

**SYMMETRIC AND PERMUTATIONAL GENERATING SET OF THE GROUPS  
 $A_{kn+1}$  AND  $S_{kn+1}$  USING  $S_n$  AND AN ELEMENT OF ORDER  $k$**

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**ABSTRACT.** In this paper we will show how to generate  $A_{kn+1}$  and  $S_{kn+1}$  using a copy of  $S_n$  and an element of order  $k$  in  $A_{kn+1}$  and  $S_{kn+1}$  respectively, for all positive integers  $n \geq 2$  and all positive integers  $k \geq 2$ . We will also show how to generate  $A_{kn+1}$  and  $S_{kn+1}$  symmetrically using  $n$  elements each of order  $k$ , for all  $n \geq 2$  and all even integers  $k \geq 2$ .

**KEY WORDS AND PHRASES:** Symmetric generators, Group presentation, Doubly transitive groups.

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**1. INTRODUCTION**

Hammas [1] showed that  $A_{2n+1}$  can be presented as

$$G = A_{2n+1} = \langle X, Y, T \mid \langle X, Y \rangle = S_n, T^2 = [T, S_{n-1}] = 1 \rangle$$

for  $n = 4, 6$ , where  $[T, S_{n-1}]$  means that  $T$  commutes with  $Y$  and with  $X^2 Y X$ , (the generators of  $S_{n-1}$ ). The relations of the symmetric group  $S_n = \langle X, Y \rangle$  of degree  $n$  are found in Coxeter and Moser [2]. Some relations must be added to the presentation that generates  $A_{2n+1}$  in order to complete the coset enumeration. Also Hammas [1] showed that, for  $n = 4, 6$ , the group  $A_{2n+1}$  can be symmetrically generated by  $n$  elements each of order 2 and of the form  $T_0, T_1, \dots, T_{n-1}$ , where  $T_i = T^{X^i} = X^{-i} T X^i$  and  $T, X$  satisfy the relations of the group  $A_{2n+1}$ . The set  $\{T_0, T_1, \dots, T_{n-1}\}$  is called the symmetric generating set of  $A_{2n+1}$  ( see the Definition 2.1 in Section 2 ).

Hammas [3] showed that  $A_{2n+1}$  can be presented as

$$A_{2n+1} = \langle X, Y, T \mid \langle X, Y \rangle = S_n, T^2 = [T, Y] = [T, X^{-2} Y X] = (X T)^{2n+1} = (Y T_{n-2})^{10} \rangle$$

when  $n$  is an even integer and  $S_{2n+1}$  can be presented as

$$S_{2n+1} = \langle X, Y, T \mid \langle X, Y \rangle = S_n, T^2 = [T, Y] = [T, X^{-2} Y X] = (X T)^{n(n+1)} = (Y T_{n-2})^{10} \rangle.$$

when  $n$  is odd. Note that the order of the third generator,  $T$ , was always 2.

Also, it has been shown by Hammam [3] that for all  $n \geq 2$  the groups  $A_{2n+1}$  and  $S_{2n+1}$  can be symmetrically generated using  $n$  elements each of order 2, and of the form  $T_0, T_1, \dots, T_{n-1}$ , where  $T_i = T^{X^i} = X^{-i} T X^i$  and  $T, X$  satisfy the relations of the groups  $A_{2n+1}$  and  $S_{2n+1}$ .

In this paper, we give a generalization of the results obtained by Hammam [1-3]. We will show that, for all  $k \geq 2$  and for all  $n \geq 2$ , the group generated by  $X, Y$  and  $T$  is the alternating group  $A_{kn+1}$  when  $n$  and  $k$  are all even integers and is the symmetric group  $S_{kn+1}$  otherwise. Moreover, relations will be given to show that, for all  $k \geq 2$  and for all  $n \geq 2$ , the group

$$G = \langle X, Y, T \mid \langle X, Y \rangle = S_n, T^k = [T, S_{n-1}] = 1 \rangle$$

is  $A_{kn+1}$  when  $n$  and  $k$  are both even and  $S_{kn+1}$  otherwise. We give permutations that generate  $A_{kn+1}$  and  $S_{kn+1}$  which satisfy the conditions given in the presentation of the group  $G$ . Further, we prove that, when  $k$  is an even integer,  $G$  can be symmetrically generated by  $n$  permutations each of order  $k$  of the form  $T_0, T_1, \dots, T_{n-1}$ , where  $T_i = T^{X^i} = X^{-i} T X^i$ , satisfying the condition that  $T_0$  commutes with the generators of the group  $S_{n-1}$ .

