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A NOTE ON COMPLEMENTARITY PROBLEM

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ABSTRACT. In this paper we prove a result of complementarity problem where compact condition is somewhat relaxed.

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Recently several interesting results have been given for complementarity problems As the complementarity problem, variational inequality and fixed point theory are closely related (equivalent to each other) that is why it has growing interest and varied applications. The applications in the field of economics, optimization, game theory, mechanics and engineering are even growing rapidly

Here we will start with implicit complementarity problem and then derive results for complementarity theory. For terminology one is referred to [1]

Let $\langle E, E^* \rangle$ be a dual system of a locally convex space. Let $K \subset E$ be a closed convex cone We denote by K^* the dual cone of K, that is

$$K^* = \{y \in E^* : \langle x, y\rangle \geq 0 \text{ for all } x \in K\}.$$

Let $f : K \to E^*$ and $g : K \to K$. Then the implicit complementarity problem is as follows.

Find an $x_0 \in K$ such that $gx_0 \in K$, $fx_0 \in K^*$ and

$$(\alpha) \qquad \langle gx_0, fx_0 \rangle = 0 \ldots .$$

The corresponding variational inequality will be as given below.

(β) Find $x_0 \in K$ such that $gx_0 \in K$ and $\langle x - gx_0, fx_0\rangle \geq 0$ for all $x \in K$

In [1] it is shown that (α) and (β) are equivalent.

Indeed, if (β) holds then by taking $gx_0 = 0$, we get $\langle x, fx_0\rangle \geq 0$ for all $x \in K$, so $fx_0 \in K^*$

Also, if $x = 0$ we get $\langle gx_0, fx_0\rangle \leq 0$ and if $x = 2gx_0$, then $\langle gx_0, fx_0\rangle \geq 0$.

Thus $\langle gx_0, fx_0\rangle = 0$ and (α) is obtained.

In case (α) holds then $gx_0 \in K$, $fx_0 \in K^*$ and $\langle gx_0, fx_0\rangle = 0$.

Since $fx_0 \in K^*$ so $\langle x, fx_0\rangle \geq 0$ for all $x \in K$.

Hence $\langle x - gx_0, fx_0\rangle \geq 0$ for all $x \in K$ and (β) holds.

The following result given in [2] extends and unifies results due to park and Kim [3], Takahashi [4] and Chitra and Subrahmanyan [5]. The proof in [2] is based on the KKM-map principle (for details see Granas [6]).

THEOREM 1. Let C be a nonempty convex subset of a topological vector space X. Let $A \subset C \times C$ and $g : C \to C$ such that the following conditions are satisfied.

i) $(x, gx) \in A$ for all $x \in C$;

ii) for each $y \in C$, the set $\{x \in C : (x, gy) \notin A\}$ is convex or empty,

iii) for each $x \in C$, the set $\{y \in C : (x, gy) \in A\}$ is closed in C;

iv) C has a nonempty compact convex subset C_0 such that the set $D = \{y \in C : (x, gy) \in A$ for all $x \in C_0\}$ is compact

Then there exists a $y_0 \in C$ such that $C \times \{gy_0\} \subset A$.

NOTE. If $g = I$, an identity function, then one gets that $C \times \{y_0\} \subset A$ [7].

In case C is compact convex and g is an identity map then we have the following

COROLLARY 1. Let C be a nonempty compact convex subset of a topological vector space X and let $A \subset C \times C$ have the following:

i) $(x, x) \in A$ for each $x \in C$,

ii) for each $y \in C$ the set $\{x \in C : (x, y) \notin A\}$ is convex or empty;

iii) for each $x \in C$ the set $\{y \in C : (x, y) \in A\}$ is closed in C.

Then there is $y_0 \in C$ such that $C \times \{y_0\} \subset A$.

