

POSITIVE SOLUTIONS OF AN ASYMPTOTICALLY PLANAR SYSTEM OF ELLIPTIC BOUNDARY VALUE PROBLEMS

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ABSTRACT. We will prove an existence result of positive solutions for an asymptotically planar system of two elliptic equations. It will be used as main tools for a Maximum Principle and a result on Bifurcation Theory.

KEY WORDS AND PHRASES: Positive solution, asymptotically planar, maximum principle, bifurcation point.

AMS SUBJECT CLASSIFICATION CODES: 35J25, 47H15.

1. INTRODUCTION

In this paper we will prove the existence of positive solutions for the elliptic system

$$-\Delta U = A(x)U + F(x, U) \text{ in } \Omega, \quad U = 0 \text{ on } \partial\Omega \quad (1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $A(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$ whose entries are continuous in $\bar{\Omega}$, $U = \begin{pmatrix} u \\ v \end{pmatrix}$, $-\Delta U = \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\Delta u \\ -\Delta v \end{pmatrix}$ and $F(x, U) = \begin{pmatrix} f(x, u, v) \\ g(x, u, v) \end{pmatrix}$ with $f, g : \bar{\Omega} \times (\mathbb{R}^+)^2 \rightarrow \mathbb{R}$ locally lipschitzian continuous satisfying

$$f(x, 0, 0) > 0 \text{ or } g(x, 0, 0) > 0 \text{ for all } x \in \bar{\Omega} \quad (2)$$

and there is a positive constant C so that

$$0 \leq f(x, u, v), g(x, u, v) \leq C \text{ for all } (x, u, v) \in \bar{\Omega} \times (IR^+)^2. \quad (3)$$

Condition (3) says that the function

$$\hat{F}(x, U) = \begin{pmatrix} a(x)u + b(x)v + f(x, u, v) \\ c(x)u + d(x)v + g(x, u, v) \end{pmatrix}$$

is of asymptotically planar type. Since we are concerned with the existence of positive solutions we will suppose through this work that system (1) is cooperative, i.e., $b(x)$ and $c(x)$ are both nonnegative for all $x \in \bar{\Omega}$. This cooperativeness is imposed in order we may use a *Maximum Principle (MP in short)*. In particular we will deal with the one due to the author of this paper in collaboration with M A S Souto [1]. Using this (MP) and a result on *Bifurcation Theory* we prove the following:

THEOREM 1. *If $a(x) < \lambda_1$, $d(x) < \lambda_1$ and if either*

$$(i) \quad |a|_\infty \leq |d|_\infty \text{ and } 1 < \frac{2\lambda_1}{2|d|_\infty + |b|_\infty + |c|_\infty}$$

or

$$(ii) \quad |d|_\infty \leq |a|_\infty \text{ and } 1 < \frac{2\lambda_1}{2|a|_\infty + |b|_\infty + |c|_\infty}$$

then problem (1) possesses a positive (classical) solution

Here $|\cdot|_\infty$ denotes the usual sup norm, that is, $|u|_\infty = \sup_{x \in \bar{\Omega}} |u(x)|$ and λ_1 is the first eigenvalue of $-\Delta$ in Ω under Dirichlet boundary condition.

To tackle this theorem we proceed as follows: Since f and g are both defined only for $u, v \geq 0$ we ought to consider the extensions of f and g , respectively

$$f_1(x, u, v) = f(x, |u|, |v|) \text{ and } g_1(x, u, v) = g(x, |u|, |v|)$$

now defined for all $(x, u, v) \in \bar{\Omega} \times \mathbb{R}^2$. We now carry on by setting

$$L = \begin{pmatrix} -\Delta - a(x) & 0 \\ 0 & -\Delta - d(x) \end{pmatrix}, \bar{F}(x, U) = \begin{pmatrix} f_1(x, u, v) \\ g_1(x, u, v) \end{pmatrix}, B(x) = \begin{pmatrix} 0 & b(x) \\ c(x) & 0 \end{pmatrix}.$$