**2. PRELIMINARY RESULTS**

**THEOREM 2.1.** Let  $1 \leq a \neq b \leq n$  be any integers. Let  $G$  be the group generated by the  $n$ -cycle  $(1, 2, \dots, n)$  and the 3-cycle  $(n, a, b)$  where the highest common factor  $\text{hcf}(n, a, b) = 1$ . If  $n$  is an odd integer then  $G = A_n$  while, if  $n$  is even, then  $G = S_n$ .

**DEFINITION 2.1.** Let  $G$  be a group and  $\Gamma = \{ T_0, T_1, \dots, T_{n-1} \}$  be a subset of  $G$  where  $T_i = T^{X^i} = X^{-i} T X^i$  for all  $i = 0, 1, \dots, n-1$ . Let  $S_n$  - a copy of the symmetric group of degree  $n$  - be the normalizer in  $G$  of the set  $\Gamma$ . We define  $\Gamma$  to be a symmetric generating set of  $G$  if and only if  $G = \langle \Gamma \rangle$  and  $S_n$  permutes  $\Gamma$  doubly transitively by conjugation, i.e.,  $\Gamma$  is realizable as an inner automorphism.

**3. PERMUTATIONAL GENERATING SET OF  $A_{kn+1}$  AND  $S_{kn+1}$**

**THEOREM 3.1.** For all  $n \geq 2$  and all  $k \geq 2$ ,  $A_{kn+1}$  can be generated using a copy of  $S_n$  and an element of order  $k$  in  $A_{kn+1}$  when  $n$  and  $k$  are both even and  $S_{kn+1}$  can be generated using a copy of  $S_n$  and an element of order  $k$  in  $S_{kn+1}$  if  $n$  or  $k$  is odd.

**PROOF.** Let  $X = (1, 2, \dots, n)(n+1, n+2, \dots, 2n) \dots ((k-1)n+1, (k-1)n+2, \dots, kn)$ ,  $Y = (n-1, n) \dots (kn-1, kn)$  and  $T = (1, n+1, 2n+1, 3n+1, \dots, (k-2)n+1, kn+1)(2, n+2, 2n+2, \dots, (k-1)n+2) \dots (n, 2n, \dots, kn)$  be three permutations; the first of order  $n$ , the second of order 2 and the third of order  $k$ . Let  $H$  be the group generated by  $X$  and  $Y$ . By a result of Burnside and Moore, ( see Coxeter and Moser[2] ), the group  $H$  is the symmetric group  $S_n$ . Let  $G$  be the group generated by  $X, Y$  and  $T$ . We have two cases :

Case 1 Let  $k$  be an odd integer. Let  $\alpha = [X, T]$ . Then  $\alpha = ( 1, (k-1)n+1, kn+1, (k-1)n+2, 2 )$ . Let  $\beta = \alpha^3 \alpha^T$ . Then

$$\beta = ( 1, (k-1)n+2 )( 2, kn+1 )( n+1, (k-1)n+1, n+2 ) .$$

Let  $\delta = \alpha X \beta^T \beta^{T^2} \dots \beta^{T^{k-3}} \alpha Y^X$ . Hence

