

ON THE CAUCHY PROBLEM FOR A DEGENERATE  
PARABOLIC DIFFERENTIAL EQUATION

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**ABSTRACT.** The aim of this work is to prove the existence and the uniqueness of the solution of a degenerate parabolic equation. This is done using H. Tanabe and P.E. Sobolevskii theory.

**KEY WORDS AND PHRASES:** Cauchy problem-Degenerate parabolic  
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## 1- INTRODUCTION

We are concerned with the Cauchy problem for the equation

$$\frac{\partial u}{\partial t} - A(x, t, D)u = f(x, t), \quad (x, t) \in R^n \times [0, T], \quad (1.1)$$

with the initial data

$$u(x, 0) = u_0(x) \quad (1.2)$$

Here we take the operator  $A(x, t, D)$  in the form

$$A(x, t, D) = \sum_{j, k=1}^n \frac{\partial}{\partial x_j} (a_{jk}(x, t)) \frac{\partial}{\partial x_k} - \sum_{j=1}^n b_j(x, t) \frac{\partial}{\partial x_j} - C(x, t) \quad (1.3)$$

Assume that  $(a_{jk}(x, t))_{1 \leq j, k \leq n}$ ,  $b_j(x, t)$  and  $C(x, t)$  are real-valued smooth functions in  $x$  and that they are Hölder continuous in  $t$ . Moreover  $(a_{jk}(x, t))$  is assumed to be symmetric and to satisfy the following condition

$$Re \sum_{j, k=1}^n a_{jk}(x, t) \xi_j \xi_k \geq 0, \quad \xi \in R^n. \quad (1.4)$$

Assume also that  $f(x, t)$  satisfies, for some  $\sigma \in (0, 1]$

$$|f(x, t) - f(x, \tau)| \leq c|t - \tau|^\sigma \quad (1.5)$$

for all  $t, \tau \in [0, T]$ , where  $c$  is a positive constant.

Historically, O.A. Oleinik has studied this problem [4]. Her method was elliptic regularization.

In [1] A. El-Fiky also studied non degenerate p-parabolic systems. Also K. Igari [5] has studied this problem by using Friedrichs mollifier.

On the other hand, H. Tanabe [3] and P.E. Sobolevskii [2] have considered the following evolution equation

$$(p) \begin{cases} \frac{dv}{dt} + A(t)v = f(t) \\ v(0) = v_0 \end{cases}$$

and the following conditions:

- 1) A is a linear closed operator acting on a Banach space E and its domain of definition D is dense and independent of t.
- 2) The operator  $(\lambda I + A)$  has a bounded inverse satisfying

$$\|(\lambda I + A)^{-1}\| \leq \frac{c_1}{|\lambda| + 1}$$

for any  $\lambda$  with  $\text{Re } \lambda \geq \beta > 0$ , where  $c_1$  and  $\beta$  are positive constants.

- 3) There exists a positive constant  $c_2$  such that, for some  $\sigma \in (0,1]$

$$\|(A(t) - A(\tau))A_p^{-1}(s)\| \leq c_2 |t - \tau|^\sigma$$

holds for some  $t, \tau, s \in [0, T]$ , where  $A_p(s) = A(s) + \beta I$ .

- 4) The function  $f(t)$  satisfies the following Hölder condition

$$\|f(t) - f(\tau)\| \leq c_3 |t - \tau|^\sigma$$

where  $c_3$  is a positive constant.

They proved that for any  $v_0 \in E$ , there exists a unique solution  $v(x,t)$  for (p) which is continuous for all  $t \in (0, T]$  and continuously differentiable for  $t > 0$ . In case  $v_0 \in D(A)$  the solution is continuously differentiable for  $t=0$  also.

In this article we shall show that the result of H. Tanabe and P.E. Sobolevskii can be applied to problem (1.1) - (1.2). Our goal is to show that the operator  $A(x,t;D)$  which is defined in (1.3) satisfies conditions 1), 2) and 3) mentioned above.

## 2. PROPOSITIONS AND THEOREM

In this section we state and prove two propositions from which our main theorem follows.