Fixing these notations we are going to pay attention to the following nonlinear eigenvalue problem

$$LU = \lambda [B(x)U + \bar{F}(x, U)] \text{ in } \Omega, U = 0 \text{ on } \partial\Omega \tag{4}$$

where $\lambda > 0$ is a real parameter and it will be proved the existence of a continuum $\Sigma \subset \mathbb{R}^+ \times [C(\bar{\Omega})]^2$ of solutions (λ, U) of (4) that begins at $(0, 0)$ and extends beyond the line $\{1\} \times [C(\bar{\Omega})]^2$ arising a solution of (1) which in view of the (MP), should be positive.

As we will show after proving Theorem 1 the motivation in studying problem (1) came of the scalar one

$$-\Delta u = f(x, u) \text{ in } \Omega, u = 0 \text{ on } \partial\Omega, \tag{5}$$

where f has a sublinear behavior.

2. PRELIMINARY RESULTS

In order to establish the (MP) we begin by fixing some notations. Let \mathbf{X} be a Banach space ordered by the positive cone $\mathbf{K} \subset \mathbf{X}$ and $\widehat{L} : \mathbf{X} \rightarrow \mathbf{X}$ a linear operator. By a (MP) to problem

$$U = \widehat{L}U + F, U \in \mathbf{X}, \tag{6}$$

we mean the statement $F \geq 0$ (i.e. $F \in \mathbf{K}$) imply $U \geq 0$ whenever that U is a solution of (6).

PROPOSITION 2 (Maximum Principle). *Let $\widehat{L} : \mathbf{X} \rightarrow \mathbf{X}$ be a positive linear compact operator (positive means $\widehat{L}(\mathbf{K}) \subset \mathbf{K}$). Then (6) satisfies the (MP) if the condition below holds true*

$$\{U \in \mathbf{X}, t \in [0, 1], U = t\widehat{L}U\} \Rightarrow U = 0. \tag{7}$$

Now we shall focus our attention on the problem

$$LU = B(x)U + \bar{F}(x, U) \text{ in } \Omega, U = 0 \text{ on } \partial\Omega, \tag{8}$$

to prove the following:

THEOREM 3. *If $a(x) < \lambda_1$, $d(x) < \lambda_1$ and if either*

$$(i) \quad |a|_\infty \leq |d|_\infty \text{ and } 1 < \frac{2\lambda_1}{2|d|_\infty + |b|_\infty + |c|_\infty}$$

or

$$(ii) \quad |d|_\infty \leq |a|_\infty \text{ and } 1 < \frac{2\lambda_1}{2|a|_\infty + |b|_\infty + |c|_\infty}$$

then every solution of (8) is positive and so is a solution of (1).

PROOF. We first observe that the extension $\bar{F}(x, U)$ is also nonnegative. Second we notice that the operator $L = \begin{pmatrix} -\Delta - a(x) & 0 \\ 0 & -\Delta - d(x) \end{pmatrix}$ has an inverse

$L = \begin{pmatrix} (-\Delta - a(x))^{-1} & 0 \\ 0 & (-\Delta - d(x))^{-1} \end{pmatrix} : [C(\bar{\Omega})]^2 \rightarrow [C(\bar{\Omega})]^2$ which is compact and positive in view of $a(x), d(x) < \lambda_1$ in $\bar{\Omega}$. So we will analyze uniqueness for the problem

$$U = tL^{-1}B(x)U, U \in [C(\bar{\Omega})]^2, t \in [0, 1]$$

that is equivalent to

$$\begin{cases} -\Delta u - a(x)u = tb(x)v & \text{in } \Omega \\ -\Delta v - d(x)v = tc(x)u & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases} \tag{9}$$