$$\begin{aligned} \delta = & ( 1, kn, 3n, n+2, \dots, 2n-1, n+1, 2n+2, \dots, 3n-1, 2n+1, 2n, 5n, 3n+2, \dots, 4n-1, 3n+1, 4n+2, \dots, \\ & 5n-1, 4n+1, 4n, 7n, 5n+2, \dots, 6n-1, 5n+1, 6n+2, \dots, 7n-1, 6n+1, 6n, 9n, \dots, (k-6)n+2, \dots, \\ & (k-5)n-1, (k-6)n+1, (k-5)n+2, \dots, (k-4)n-1, (k-5)n+1, (k-5)n, (k-2)n, (k-4)n+2, \dots, \\ & (k-3)n-1, (k-4)n+1, (k-3)n+2, \dots, (k-2)n-1, (k-3)n+1, (k-3)n, (k-2)n+2, \dots, \\ & (k-1)n-1, (k-2)n+1, (k-1)n, kn+1, (k-1)n+3, \dots, \\ & kn-1, (k-1)n+1, n, 2, (k-1)n+2, 3, \dots, n-1 ) \end{aligned}$$

which is a cycle of length  $kn+1$ . Let  $K = \langle \delta, \beta^2 \rangle$ . We claim that  $K$  is either  $A_{kn+1}$  or  $S_{kn+1}$ . To show this, let  $\theta$  be the mapping which takes the element in the position  $i$  of the permutation  $\beta$  into the element  $i$  in the permutation  $(1, 2, \dots, kn+1)$ . Under the mapping  $\theta$ , the group  $K$  will be mapped into the group

$$\theta(K) = \langle (1, 2, \dots, kn+1), (n+1, 4, (k-1)n+1) \rangle.$$

Since  $k$  is an odd integer the highest common factor  $\text{hcf}(n+1, 4, (k-1)n+1) = 1$ . Hence by Theorem 2.1, if  $n$  is an odd integer then  $\theta(K)$  is  $S_{kn+1}$ . Hence  $G$  is  $S_{kn+1}$ . But if  $n$  is an even integer then  $\theta(K)$  is  $A_{kn+1}$ . Since  $k$  is an odd integer,  $Y$  is an odd permutation. The action of the generators of  $A_{kn+1}$  on  $Y$  is not trivial and therefore  $G$  is the symmetric group  $S_{kn+1}$ .

Case 2 Let  $k$  be an even integer. Let  $\alpha = [X, T]$ . Then  $\alpha = (1, (k-1)n+1, kn+1, (k-1)n+2, 2)$ . Let  $\beta = \alpha^3 \alpha^T$ . Then

$$\begin{aligned} \beta &= (1, (k-1)n+2)(2, kn+1)(n+1, (k-1)n+1, n+2). \\ \text{Let } \delta &= \alpha X \beta^T \beta^{T^3} \dots \beta^{T^{(k-4)/2}} \alpha Y^X. \text{ Hence} \\ \delta &= (1, 2, 2n, n, n+2, \dots, 2n-1, n+1, kn, 4n, 2n+2, \dots, 3n-1, 2n+1, 3n+2, \dots, \\ &4n-1, 3n+1, 3n, 6n, 4n+2, \dots, 5n-1, 4n+1, 5n+2, \dots, 6n-1, 5n+1, 5n, 8n, \dots, \\ &(k-6)n+2, \dots, (k-5)n-1, (k-6)n+1, (k-5)n+2, \dots, (k-4)n-1, (k-5)n+1, (k-5)n, \\ &(k-2)n, (k-4)n+2, \dots, (k-3)n-1, (k-4)n+1, (k-3)n+2, \dots, (k-2)n-1, (k-3)n+1, \\ &(k-3)n, (k-2)n+2, \dots, (k-1)n-1, (k-2)n+1, (k-1)n, kn+1, (k-1)n+3, \dots, \\ &kn-1, (k-1)n+1, (k-1)n+2, 3, \dots, n-1) \end{aligned}$$

which is a cycle of length  $kn+1$ . Let  $K = \langle \delta, \beta^2 \rangle$ . Using the same method used above we can easily show that  $K$  is the alternating group  $A_{kn+1}$ . Now, since  $k$  is an even integer, then, if  $n$  is an even integer too,  $G$  has to be the alternating group  $A_{kn+1}$  or a proper subgroup of it. Since  $K$  is the alternating group  $A_{kn+1}$  then  $G$  is the alternating group  $A_{kn+1}$ . But if  $n$  is an odd integer then  $T$ , the third generator of  $G$ , is an odd permutation. Since the action of the generators of the group  $K$  on the element  $T$  is not trivial, the group  $\langle \delta, \beta^2, T \rangle$  is the symmetric group  $S_{kn+1}$ . Hence  $G$  is the symmetric group  $S_{kn+1}$ .

