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ON A NEW ABSOLUTE SUMMABILITY METHOD

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ABSTRACT. A theorem concerning some new absolute summability method is proved. Many other results, some of them known, are deduced.

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1. INTRODUCTION

Let $\Sigma\, a_n$ be an infinite series with partial sums s_n Let σ_n^δ and η_n^δ denote the nth Cesàro mean of order $\delta(\delta > -1)$ of the sequences $\{s_n\}$ and $\{na_n\}$ respectively. The series $\Sigma\, a_n$ is said to be summable $|C,\delta|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}\left|\sigma_n^\delta - \sigma_{n-1}^\delta\right|^k < \infty,$$

or equivalently

$$\sum_{n=1}^{\infty} n^{-1}\left|\eta_n^\delta\right|^k < \infty.$$

Let $\{p_n\}$ be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty\ (P_{-i} = p_{-i} = 0, i \geq 1).$$

The series $\Sigma\, a_n$ is said to be summable $|\overline{N}, p_n|_k$, $k \geq 1$, if (Bor [1]),

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1}|t_n - t_{n-1}|^k < \infty\,,$$

where

$$t_n = P_n^{-1}\sum_{v=0}^{n} p_v s_v.$$

The series $\Sigma\, a_n$ is said to be summable $|R, p_n|_k$, $k \geq 1$, if Bor [2],

$$\sum_{n=1}^{\infty} n^{k-1}|t_n - t_{n-1}|^k < \infty,$$

If we take $p_n = 1$, each of the two summabilities $|\overline{N}, p_n|_k$ and $|R, p_n|_k$ is the same as $|C, 1|_k$ summabililty Let $\{\phi_n\}$ be any sequence of positive numbers. The series $\Sigma\, a_n$ is said to be summable $|\overline{N}, p_n, \phi_n|_k$, $k \geq 1$, if (Sulaiman [3]),

$$\sum_{n=1}^{\infty} \phi_n^{k-1} |t_n - t_{n-1}|^k < \infty .$$

It is clear that

$$|\overline{N}, p_n, n|_k = |R, p_n|_k, |\overline{N}, p_n, P_n/p_n|_k = |\overline{N}, p_n|_k, |\overline{N}, p_n, 1|_1 = |\overline{N}, p_n|, \text{ and } |\overline{N}, 1, n|_k = |C, 1|_k.$$

We assume $\{\alpha_n\}$, $\{\beta_n\}$ and $\{q_n\}$ be sequences of positive numbers such that

$$Q_n = \sum_{v=0}^{n} q_v \to \infty.$$

We prove the following.

THEOREM 1. Let t_n denote the $(\overline{N}, p_n)$-mean of the series $\Sigma\, a_n$ and write $T_n = \beta_n^{1-1/k} \Delta\, t_{n-1}$. If

$$\sum_{n=v+1}^{\infty} \frac{\alpha_n^{k-1} q_n^k}{Q_n^k Q_{n-1}} = O\left\{ \frac{\alpha_v^{k-1} q_v^{k-1}}{Q_v^k} \right\}, \tag{I}$$

$$\sum_{n=1}^{\infty} \left(\frac{\alpha_n}{\beta_n}\right)^{k-1} \left(\frac{P_n}{p_n}\right)^k \left(\frac{q_n}{Q_n}\right)^k |\epsilon_n|^k\, |T_n|^k < \infty,$$

$$\sum_{n=1}^{\infty} \left(\frac{\alpha_n}{\beta_n}\right)^{k-1} |\epsilon_n|^k\, |T_n|^k < \infty,$$

and

$$\sum_{n=1}^{\infty} \left(\frac{\alpha_n}{\beta_n}\right)^{k-1} \left(\frac{P_{n-1}}{p_n}\right)^k |\Delta\, \epsilon_n|^k\, |T_n|^k < \infty,$$

then the series $\Sigma\, a_n\, \epsilon_n$ is summable $|\overline{N}, q_n, \alpha_n|_k$, $k \geq 1$.

2. PROOF OF THEOREM 1

Let τ_n be the $(\overline{N}, q_n)$-mean of the series $\Sigma\, a_n\, \epsilon_n$. Then