By multiplying both sides of the first equation in (9) by u and both sides of the second one by v and integrating by parts we obtain

$$\int |\nabla u|^2 = \int a(x)u^2 + t \int b(x)uv$$

and

$$\int |\nabla v|^2 = \int c(x)uv + \int d(x)v^2$$

Since a, b, c and d belong to $C(\bar{\Omega})$ one gets, thanks to both Holder's and Pioncare's inequalities,

$$\int |\nabla u|^2 \leq \frac{|a|_\infty}{\lambda_1} \int |\nabla u|^2 + \frac{|b|_\infty}{2\lambda_1} \left(\int [|\nabla u|^2 + |\nabla v|^2] \right)$$

and

$$\int |\nabla v|^2 \leq \frac{|c|_\infty}{2\lambda_1} \left(\int [|\nabla u|^2 + |\nabla v|^2] \right) + \frac{|d|_\infty}{\lambda_1} \int |\nabla v|^2.$$

Summing up these two inequalities and assuming that $|a|_\infty \leq |d|_\infty$ one has

$$\int |\nabla u|^2 + \int |\nabla v|^2 \leq \frac{1}{\lambda_1} \left[\frac{2|d|_\infty + |b|_\infty + |c|_\infty}{2} \right] \left[\int (|\nabla u|^2 + |\nabla v|^2) \right].$$

Since $1 < \frac{2\lambda_1}{2|d|_\infty + |b|_\infty + |c|_\infty}$ we conclude that $U = 0$. We arrive at the same conclusion by assuming assumption (ii). Thus system (8) enjoys the (MP) and in view of $\bar{F}(x, U) \geq 0$ we have $U \geq 0$ and so it is a solution of (1). \square

We now enunciate a proposition, due to Rabinowitz [5], which is another tool in proving Theorem 1.

PROPOSITION 4. *Let X be a Banach space and suppose that $T : \mathbb{R}^+ \times X \rightarrow X$ is a continuous map. Then the nonlinear eigenvalue problem $u = T(\lambda, u)$ possesses an unbounded continuum of solutions meeting $(0, 0) \in \mathbb{R} \times X$, if in addition, we suppose $T(0, u) = 0$ for all $u \in X$.*

3. MAIN RESULTS AND REMARKS

We start this section proving Theorem 1.

PROOF OF THEOREM 1. Set $X = [C(\bar{\Omega})]^2$ endowed with the usual norm $|U|_\infty = |u|_\infty + |v|_\infty$. Hence X is a Banach space and, as we said before, $L^{-1} : X \rightarrow X$ is linear, compact and positive. So problem (4) is equivalent to the following functional equation in $\mathbb{R}^+ \times X$:

$$U = \lambda[L^{-1}A(x)U + L^{-1}\tilde{F}(x, U)], \lambda \geq 0, U \in X, \tag{10}$$

where $\tilde{F}(x, U)$ is the Nemytskii operator associated with the function \bar{F} , i.e., for each $U \in X$ one has

$$\tilde{F}(\cdot, U(\cdot))(x) = \bar{F}(x, U(x)).$$

Since $L^{-1}A$ and $L^{-1}\tilde{F}$ are compact operators we are able to conclude the existence of an unbounded continuum Σ of solutions of (10) beginning at $(0, 0) \in \mathbb{R}^+ \times X$. If $(\lambda, 0) \in \Sigma$ then $\lambda = 0$ because $f(x, 0, 0) > 0$ or $g(x, 0, 0) > 0$. Plainly $(0, U) \in \Sigma$ implies $U = 0$. Thus Σ meets $\{0\} \times [C(\bar{\Omega})]^2$ and $\mathbb{R}^+ \times \{0\}$ only at $(0, 0)$. Note that bootstrapping these solutions, that at first sight belong only to $[C(\bar{\Omega})]^2$, we obtain classical solutions.

