Internat. J. Math. & Math. Sci.
Vol. 22, No. 1 (1999) 75–79
S 0161-1712(99)22075-3


# CONVEX AND STARLIKE CRITERIA

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(Received 30 April 1997)

ABSTRACT. We investigate an expression involving the quotient of the analytic representations of convex and starlike functions. Sufficient conditions are found for functions to be starlike of a positive order and convex.

Keywords and phrases. Univalent, starlike, convex.

1991 Mathematics Subject Classification. 30C45.

## 1. Introduction.

Let $S$ denote the class of functions $f$ normalized by $f(0)=f'(0)-1=0$ that are analytic and univalent in the unit disk $\Delta=\{z:|z|<1\}$. A function $f$ in $S$ is said to be starlike of order $\alpha, 0\le\alpha<1$, and is denoted by $S^*(\alpha)$ if $\operatorname{Re}\{zf'(z)/f(z)\}>\alpha, z\in\Delta$, and is said to be convex and is denoted by $K$ if $\operatorname{Re}\{1+zf''(z)/f'(z)\}>0, z\in\Delta$. Mocanu [9] studied linear combinations of the representations of convex and starlike functions and defined the class of $\alpha$-convex functions. In [8], it was shown that if

$$\operatorname{Re}\left[\alpha(1+zf''(z)/f'(z))+(1-\alpha)zf'(z)/f(z)\right]>0 \tag{1.1}$$

for $z\in\Delta$, then $f$ is starlike for $\alpha$ real and convex for $\alpha\ge1$.

In this note, we investigate the properties of functions defined in terms of the quotient of the analytic representations of convex and starlike functions. In particular, we consider the class $G_b$ consisting of normalized functions $f$ defined by

$$G_b=\left\{f:\left|\left(\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}\right)-1\right|<b,\ z\in\Delta\right\}. \tag{1.2}$$

We determine sharp values of $b$ for which $G_b\subset S^*(\alpha), 1/2\le\alpha<1$, and also find values of $b$ for which $G_b\subset K$. It is known ([7, 10]) that $K\subset S^*(1/2)$. We show that $G_1\subset S^*(1/2)-K$. We also find values of $b$ for which $G_b$ is not starlike and not univalent.

We make use of the following lemma obtained by Jack in [4].

**Lemma A.** *Suppose $\omega$ is analytic for $|z|\le r$, $\omega(0)=0$ and $|\omega(z_0)|=\max_{|z|=r}|\omega(z)|$. Then $z_0\omega'(z_0)=k\omega(z_0), k\ge1$.*

## 2. Main results

**Theorem 1.** *If $0<b\le1$ and $G_b$ is defined by (1.2), then $G_b\subset S^*(2/(1+\sqrt{1+8b}))$. The result is sharp for all $b$.*

We prove this theorem in an equivalent form, which we write as

**Theorem 1a.** *Set $b = (1-\alpha)/2\alpha^2, 1/2 \le \alpha < 1$. Then $G_b \subset S^*(\alpha)$, with extremal function $z/(1-z)^{2(1-\alpha)}$.*

**Proof of Theorem 1a.** It is well known that if $\omega(z)$ is analytic in $\Delta$ with $\omega(0) = 0$, then $\operatorname{Re}\left(\frac{1+(1-2\alpha)\omega(z)}{1-\omega(z)}\right) > \alpha, z \in \Delta$, if and only if $\omega(z)$ is a Schwarz function, i.e., $|\omega(z)| < 1$ for $z \in \Delta$ with $\omega(0) = 0$. Set

$$p(z) = \frac{zf'(z)}{f(z)} = \frac{1+(1-2\alpha)\omega(z)}{1-w(z)} \tag{2.1}$$

Then

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)} \tag{2.2}$$

and

$$\left| \left( \frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)} \right) - 1 \right| = \left| \frac{zp'(z)}{(p(z))^2} \right| = \left| \frac{2(1-\alpha)z\omega'(z)}{(1+(1-2\alpha)\omega(z))^2} \right|. \tag{2.3}$$

