

NECESSARY AND SUFFICIENT CONDITIONS FOR THE OSCILLATION OF DELAY DIFFERENTIAL EQUATION WITH A PIECEWISE CONSTANT ARGUMENT

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ABSTRACT: The characteristic equation for an equation with continuous and piecewise constant argument in the form

$$\dot{x}(t) + px(t - \tau) + qx([t - k]) = 0 \text{ where } p, q \in \mathfrak{R}, \tau \in \mathfrak{R}^+ \text{ and } k \in \mathfrak{N}.$$

is presented, which when $q=0$ reduces to

$$f(\lambda) = \lambda + e^{-\lambda\tau} = 0$$

and when $p=0$ reduces to

$$\lambda - 1 + q\lambda^{-k} = 0.$$

Also, the necessary and sufficient conditions for oscillation are obtained.

KEY WORDS: Oscillations, Delay differential equations.

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1. INTRODUCTION

The study of equations with piecewise constant argument was originated by the work of Wiener and his collaborators. See [1,2,3,4,5 and 6] and the references cited therein. In addition to its own interest this area has stimulated much activity in the study of delay difference equations.

As usual, a solution $x(t)$ is called oscillatory if it has arbitrarily large zeros. Otherwise, the solution is called nonoscillatory. An equation is called oscillatory if all its solutions are oscillatory.

Let $[.]$ denote the greatest-integer function, \mathfrak{N} the set of non-negative integers and \mathfrak{R} the set of real numbers.

Consider

$$\dot{x}(t) + px(t - \tau) + qx([t - k]) = 0 \tag{1.1}$$

where $p, q \in \mathfrak{R}$, $\tau \in \mathfrak{R}^+$ and $k \in \mathfrak{N}$.

By a solution of Eqn.(1.1) , we mean a function x which is defined on the set $\{-k, \dots, -1, 0\} \cup [-\tau, \infty)$ and satisfies the following properties:

- (a) x is continuous on $[-\tau, \infty)$.
- (b) the derivative \dot{x} exists at each point $t \in (0, \infty)$ with the possible exception of the points $t \in \mathbb{N}^-$, where one side derivatives exist.
- (c) Eqn.(1.1) is satisfied on each interval $[n, n+1)$ for $n \in \mathbb{N}^+$.

Let $\phi \in C([-\tau, 0], \mathbb{R})$ and $a_{-k}, \dots, a_{-1}, a_0$ be given real numbers such that

$$a_{-j} = \phi(-j) \text{ for } j \leq \tau, j=0, 1, 2, \dots, k, \tag{1.2}$$

then one can show that Eqn. (1.1) has a unique solution satisfying the initial conditions

$$x(t) = \phi(t) \quad -\tau \leq t \leq 0 \tag{1.3a}$$

$$x(-j) = a_{-j} \quad j=0, 1, \dots, k. \tag{1.3b}$$

When $q=0$, Eqn. (1.1) reduces to

$$\dot{u}(t) + pu(t - \tau) = 0 \tag{1.4}$$

which is oscillatory if and only if its characteristic equation

$$f(\lambda) = \lambda + e^{-\lambda\tau} = 0 \tag{1.5}$$

has no real roots, or equivalently, to

$$p\tau > \frac{1}{e}. \tag{1.6}$$

On the other hand , when $p=0$, Eqn.(1.1) reduces to

$$\dot{v}(t) + qv([t - k]) = 0 \tag{1.7}$$

which is oscillatory if and only if the following equation

$$\lambda - 1 + q\lambda^{-k} = 0 \tag{1.8}$$

has no positive real roots , or equivalently,

$$q > \frac{k^k}{(k+1)^{k+1}} \quad , k \geq 1 \tag{1.9a}$$

$$q \geq 1 \quad , k=0 \tag{1.9b}$$

An open question arises (see [4] ,p. 223) for obtaining a characteristic equation for equation (1.1) which reduces to Eqn.(1.5) when $q=0$ and reduces to Eqn. (1.8) when $p=0$ and also obtaining the necessary and sufficient conditions for oscillation of all solutions of

$$\dot{x}(t) + px(t - 1) + qx([t - 1]) = 0 \tag{1.10}$$

2. THE MAIN RESULTS

In the following , a characteristic equation associated with equation (1.1) will be presented in Theorem 2.1 . Also the necessary and sufficient conditions for oscillation are obtained through Theorems 2.2 and 2.3 .