It is worthy to say that hitherto we cannot affirm that Σ contains positive solutions. In spite of this we can say that a piece (or perhaps pieces) of Σ contains only positive solutions. Indeed, if $\lambda \leq 1$ we may prove, reasoning as in the proof of Theorem 3, that every solution U of problem (10) is positive. It rests to show that in fact Σ reaches $\lambda = 1$

Since Σ is unbounded it may be unbounded with respect to λ , or with respect to U or with respect to both λ and U . If Σ is unbounded in λ then it crosses the line $\{1\} \times X$ and so we find a solution U of the problem (4) and in view of assumptions (i) and (ii) of Theorem 3 is positive and so is a solution of (1). We now suppose that if $(\lambda, U) \in \Sigma$ then $\lambda \leq 1$. Hence there is a sequence $(\lambda_n, U_n) \in \Sigma$ with $\lambda_n \leq 1$ and $|U_n|_\infty \rightarrow \infty$. Thus

$$LU_n = \lambda[B(x)U_n + \bar{F}(x, U_n)] \text{ in } \Omega, U_n = 0 \text{ on } \partial\Omega.$$

Setting $W_n = \frac{U_n}{|U_n|_\infty}$ we obtain

$$LW_n = \lambda_n \left[B(x)W_n + \frac{\bar{F}(x, U_n)}{|U_n|_\infty} \right] \text{ in } \Omega, W_n = 0 \text{ on } \partial\Omega.$$

Passing to a subsequence if necessary we obtain $\lambda_n \rightarrow \lambda_0 \in [0, 1]$, $W_n \rightarrow W$ in $[C(\bar{\Omega})]^2$ and $LW_n \rightarrow \lambda_0 B(x)W$. As $L : D(L) \rightarrow [C(\bar{\Omega})]^2$, where $D(L) = \{U \in [C(\bar{\Omega})]^2; LU \in [C(\bar{\Omega})]^2 \text{ and } U = 0 \text{ on } \partial\Omega\}$, is closed one has that $W \in D(L)$ and

$$LW = \lambda_0 A(x)W \text{ in } \Omega, W = 0 \text{ on } \partial\Omega.$$

Because $W_n \rightarrow W$ in $[C(\bar{\Omega})]^2$ and $|W_n|_\infty = 1$ then $|W|_\infty = 1$, i.e., W is a nontrivial solution of the above problem. But, in view of (MP) and $\lambda_0 \leq 1$, $W \equiv 0$ which is absurd. Thus Σ crosses $\{1\} \times X$ and, by Theorem 3, such solution is positive and the proof of Theorem 1 is over. \square

REMARK 1. The proof of Theorem 1 rests heavily on the existence of a (MP) like the one contained in [1]. We must observe that this (MP) is valid for a more general elliptic operator. Indeed, if we consider uniformly elliptic operators in the divergence form

$$L_k = -D_j(a_{ij}^k D_i u) + D_j(a_j^k u), k = 1, 2$$

(the symbols of summation are implicit in the expressions) where coefficients are regular enough, $a_{ij}^k = a_{ji}^k$ and setting $\lambda_1(L_k)$ as being the first eigenvalue of $(L_k, H_0^1(\Omega))$, the system

$$\mathcal{L}U = A(x)U + F(x) \text{ in } \Omega, U = 0 \text{ on } \partial\Omega$$

where $\mathcal{L} = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$, enjoys the (MP) if

$$(B(x)\xi, \xi) < \lambda_1(L_k)(\xi_1^2 + \xi_2^2)$$

for all $\xi = (\xi_1, \xi_2)$ and $x \in \bar{\Omega}$. Here (\cdot, \cdot) is the usual inner product in \mathbb{R}^2 and $B(x) = \begin{pmatrix} 0 & b(x) \\ c(x) & 0 \end{pmatrix}$

Note that the above condition provides the uniqueness required by the (MP) in [1]. So Theorems 1 and 3 remain valid, with slight modifications, if system (1) is considered with $-\Delta$ substituted by the nonselfadjoint operators L_1 and L_2 .