If $f \notin S^*(\alpha)$, then by Lemma A there is a $z_0 \in \Delta$ for which $|\omega(z_0)| = 1$ and $z_0\omega'(z_0) \ge \omega(z_0)$. It then follows from (2.3) that $\left| \frac{z_0 p'(z_0)}{(p(z_0))^2} \right| \ge \frac{2(1-\alpha)}{(2\alpha)^2}$ which contradicts our hypothesis. This completes the proof. □

**Corollary 1.** *$G_1 \subset S^*(1/2)$.*

**Proof.** Set $b = 1$ in Theorem 1. □

**Corollary 2.** *If $\operatorname{Re}\left(\frac{zf'(z)/f(z)}{1+zf''(z)/f'(z)}\right) > 1/2$ for $z \in \Delta$, then $f \in S^*(1/2)$.*

**Proof.** This follows from Corollary 1 upon noting that for any complex value $w$, $|w-1| < 1 \iff \operatorname{Re}(1/w) > 1/2$. □

We next give a partial converse to Corollary 1.

**Theorem 2.** *If $f \in S^*(1/2)$, then $\left| \left(\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}\right) - 1 \right| < 1$ for $|z| < \left(2\sqrt{3}-3\right)^{1/2} = 0.68\ldots$. The result is sharp.*

**Proof.** Set $p(z) = zf'(z)/f)(z) = 1/(1-\omega(z))$, where $\omega(z)$ is a Schwarz function. We need to find the largest disk $|z| < R$ for which $|zp'(z)/p(z))^2| = |z\omega'(z)| < 1$. Dieudonné [2] found the region of values for the derivative of Schwarz functions. This led to the sharp bound [3],

$$|\omega'(z)| \le \begin{cases} 1, & r = |z| \le \sqrt{2}-1 \\ \dfrac{(1+r^2)^2}{4r(1-r^2)}, & r \ge \sqrt{2}-1. \end{cases} \tag{2.4}$$

Since $|z\omega'(z)| \le (1+r^2)^2/4(1-r^2) = 1$ for $r = \left(2\sqrt{3}-3\right)^{1/2}$, the proof is complete. □

## 3. A counterexample.

The extreme points of the closed convex hull of convex functions and functions starlike of order $1/2$ are identical. See [1]. Since $G_1 \subset S^*(1/2)$, one might, also, expect to have $G_1 \subset K$. Surprisingly, this is not the case. We now construct a function $f \in G_1 - K$.

**Theorem 3.** $G_1 \not\subset K$.

**Proof.** $G_1 \subset S^*(1/2)$. Any of $f \in G_1$ satisfies $zf'(z)/f(z) = 1/(1-\omega(z))$ for some Schwarz function $\omega(z)$. Setting $\alpha = 1/2$ in (2.3), we see that $f \in G_1 \iff |z\omega'(z)| < 1$ for $z \in \Delta$, which means that $z\omega'(z)$ must, also, be a Schwarz function. Since $1 + zf''(z)/f(z) = (1+z\omega'(z))/(1-\omega(z))$, it suffices to construct a Schwarz function $\Omega(z) = z\omega'(z)$ for which

$$\operatorname{Re}\left\{\frac{1+\Omega(z)}{1-\omega(z)}\right\} < 0 \tag{3.1}$$

at some point $z \in \bar{\Delta}$. Let

$$A = \{z \in \Delta : |z - z_0| < 10^{-5}, z_0 = e^{\pi i/4} = e^{i\theta_0}\}, \tag{3.2}$$

and set

$$\phi(z) = (z_0 + \bar{z}_0)\left[(1 - \bar{z}_0 z)^{1/N} - 1\right], \tag{3.3}$$

where $N$ is large enough so that $|\phi(z)/z| < 10^{-4}$ for $z \in \Delta - A$ and $|\operatorname{Im}\phi(z)| < 10^{-8}$ for $z \in A$. Define $\Omega$ by $\Omega(z) = 0.9999(z + \phi(z))$.

We first show that $\Omega(z)$ (and, consequently, $\omega(z)$) is a Schwarz function and then show that inequality (3.1) holds when $z = z_0$.