**4. SYMMETRIC PERMUTATIONAL GENERATING SET OF  $A_{kn+1}$  and  $S_{kn+1}$**

**THEOREM 4.1.** Let  $X, Y$  and  $T$  be the permutations described in Theorem 3.1 where  $T^k = 1$ . Let  $\Gamma = \{T_0, T_1, \dots, T_{n-1}\}$ , where  $T_i = T^{X^i}$ . Let  $k$  be an even integer. If  $n$  is an even integer too, then the set  $\Gamma$  generates the alternating group  $A_{kn+1}$  symmetrically, while, if  $n$  is an odd integer, then the set  $\Gamma$  generates the symmetric group  $S_{kn+1}$  symmetrically.

**PROOF.** Let  $T_0 = (1, n+1, 2n+1, 3n+1, \dots, kn+1)(2, n+2, 2n+2, \dots, (k-1)n+2) \dots (n, 2n, \dots, kn)$ ,  $T_1 = T^X = (1, n+1, 2n+1, \dots, (k-1)n+1)(2, n+2, 2n+2, \dots, kn+1) \dots (n, 2n, \dots, kn)$ ,  $\dots$ ,  $T_{n-1} = T^{X^{n-1}} = (1, n+1, 2n+1, \dots, (k-1)n+1)(2, n+2, 2n+2, \dots, (k-1)n+2) \dots (n, 2n, \dots, kn+1)$ . Let  $H = \langle \Gamma \rangle$ . We claim that  $H \cong A_{kn+1}$  or  $S_{kn+1}$ . To show this, suppose first that  $n$  is an odd integer. Let  $\ell = \frac{k}{2}$  if  $\frac{k}{2}$  is even and  $\ell = \frac{k}{2} + 1$  if  $\frac{k}{2}$  is odd and let  $r = \ell - 1$  if  $\frac{k}{2}$  is odd and  $r = \ell + 1$  if  $\frac{k}{2}$  is even. Consider the element  $\alpha = (T_0^{T_1 T_2 \dots T_{n-2}})_{T_{n-1}}$ . We find that

$$\begin{aligned} \alpha &= (1, 2n+1, 4n+1, \dots, \ell n+1, (\ell+1)n+1, (\ell+3)n+1, \dots, (k-1)n+1, n+1, 3n+1, 5n+1, \dots, \\ &n+1, (r+1)n+2, (r+3)n+2, \dots, (k-1)n+2, n+2, 3n+2, 5n+2, \dots, rn+2, (r+3)n+2, (r+5)n+2, \dots, \end{aligned}$$