REMARK 2. At the outset of our motivations in studying problem (1) we had considered the following

$$Lu := - \sum_{i,j=1}^N a_{i,j}(x)D_{i,j}u + \sum_{i,j=1}^N a_i(x)D_i u = f(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{11}$$

where L is a second order uniformly elliptic operator in Ω with real smooth coefficients satisfying $a_{ij} = a_{ji}$ in $\bar{\Omega}$, for all $1 \leq i, j \leq N$, and $f : \bar{\Omega} \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a sublinear nonlinearity. It is to say, setting

$$a_0(x) = \lim_{t \rightarrow 0^+} \frac{f(x, t)}{t}, \quad a_\infty(x) = \limsup_{t \rightarrow \infty} \frac{f(x, t)}{t} \tag{12}$$

one must have

$$\lambda_1(a_0) < 1 < \lambda_1(a_\infty) \tag{13}$$

where $\lambda_1(a_i)$, $i = 0, \infty$, is the first eigenvalue of the linear eigenvalue problem

$$Lu = \lambda a_i(x)u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Condition (13) says that we are working with a sublinear problem, i.e, in case, for instance, a_0 and a_∞ are constants the nonlinearity f begins, above the straight line $\lambda_1 t$ and at the end it remains below the same line.

In Brezis-Oswald [2] the authors consider $L = -\Delta$ and use Variational Methods by exploring the selfadjointness of $-\Delta$ and f is not necessarily a smooth function. In fact $a_0(x)$ and $a_\infty(x)$ may take values $+\infty$ and $-\infty$, respectively, so we address the reader to Section 3 of [2] for the precise meaning of (13).

In de Figueiredo [3] problem (11) is studied under condition (13) where L is a selfadjoint operator more general than $-\Delta$ but f is a C^α -function, $0 < \alpha < 1$, and $f(x, t) + Kt$ is nondecreasing in t for some $K \geq 0$. In this case the sub and supersolution method is used.

If L is not necessarily selfadjoint problem (11) was studied by Costa-Gonçaves [4] under condition (13), still using the sub and supersolution technique. In the works quoted above the authors always show existence of a positive solution as well as give sufficient condition for uniqueness.

This scalar problem arises a very natural question: How can we formulate a sublinear problem like before when we take a system into account?

We think that the best motivation towards a more general situation is to consider the biharmonic problem because it brings up for attention a very simple system and from it we would deal a more general sublinear problem. More precisely we first analyze the simplest biharmonic problem

$$\Delta^2 u = mu + g(u) \text{ in } \Omega, \quad u = \Delta u = 0 \text{ on } \partial\Omega, \tag{14}$$

that is, the biharmonic equation under the so called Navier boundary conditions. Here m is a positive constant. Setting $v = -\Delta u$, $A = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$, $G(U) = \begin{pmatrix} 0 \\ g(u) \end{pmatrix}$ we get the system

$$-\Delta U = AU + G(U) \text{ in } \Omega, \quad U = 0 \text{ on } \partial\Omega. \tag{15}$$

Taking g a bounded function then $f(u) = mu + g(u)$ would be sublinear if it begins at $t = 0$ zero above $\lambda_1^2 t$ and remains below $\lambda_1^2 t$ for t large enough. Note that this is the counterpart of condition (13) when we are dealing with Δ^2 . Observe that λ_1^2 is the first eigenvalue of Δ^2 in Ω under Navier boundary conditions and the situation described above occurs, for instance, if $g(0) > 0$ and $m < \lambda_1^2$.

REMARK 3. Now we are going to analyze the condition given in Theorem 1 for the system (15). In this case one has that $\frac{1+m}{2} < \lambda_1$ is a sufficient condition in order system (15) enjoys the (MP).

Next we will show that this condition leads to a sublinear problem related to (14). Let us suppose that $\frac{1+m}{2} < \lambda_1$

- a) If $m \neq 1$ one has $(m-1)^2 > 0$ which implies $\frac{(1+m)^2}{4} > m$ and since $\lambda_1 > \frac{1+m}{2}$ we get $\lambda_1^2 > \frac{(1+m)^2}{4} > m$ and so we have a sublinear problem.
- b) If $m = 1$ then $\lambda_1 > \frac{1+m}{2} = 1$ and hence $\lambda_1^2 > \lambda_1 > 1 = m$. In this case we still have a sublinear problem.

Therefore we believe that conditions (i) and (ii) are two kinds of sublinearity conditions when we deal with a system of two equations.

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