1$ and all $x \in X$. Let $n > m > 0$. It then follows from (2.1) and (1.2) that

$$\begin{aligned} \|z^{-nk} f(z^n x) - z^{-mk} f(z^m x)\| &= \|z\|^{-mk} \|z^{-(n-m)k} f(z^{n-m} z^m x) - f(z^m x)\| \\ &\leq \|z\|^{-mk} \sum_{j=0}^{n-m-1} \|z\|^{-(j+1)k} \varphi(z^j z^m x, z) \\ &= \sum_{j=m}^{n-1} \|z\|^{-(j+1)k} \varphi(z^j x, z) \rightarrow 0 \text{ as } m \rightarrow \infty. \end{aligned}$$

Therefore, $(z^{-nk} f(z^n x))$ is a Cauchy sequence. Since X is a Banach space, we may define

$$T(x) = \lim_{n \rightarrow \infty} z^{-nk} f(z^n x)$$

for all $x \in X$. From the definition of T , (1.2) and (2.1) we can easily verify the truth of the first relation in (1.7).

Suppose x and $y \neq 0$ to be arbitrary elements of X . By (2.1) we have

$$\|y^{-k} f(yz^n x) - f(z^n x)\| \leq \|y\|^{-k} \varphi(z^n x, y).$$

It follows from the inequality just above and (1.6) that

$$\|f(z^n x)^{-1} y^{-k} f(yz^n x) - 1\| \leq \|y\|^{-k} \|f(z^n x)\|^{-1} \varphi(z^n x, y) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence, it holds

$$\lim_{n \rightarrow \infty} f(z^n x)^{-1} y^{-k} f(yz^n x) = 1. \tag{2.2}$$

By (2.2) we can show that for all x and $y \neq 0$ in X

$$\begin{aligned} T(yx) &= \lim_{n \rightarrow \infty} z^{-nk} f(z^n yx) \\ &= y^k \lim_{n \rightarrow \infty} z^{-nk} f(z^n x) \lim_{n \rightarrow \infty} f(z^n x)^{-1} y^{-k} f(z^n yx) \\ &= y^k T(x). \end{aligned}$$

Besides, it is not difficult to show that $T(0) = 0$. Hence, $T(yx) = y^k T(x)$ holds true for all $x, y \in X$

Let $U : X \rightarrow X$ be another mapping which fulfills (1.1) and (1.7). By using (1.1), (1.7) and (1.4) we get

$$\|T(x) - U(x)\| = \|w\|^{-nk} \|T(w^n x) - U(w^n x)\| \leq 2\|w\|^{-nk} \Phi_z(w^n x) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence, it is clear that $T(x) = U(x)$ for all $x \in \bar{X}$.

(b) Now, we consider the case $\tilde{\Phi}_z(x) < \infty$ for some $z \in X$ with $\|z\| > 1$ and all $x \in X$. By replacing x in (2.1) with $y^{-n}x$ we get

$$\|f(x) - y^{nk} f(y^{-n}x)\| \leq \sum_{j=0}^{n-1} \|y\|^{jk} \varphi(y^{-(j+1)}x, y) \tag{2.3}$$

for any $n \in \mathbb{N}$. As in part (a), if $n > m > 0$ then we obtain

$$\|z^{nk} f(z^{-n}x) - z^{mk} f(z^{-m}x)\| \leq \sum_{j=m}^{n-1} \|z\|^{jk} \varphi(z^{-(j+1)}x, z) \rightarrow 0 \text{ as } m \rightarrow \infty,$$

by using (2.3) and (1.3). We may define

$$T(x) = \lim_{n \rightarrow \infty} z^{nk} f(z^{-n}x)$$

for all $x \in X$. Hence, the second inequality in (1.7) is obvious in view of (2.3).

For arbitrary x and $y \neq 0$ in X , it follows from (2.1) and (1.6) that

$$\lim_{n \rightarrow \infty} f(z^{-n}x)^{-1} y^{-k} f(yz^{-n}x) = 1 \tag{2.4}$$

as in part (a) above. By using (2.4), we get for x and $y \neq 0$ in X

$$\begin{aligned} T(yx) &= \lim_{n \rightarrow \infty} z^{nk} f(z^{-n}yx) \\ &= y^k \lim_{n \rightarrow \infty} z^{nk} f(z^{-n}x) \lim_{n \rightarrow \infty} f(z^{-n}x)^{-1} y^{-k} f(yz^{-n}x) \\ &= y^k T(x). \end{aligned}$$

Since $f(0) = 0$ is assumed in the case of $\tilde{\Phi}_z(x) < \infty$, it also holds $T(yx) = y^k T(x)$ for $y = 0$

The uniqueness of T can be proved as in (a).

3. EXAMPLES

EXAMPLE 1. Let $\varphi(x, y) = \delta + \theta \|x\|^a \|y\|^b$ ($\delta \geq 0, \theta \geq 0, 0 \leq a < k, b \geq 0$) be given in the functional inequality (1.5). If a mapping $f : X \rightarrow X$ satisfies the first condition in (1.6) then there exists a unique mapping $T : X \rightarrow X$ fulfilling (1.1) and

$$\|T(x) - f(x)\| \leq \delta(\|z\|^k - 1)^{-1} + \theta\|z\|^b(\|z\|^k - \|z\|^a)^{-1}\|x\|^a$$

for any $x, z \in X$ with $\|z\| > 1$. In particular, if $\delta > 0$ and $\theta = 0$ then f itself satisfies (1.1).

EXAMPLE 2. Assume that $\varphi(x, y) = \theta\|x\|^a\|y\|^b$ ($\theta \geq 0, a > k, b \geq 0$) is given in the functional inequality (1.5). If a mapping $f : X \rightarrow X$ satisfies the second condition in (1.6) then there exists a unique mapping $T : X \rightarrow X$ which satisfies (1.1) and

$$\|T(x) - f(x)\| \leq \theta\|z\|^b(\|z\|^a - \|z\|^k)^{-1}\|x\|^a$$

for all $x, z \in X$ with $\|z\| > 1$.

If $\varphi(x, y) = \theta\|x\|^k g(\|y\|)$ for some mapping $g : [0, \infty) \rightarrow [0, \infty)$ then our method to get stability for the homogeneous equations (1.1) cannot be applied. By modifying an example in the paper of Rassias and Šemrl [2] we shall introduce a mapping $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying (1.5) and (1.6) with some φ and such that $|f(x)| |x|^{-k}$ (for $x \neq 0$) is unbounded.

EXAMPLE 3. Let us define $f(x) = x^k \log |x|$ for $x \neq 0$ and $f(0) = 0$. Then f satisfies (1.5) and both conditions of (1.6) with $\varphi(x, y) = |x|^k |y|^k |\log |y||$ ($y \neq 0$) and $\varphi(x, 0) = 0$, even though φ satisfies neither (1.2) nor (1.3). In this case we can expect no analogy to the results of Example 1 and 2. Really, it holds

$$\lim_{n \rightarrow \infty} |T(x) - f(x)| |x|^{-k} = \infty$$

for each mapping $T : \mathbb{R} \rightarrow \mathbb{R}$ fulfilling (1.1).

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