THEOREM 2.1. The characteristic equation associated with equation (1.1) is

$$f(\lambda) = \lambda - 1 + \frac{p\lambda^{-\tau}}{\ln \lambda} (\lambda - 1) + q\lambda^{-k} = 0 \tag{2.1}$$

which reduces to Eqn. (1.5) when $q=0$ and reduces to Eqn. (1.8) when $p=0$.

PROOF: Consider Eqn.(1.1) and assume that the initial conditions (1.3a) and (1.3b) are satisfied .For $t \in [n, n+1)$, we have $[t-k] = n-k$ and one can write

$$\dot{x}(t) + p x(t - \tau) + q a_{n-k} = 0 \quad , \quad t \in [n, n+1) \tag{2.2a}$$

$$x(n) = a_n \quad , \quad n \in \mathbb{N} \tag{2.2b}$$

Integrating (2.2a) from n to t , we get

$$x(t) - a_n + p \int_n^t x(s - \tau) ds + q a_{n-k} (t - n) = 0 . \tag{2.3}$$

By using the continuity of $x(t)$ as $t \rightarrow n+1$, we find

$$a_{n+1} - a_n + p \int_n^{n+1} x(s - \tau) ds + q a_{n-k} = 0 . \tag{2.4}$$

Assume that $x(t) = e^{\lambda t}$, $t \in [n, n+1)$, then from (2.4) ,we get

$$f(\lambda) = e^{\lambda} - 1 + \frac{p e^{-\lambda \tau}}{\lambda} (\lambda - 1) + q e^{-\lambda k} = 0 \tag{2.5}$$

Putting $e^{\lambda} = \gamma$ in Eqn.(2.5),then

$$F(\gamma) = \gamma - 1 + \frac{p \gamma^{-\tau}}{\ln \gamma} (\gamma - 1) + q \gamma^{-k} = 0 . \tag{2.6}$$

and consequently Eqn.(2.6) has no positive real roots if and only if Eqn.(2.5) has no real roots. Assume that Eqn.(2.5) has no real roots , then $\lambda \neq 0$, and consequently $\gamma \neq 1$. If $p=0$, then Eqn. (2.6) reduces to Eqn.(1.8),also if $q=0$,then Eqn.(2.6) reduces to Eqn.(1.5) .

THEOREM 2.2. Equation (1.1) is oscillatory if and only if its characteristic equation (2.6) has no positive real roots.

PROOF: Assume that the characteristic equation (2.6) has a positive real root γ_0 , then γ_0 is a solution of Eqn.(2.4) which is a nonoscillatory solution and consequently Eqn.(1.1) is not oscillatory .On the other hand , assume that $x(t) > 0 \forall t \in [n, n+1)$ for sufficiently large n and $F(\gamma)$ has no positive real roots .As $F(\infty) = \infty$, it follows that $F(\gamma) > 0 \forall \gamma \in (0, \infty)$.For seeking the contradiction , choose :

- (i) $p \leq 0$ and $q \leq 0$ then $F(\gamma) < 0 \forall \gamma \in (0, 1)$,
- (ii) $p \geq 0, q < 0$ with $p < |q|$ and $\tau \leq k$ then $F(\gamma) < 0 \forall \gamma \in (0, 1)$,
- (iii) $p < 0, q \geq 0$ with $q < |p| (1 - 1/e)$ and $\tau = k$ then $F(1/e) < 0$,
- (iv) $p \geq 0, q \geq 0$ with $p + q \leq 1/8e^k$ and $\tau \leq k$ then $F(1/e) < 0$,

which is a contradiction.