$(k-2)n+2, 2, 2n+2, 4n+2, \dots, (\ell-2)n+2, \ell n+3, (\ell+2)n+3, \dots, (k-2)n+3, 3, 2n+3, 4n+3, \dots,$   
 $(\ell-2)n+3, (\ell+1)n+3, (\ell+3)n+3, \dots, (k-1)n+3, n+3, 3n+3, 5n+3, \dots, (r-2)n+3, rn+4, (r+2)n+4, \dots,$   
 $(k-1)n+4, n+4, 3n+4, 5n+4, \dots, (r-2)n+4, (r+1)n+4, (r+3)n+4, \dots, (k-2)n+4, 4, 2n+4, 4n+4, \dots,$   
 $(\ell-4)n+4, (\ell-2)n+5, \ell n+5, (\ell+2)n+5, \dots, (k-2)n+5, 5, 2n+5, 4n+5, \dots, (\ell-4)n+5, (\ell-1)n+5,$   
 $(\ell+1)n+5, (\ell+3)n+5, \dots, (k-1)n+5, n+5, 3n+5, 5n+5, \dots, (r-4)n+5, (r-2)n+6, rn+6, (r+2)n+6, \dots,$   
 $(k-1)n+6, n+6, 5n+6, 7n+6, \dots, (r-4)n+6, (r-1)n+6, (r+1)n+6, (r+3)n+6, \dots, (k-2)n+6, 6, 2n+6,$   
 $4n+6, \dots, (\ell-6)n+6, (\ell-4)n+7, (\ell-2)n+7, \ell n+7, (\ell+2)n+7, \dots, (k-2)n+7, 7, 2n+7, 4n+7, \dots,$   
 $(\ell-6)n+7, (\ell-3)n+7, (\ell-1)n+7, (\ell+1)n+7, \dots, (k-1)n+7, n+7, 3n+7, 5n+7, \dots, (r-6)n+7,$   
 $(r-4)n+8, (r-2)n+8, rn+8, \dots, (k-1)n+8, n+8, 5n+8, 7n+8, \dots, (r-6)n+8, (r-3)n+8, (r-1)n+8,$   
 $(r+1)n+8, (r+3)n+8, \dots, (k-2)n+8, \dots, n-2, \dots, n, 3n, \dots, (k-1)n, kn, 2n, 4n, \dots, (k-2)n, kn+1, n-1,$   
 $2n-2, 3n-3, \dots, (n-2)n-(n-2), (n-1)n+1, (n+1)n+1, (n+3)n+1, \dots, (k-2)n+1$

which is a cycle of length  $kn+1$ . Let  $\beta = T^{-1}T^X$ . Therefore  $\beta = (1, 2, (k-1)n+2, kn+1, (k-1)n+1)$  which is a cycle of length 5. Let  $\gamma = \beta^3\beta^T$ . Since  $\gamma = (2, n+2, n+1)$  then using the same method used in Theorem 3.1' above we get  $H_1 = \langle \alpha, \gamma, T \rangle \cong S_{kn+1}$ . Hence  $H \cong H_1 \cong \Theta(H_1) \cong S_{kn+1}$ . In the same way we can show that, when  $n$  is an even integer,  $H \cong A_{kn+1}$ .

The above results can be summarised in the following table:

	$n$	$k$	$G \cong \langle X, Y, T \rangle$	$\langle X, T \rangle$	$\langle \Gamma \rangle$
1	even	even	$A_{kn+1}$	$A_{kn+1}$	$A_{kn+1}$
2	even	odd	$S_{kn+1}$	$S_{kn+1}$	$A_{kn+1}$
3	odd	even	$S_{kn+1}$	$S_{kn+1}$	$S_{kn+1}$
4	odd	odd	$S_{kn+1}$	$A_{kn+1}$	$A_{kn+1}$

where

$$G = \langle X, Y, T \mid \langle X, Y \rangle = S_n, T^k = [T, Y] = [T, X^2 Y X] = ([X, T] X)^n = (Y X)^{n-1}, [X, T]^5 = (T^X T^{-1})^5 = (T T^{-1} [X, T]^2 [X, T]^T)_{T^{-1}}^k = (T^{([X, T]^2 [X, T]^T)_{T^{-1}}})^6 = (T [X, T]^X)^r = 1 >$$

where  $r = k(k-1)$  when  $k$  is odd and  $r = 2k(k+1)$  when  $k$  is even for all  $n, k \geq 3$ .

From the above table we can see that in the case when  $k$  is an odd integer the set  $\Gamma = \{T_0, T_1, \dots, T_{n-1}\}$  cannot generate the symmetric group  $S_{kn+1}$  symmetrically. As a matter of fact, as we verified using the GAP package, the set  $\Gamma$  generates the alternating group  $A_{kn+1}$  symmetrically. But unfortunately we haven't found a hand proof of this case yet.

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