**Proposition 1.** Take the domain of definition  $D(A)$  of the operator A as follows:

$$D(A) = \{u; u \in L^2, Au \in L^2\} \quad (2.1)$$

Then, for large  $\lambda$ ,  $(\lambda I - A)$  defines a one-to-one surjective mapping of  $D(A)$  onto  $L^2$ . Moreover there exists a constant  $\alpha$  such that

$$\| (\lambda I - A)^{-1} \| \leq \frac{1}{\lambda - \alpha} \text{ for any } \lambda > \alpha, \tag{2.2}$$

**Proof.** For any  $u \in D(A)$  it holds that

$$\| (\lambda I - A) u \| \geq (\lambda^2 - \text{const. } \lambda) \| u \|^2 + \| A u \|^2 \tag{2.3}$$

Indeed

$$\begin{aligned} \| (\lambda I - A) u \|^2 &= ( (\lambda I - A) u, (\lambda I - A) u ) \\ &= \lambda^2 \| u \|^2 + \| A u \|^2 - 2 \lambda \text{Re} ( A u, u ) \end{aligned} \tag{2.4}$$

Using the condition (1.4), we have

$$2 \text{Re} \left( \frac{\partial}{\partial x_j} ( a_{jk} \frac{\partial}{\partial x_k} ) u, u \right) = -2 \left( a_{jk} \frac{\partial u}{\partial x_k}, \frac{\partial u}{\partial x_j} \right) \leq 0 \tag{2.5}$$

Similar arguments can be applied to the remaining two terms of the operator  $A$ , under the condition that  $C$  is uniformly bounded. Hence we obtain (2.3).

The inequality (2.3) shows that, for large  $\lambda$ ,  $(\lambda I - A)$  defines a one-to-one closed mapping of  $D(A)$  into  $L^2$ . Therefore we have only to show that the image  $(\lambda I - A) D(A)$  is dense in  $L^2$ . We show this by contradiction. Assume  $(\lambda I - A) D(A)$  is not dense in  $L^2$ . There exists  $\psi (\neq 0)$  in  $L^2$  such that

$$( (\lambda I - A) u, \psi ) = 0 \text{ for every } u \in D(A).$$

Hence, as  $D(A)$  is dense in  $L^2$ ,

$$(\lambda I - A^*) \psi = 0, \tag{2.6}$$

where  $A^*$  is the formal adjoint of  $A$ .

Since  $\psi \in L^2$ , (2.6) shows  $A^* \psi \in L^2$ . If we note that  $A^*$  satisfies the same conditions as  $A$ , we can use the inequality (2.3) to obtain

$$0 = \| (\lambda I - A^*) \psi \|^2 \geq (\lambda^2 - \text{const. } \lambda) \| \psi \|^2 \tag{2.7}$$

For large  $\lambda$ , this inequality requires the  $\psi = 0$ . This is contradictory to our assumption  $\psi \neq 0$ .

Thus the proof is complete

**Proposition 2.** Assume all the coefficients in (1.1) are smooth in  $x$  and Hölder continuous in  $t$ . Then

$$\| [A(t) - A(\tau)] A_\beta^{-1}(s) \| \leq c |t - \tau|^\sigma$$

holds for any  $t, \tau, s \in (0, T]$ .

**Proof.** For any  $\beta > \alpha$  and from proposition 1,  $A_\beta(s)$  is a one-to-one linear mapping from  $D(A)$  onto  $L^2$ . Moreover, it satisfies.

$$\| A_\beta(x, s, d) u \| \geq c_4 \| u \| \tag{2.8}$$

where  $c_4$  is a positive constant. This implies that

$$|V| \geq c_4 |A_p^{-1}(x, s, D) V|$$

Since all the coefficients appearing in (1.1) are assumed to be smooth in  $x$  and Hölder continuous in  $t$ . So, we have

$$\begin{aligned} & \left| [A(x, t, D) - A(x, \tau, D)] A_p^{-1}(x, s, D) V \right| \\ & \leq c_2 |t - \tau|^\sigma |A_p^{-1}(x, s, D) V| \\ & \leq c_2 c_4^{-1} |t - \tau|^\sigma |V|. \end{aligned}$$

Thus the proof is complete

The above propositions show that all condition of H. Tanabe and P.E. Sobolevskii are satisfied. Therefore, we have the following theorem.

**THEOREM:** For any initial data  $u_0 \in L_2$  and any right-hand side  $f(t)$  satisfying Hölder condition (1.5), there exists a unique solution  $u(x, t)$  for the Cauchy problem (1.1)-(1.2) belonging to the space  $C_1^0([0, T], L^2) \cap C_1^1([0, T], L^2)$ .

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