$$\begin{aligned}
\Delta \tau_{n-1} &= -\frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} a_v \epsilon_v \\
&= -\frac{q_n}{Q_n Q_{n-1}} \left\{ \sum_{v=1}^{n-1} \sum_{r=1}^{n} P_{r-1} a_r \Delta (P_{v-1}^{-1} Q_{v-1} \epsilon_v) + \left(\sum_{r=1}^{n} P_{r-1} a_r \right) P_{n-1}^{-1} Q_{n-1} \epsilon_n \right\} \\
&= -\frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v P_{v-1}}{p_v} \beta_v^{1/k-1} T_v \right) \left(-q_v P_{v-1}^{-1} \epsilon_v + \frac{p_v}{P_v P_{v-1}} Q_v \epsilon_v + Q_v P_v^{-1} \Delta\, \epsilon_v \right) \\
&\quad + \left(\frac{P_n}{p_n}\right) \left(\frac{q_n}{Q_n}\right) \epsilon_n \beta_n^{1/k-1} T_n \\
&= -\frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \left(-\frac{P_v}{p_v} q_v \epsilon_v \beta_v^{1/k-1} T_v + Q_v \epsilon_v \beta_v^{1/k-1} T_v + \frac{P_{v-1}}{p_v} Q_v\, \Delta\, \epsilon_v \beta_v^{1/k-1} T_v \right) \\
&\quad + \left(\frac{P_n}{p_n}\right) \left(\frac{q_n}{Q_n}\right) \epsilon_n \beta_n^{1/k-1} T_n \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad \text{say.}
\end{aligned}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \alpha_n^{k-1} |T_{n,r}|^k < \infty, \quad r = 1,2,3,4.$$

Applying Hölder's inequality, with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \alpha_n^{k-1} |T_{n,1}|^k &= \sum_{n=2}^{m+1} \alpha_n^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}} \right)^k \left| \sum_{v=1}^{n-1} - \frac{P_v}{p_v} q_v \epsilon_v \beta_v^{1/k-1} T_v \right|^k \\
&\leq \sum_{n=2}^{m+1} \alpha_n^{k-1} \left(\frac{q_n}{Q_n} \right)^k Q_{n-1}^{-1} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \right)^k q_v |\epsilon_v|^k \beta_v^{1-k} |T_v|^k \left\{ \frac{1}{Q_{n-1}} \sum_{v=1}^{n-1} q_v \right\}^{k-1} \\
&\leq 0(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^k q_v |\epsilon_v|^k \beta_v^{1-k} |T_v|^k \sum_{n=v+1}^{m+1} \alpha_n^{k-1} \left(\frac{q_n}{Q_n} \right)^k Q_{n-1}^{-1} \\
&\leq 0(1) \sum_{v=1}^{m} \left(\frac{\alpha_v}{\beta_v} \right)^{k-1} \left(\frac{P_v}{p_v} \right)^k \left(\frac{q_v}{Q_v} \right)^k |\epsilon_v|^k |T_v|^k.
\end{aligned}$$

$$\begin{aligned}
\sum_{n=2}^{m+1} \alpha_n^{k-1} |T_{n,2}|^k &= \sum_{n=2}^{m+1} \alpha_n^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}} \right)^k \left| \sum_{v=1}^{n-1} \frac{Q_v}{q_v} q_v \epsilon_v \beta_v^{1/k-1} T_v \right|^k \\
&\leq \sum_{n=2}^{m+1} \alpha_n^{k-1} \left(\frac{q_n}{Q_n} \right)^k Q_{n-1}^{-1} \sum_{v=1}^{n-1} \left(\frac{Q_v}{q_v} \right)^k q_v |\epsilon_v|^k \beta_v^{1-k} |T_v|^k \left\{ \frac{1}{Q_{n-1}} \sum_{v=1}^{n-1} q_v \right\}^{k-1} \\
&\leq 0(1) \sum_{v=1}^{m} \left(\frac{Q_v}{q_v} \right)^k q_v |\epsilon_v|^k \beta_v^{1-k} |T_v|^k \sum_{n=v+1}^{m+1} \alpha_n^{k-1} \left(\frac{q_n}{Q_n} \right)^k Q_{n-1}^{-1} \\
&\leq 0(1) \sum_{v=1}^{m} \left(\frac{\alpha_v}{\beta_v} \right)^{k-1} |\epsilon_v|^k |T_v|^k.
\end{aligned}$$