THEOREM 2.3. If $p, q \in \mathbb{R}^+$, then all solutions of equation (1.1) are oscillatory if and only if

$$p e \tau + q \frac{(k+1)^{k+1}}{k^k} > 1 \quad , \quad k \geq 1 \tag{2.7}$$

PROOF: Assume that Eqn.(1.1) has a nonoscillatory solution ,then the characteristic Eqn.(2.6) has a positive real root $\gamma_0 \in (0, 1)$. Otherwise $F(\gamma_0) > 0 \forall \gamma_0 \in [1, \infty)$ and therefore , we have

$$F(\gamma_0) = \gamma_0 - 1 + \frac{p\gamma_0^{-\tau}}{\ln \gamma_0}(\gamma_0 - 1) + q\gamma_0^{-k} = 0, \gamma_0 \in (0,1)$$

and then

$$0 = (\gamma_0 - 1) \left\{ 1 + \frac{p\gamma_0^{-\tau}}{\ln \gamma_0} + q\gamma_0^{-k} / (\gamma_0 - 1) \right\}$$

$$0 = 1 + \frac{p\gamma_0^{-\tau}}{\ln \gamma_0} + q\gamma_0^{-k} / (\gamma_0 - 1)$$

$$\leq 1 - pe\tau - q \frac{(k+1)^{k+1}}{k^k}$$

which is a contradiction. On the other hand, assume that

$$pe\tau + q \frac{(k+1)^{k+1}}{k^k} \leq 1, k \geq 1.$$

Now, we study the following cases:

(1) $q=0, p>0$.

Since $F(\gamma) > 0, \forall \gamma \in (1, \infty)$ and $F(e^{-\frac{1}{\tau}}) \leq 0$, then there exists $\gamma_1 \in \mathfrak{R}^+$ such that $F(\gamma_1) = 0$. i.e. the characteristic equation has a positive real root and consequently equation (1.1) is not oscillatory.

(2) $p=0, q>0$.

In this case, $F(\gamma) > 0, \forall \gamma \in (1, \infty)$ and $F(\frac{k}{k+1}) \leq 0$. Therefore, the characteristic equation has a positive real root and then equation (1.1) has a nonoscillatory solution.

(3) $p>0, q>0$.

Since $pe\tau + q \frac{(k+1)^{k+1}}{k^k} \leq 1$,

then,

$$q \frac{(k+1)^{k+1}}{k^k} + \frac{p \frac{(k+1)^{\tau+1}}{k^\tau}}{\ln(\frac{k}{k+1})} < pe\tau + q \frac{(k+1)^{k+1}}{k^k} \leq 1, k \geq 1. \tag{2.8}$$

It is clear that the characteristic equation has no real roots in $(1, \infty)$ and $F(\gamma) > 0$, but

$$F(\frac{k}{k+1}) = \frac{k}{k+1} - 1 + \frac{p \frac{(k+1)^\tau}{k^\tau}}{\ln(\frac{k}{k+1})} + q \frac{(k+1)^k}{k^k}$$

$$= -\frac{1}{k+1} + \frac{q}{(k+1)} \frac{(k+1)^{k+1}}{k^k} + \frac{p}{k+1} \frac{(k+1)^{\tau+1}}{\ln(\frac{k}{k+1})}$$

From (2.8), it follows that $F(\frac{k}{k+1}) \leq 0$ and

consequently equation (1.1) has a nonoscillatory solution.

REMARK. If $\tau = k = 1$ and $p, q \in \mathfrak{R}^+$, then $pe + 4q > 1$ is a necessary and sufficient condition for oscillation of

$$\dot{x}(t) + px(t-1) + qx([t-1]) = 0.$$

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