We will make use of Theorem 1 to prove the following result in complementary problem. First we give our result for variational inequality and then as a consequence derive complementarity problem

THEOREM 2. Let D be a nonempty convex subset of a topological vector space E If $f : D \to E^*$ and $g : D \to D$ are continuous functions such that

$$\langle gx, fx \rangle \le \langle x, fx \rangle \quad \text{for all} \quad x \in D.$$

Let D have a nonempty compact convex subset D_0 such that the set

$$B = \{y \in D : \langle gy, fy \rangle \ge 0 \quad \text{for all} \quad x \in D_0\}$$

is compact

Then there is $y_0 \in D$ such that

$$\langle x - gy_0, fy_0 \rangle \ge 0 \quad \text{for all} \quad x \in D.$$

PROOF. Let $A = \{(x, y) \in D \times D : \langle x - gy, fy \rangle \ge 0\}$. Then $(x, x) \in A$ by hypothesis

Since f and g are continuous and the bilinear form $\langle\, , \rangle$ is continuous therefore the set

$$\{y \in D : \langle x - gy, fy \rangle \ge 0\}$$

is closed for every $x \in D$

The set

$$A_x = \{x \in D : (x, y) \notin A\} = \{x \in D : \langle x - gy, fy \rangle < 0\}$$

is convex.

Indeed, if x_1 and x_2 are in A_x then $z = \lambda x_1 + (1 - \lambda)x_2 \in A_x$, where $0 \le \lambda \le 1$.

We write

$$\begin{aligned}\langle z - gy, fy \rangle &= \langle \lambda x_1 + (1 - \lambda)x_2 - gy, fy \rangle \\ &= \langle \lambda x_1 - gy, fy \rangle + \langle (1 - \lambda)x_2 - gy, fy \rangle \\ &= \lambda \langle x_1 - gy, fy \rangle + (1 - \lambda)\langle x_2 - gy, fy \rangle \\ &< \lambda .0 + (1 - \lambda) \cdot 0 = 0.\end{aligned}$$

So $\langle z - gy, fy \rangle < 0$; and A_x is convex.

Thus conditions of Theorem 1 are satisfied and there is a $y_0 \in D$ such that

$$\langle x - gy_0, fy_0 \rangle \ge 0 \quad \text{for all} \quad x \in D.$$

Noting that the variational inequality is equivalent to the complementarity problem we derive the following

Under the hypotheses of Theorem 2 we get that there is a $y_0 \in D$ such that $gy_0 \in D, fy_0 \in D^*$ and $\langle gy_0, fy_0 \rangle = 0$

In case $g = I$, an identity function, then we get a complementarity problem, that is, there is a $y_0 \in D$ such that $fy_0 \in D^*$ and $\langle y_0, fy_0 \rangle = 0$.

In case D is also a compact set then the following result is obtained as a corollary This is due to Isac [8].

COROLLARY 2. Let D be a nonempty compact convex subset of E and $f : D \to E^*, g : D \to D$ continuous maps. If $\langle x - gx, fx \rangle \geq 0$ for all $x \in D$, then there is a $y_0 \in D$ such that

$$\langle x - gy_0, fy_0 \rangle \geq 0 \quad \text{for all} \quad x \in D.$$

We note that E need not be a Hausdorff space.

In case $\langle E, E^* \rangle$ is a dual system of Banach space E, K is a closed convex cone in E and D is a nonempty compact convex subset of K, then we obtain the following

THEOREM 3. Let $f, g : K \to E^*$ satisfy the following:

i) $\langle gx, fx \rangle \geq \langle x, fx \rangle$ for every $x \in D$;

ii) for each sequence $\{y_n\}$ in D weakly converging to y_0, then

$$\liminf \langle x, fx_n \rangle \leq \langle x, fy_0 \rangle \quad \text{for every} \quad x \in D;$$

iii) $x \to \langle gx, fx \rangle$ is sequentially weakly lower semicontinuous on D. Then there is an $x_0 \in D$ such that

$$\langle x - gx_0, fx_0 \rangle \geq 0 \quad \text{for all} \quad x \in D.$$

In the proof of this result one can use Corollary 1 in the setting of weak topology.

PROOF. Let $a = \{(x, y) \in D \times D : \langle gy - x, fy \rangle \leq 0\}$.

Then, on the same lines as given in [1] one gets that

i) $(x, x) \in A$ for every $x \in D$;

ii) the set $\{x \in D : (x, y) \notin A\}$ is convex for every $y \in D$;

iii) the set $\{y \in D : (x, y) \in D\}$ is weakly closed.

Hence the result follows from Corollary 1 in the weak topology sense.

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