$$\begin{aligned}
\sum_{n=2}^{m+1} \alpha_n^{k-1} |T_{n,3}|^k &= \sum_{n=2}^{m+1} \alpha_n^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}} \right)^k \left| \sum_{v=1}^{n-1} - \frac{P_{v-1}}{p_v} \frac{Q_v}{q_v} q_v \Delta \epsilon_v \beta_v^{1/k-1} T_v \right|^k \\
&\leq \sum_{n=2}^{m+1} \alpha_n^{k-1} \left(\frac{q_n}{Q_n} \right)^k Q_{n-1}^{-1} \sum_{v=1}^{n-1} \left(\frac{P_{v-1}}{p_v} \right)^k \left(\frac{Q_v}{q_v} \right)^k q_v |\Delta \epsilon_v|^k \beta_v^{1-k} |T_v|^k \left\{ \frac{1}{Q_{n-1}} \sum_{v=1}^{n-1} q_v \right\}^{k-1} \\
&\leq 0(1) \sum_{v=1}^{m} \left(\frac{P_{v-1}}{p_v} \right)^k \left(\frac{Q_v}{q_v} \right)^k q_v |\Delta \epsilon_v|^k \beta_v^{1-k} |T_v|^k \sum_{n=v+1}^{m+1} \alpha_n^{k-1} \left(\frac{q_n}{Q_n} \right)^k Q_{n-1}^{-1} \\
&\leq 0(1) \sum_{v=1}^{m} \left(\frac{\alpha_v}{\beta_v} \right)^{k-1} \left(\frac{P_{v-1}}{p_v} \right)^k |\Delta \epsilon_v|^k |T_v|^k.
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{m} \alpha_n^{k-1} |T_{n,4}|^k &= \sum_{n=1}^{m} \alpha_n^{k-1} \left| \left(\frac{P_n}{p_n} \right) \left(\frac{q_n}{Q_n} \right) \epsilon_n \beta_n^{1/k-1} T_n \right|^k \\
&\leq 0(1) \sum_{n=1}^{m} \left(\frac{\alpha_n}{\beta_n} \right)^{k-1} \left(\frac{P_n}{p_n} \right)^k \left(\frac{q_n}{Q_n} \right)^k |\epsilon_n|^k |T_n|^k.
\end{aligned}$$

3. APPLICATIONS

THEOREM 2. If

$$\alpha_n = 0(\beta_n), \quad P_n q_n = 0(p_n Q_n),$$

and (I) is satisfied, then the series $\Sigma\, a_n$ is summable $|\overline{N}, q_n, \alpha_n|_k$, whenever it is summable $|\overline{N}, p_n, \beta_n|_k$, $k \geq 1$, (that is $|\overline{N}, p_n, \beta_n|_k \Rightarrow |\overline{N}, q_n, \alpha_n|_k$).

PROOF. Follows from Theorem 1 by putting $\epsilon_n = 1$.

COROLLARY 1 (Bor [1] and [4]). If

$$nq_n = 0(Q_n), \quad Q_n = 0(nq_n),$$

then the series Σa_n is summable $|\overline{N}, q_n|_k$ iff it is summable $|C, 1|_k$, $k \geq 1$

PROOF. Applying Theorem 2 with $\alpha_n = Q_n/q_n$, $\beta_n = n$, and $p_n = 1$. We have $\alpha_n = 0(\beta_n)$, $P_n q_n = 0(p_n Q_n)$, and (I) is satisfied. Therefore $|C, 1|_k \Rightarrow |\overline{N}, q_n|_k$. Now the same application of Theorem 2 with $\alpha_n = n$, $\beta_n = P_n/p_n$, we obtain the other way round.

COROLLARY 2 (Bor and Thorpe [5]). If

$$P_n q_n = 0(p_n Q_n), \quad p_n Q_n = 0(P_n q_n) \tag{II}$$

then the series Σa_n is summable $|\overline{N}, q_n|_k$ iff it is summable $|\overline{N}, p_n|_k$, $k \geq 1$.

PROOF. Applying Theorem 2 with $\alpha_n = Q_n/q_n$, $\beta_n = P_n/p_n$. Clearly $\alpha_n = 0(\beta_n)$ and (I) is satisfied. Therefore $|\overline{N}, p_n|_k \Rightarrow |\overline{N}, q_n|_k$. The result is still valid if we interchange $\{p_n\}$ and $\{q_n\}$

COROLLARY 3. Suppose that (I) is satisfied for p and q, (II) is also satisfied and that $\{q_n/p_n\}$ is nonincreasing, then the series Σa_n is summable $|R, q_n|_k$ iff it is summable $|R, p_n|_k$, $k \geq 1$.

PROOF. Applying Theorem 2 with $\alpha_n = \beta_n = n$. It is clear that $|R, p_n|_n \Rightarrow |R, q_n|_k$. For the other direction, it needs to be shown that (I) is satisfied if we are replacing q_n by p_n Since $\{q_n/p_n\}$ is nonincreasing, we have, using (II),

$$\begin{aligned}
\sum_{n=v+1}^{\infty} \frac{n^{k-1} p_n^k}{P_n^k P_{n-1}} &= 0(1) \sum_{n=v+1}^{\infty} n^{k-1} \left(\frac{q_n}{Q_n}\right)^k Q_{n-1}^{-1} \left(\frac{q_{n-1}}{p_{n-1}}\right) \\
&= 0(1) \left(\frac{q_v}{p_v}\right) \sum_{n=v+1}^{\infty} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}} \\
&= 0(1) \left(\frac{q_v}{p_v}\right) \frac{v^{k-1} q_v^{k-1}}{Q_v^k} = 0\left\{\frac{v^{k-1} p_v^{k-1}}{P_v^k}\right\}.
\end{aligned}$$

It may be mentioned that Corollary 3 gives an alternative proof to the sufficiency part of the theorem in [2].

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