If

$$z \in \Delta - A, \tag{3.4}$$

then

$$|\Omega(z)| \le 0.9999(|z| + |\phi(z)|) \le 0.9999(1.0001) < 1. \tag{3.5}$$

If $z \in A$, set $z = z_0 - \epsilon e^{i\beta}$, $0 < \epsilon < 10^{-5}$, and note that $-2\cos\theta_0 \le \operatorname{Re}\phi(z) \le 0$. If $\operatorname{Re}(z + \phi(z)) \ge 0$, then $|z + \operatorname{Re}\phi(z)| \le |z| < 1$. If $\operatorname{Re}(z + \phi(z)) < 0$, then

$$|z + \operatorname{Re}\phi(z)| \le \sqrt{(\cos\theta_0 + \epsilon)^2 + (\sin\theta_0 + \epsilon)^2} < \sqrt{1 + 4\epsilon} < 1 + 2\epsilon < 1.0001. \tag{3.6}$$

Thus, if $z \in A$,

$$|\Omega(z)| \le 0.9999|z + \operatorname{Re}\phi(z)| + |\operatorname{Im}\phi(z)| < 0.9999(1.0001) + 10^{-8} = 1. \tag{3.7}$$

Therefore, $\Omega(z)$ is a Schwarz function.

We now show that (3.1) holds at $z = z_0$ for this choice of $\Omega(z)$. Since

$$\left|\frac{\Omega(z)}{z} - 1\right| = |\omega'(z) - 1| < 0.0002 \quad \text{for } z \in \Delta - A, \tag{3.8}$$

we may write $\omega(z) = z + \eta(z)$, where $|\eta(z)| < 0.0003$ for $z \in A$. Note that

$$\begin{aligned}(|1-\omega(z_0)|^2)\operatorname{Re}\left(\frac{1+\Omega(z_0)}{1-\Omega(z_0)}\right) &= \operatorname{Re}\left\{(1-\Omega(z_0))(1+\overline{\omega(z_0)})\right\}\\ &= \operatorname{Re}\left\{(1-0.9999\bar{z}_0)(1-\bar{z}_0-\overline{\eta(z_0)})\right\}\\ &\le 1 - 1.9999\cos\theta_0 + 0.9999\cos 2\theta_0 + 2|\eta(z_0)|\\ &< 1 - 1.9999\cos(\pi/4) + 0.0006 < 0.\end{aligned} \tag{3.9}$$

Hence, the function $f$ for which $1 + zf''(z)/f'(z) = (1+\Omega(z))/(1-\omega(z))$ must be in $G_1 - K$. □

**4. Convexity.** Since $G_1 \not\subset K$, we can ask if $G_b \subset K$ for some $b < 1$. In general, $S^*(\alpha) \not\subset K$ even for $\alpha$ arbitrary close to 1 ($b$ close to 0). To see this, we note that $f_n(z) = z + a_n z^n$ is in $S^*(\alpha)$ if and only if $|a_n| \le (1-\alpha)/(n-\alpha)$ and $f_n(z) \in K$ if and only if $|a_n| \le 1/n^2$. Thus, $f(z) = z + (1-\alpha)/(n-\alpha) z^n \in S^*(\alpha) - K$ for $n > 2/(1-\alpha)$.

We next show that there are values of $b$ for which the functions in $G_b$ must be convex.

**Theorem 4.** *$G_b \subset K$ for $b \le \sqrt{2}/2$.*

**Proof.** Since $f \in G_b \subset G_1 \subset S^*(1/2)$, we may write $zf'(z)/f(z) = 1/(1-\omega(z))$, where $\omega$ is a Schwarz function. For $f \in G_b$, we take $\alpha = 1/2$ in (2.3) to obtain $|z\omega'(z)| < \sqrt{2}/2$ and, consequently, $|\omega(z)| < \sqrt{2}/2$, $z \in \Delta$. We need to show that

$$\operatorname{Re}\{1 + zf''(z)/f'(z)\} = \operatorname{Re}\left\{\frac{(1 + z\omega'(z))}{(1-\omega(z))}\right\} > 0. \tag{4.1}$$

Since

$$\begin{aligned} \left|\arg\left(\frac{1 + z\omega'(z)}{1-\omega(z)}\right)\right| &\le |\arg(1 + z\omega'(z))| + |\arg(1-\omega(z))| \\ &\le \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}, \end{aligned} \tag{4.2}$$

the result follows. □

In [6], MacGregor found the radius of convexity for $S^*(1/2)$ to be $(2\sqrt{3}-3)^{1/2} = 0.68\ldots$. Since $G_1 \subset S^*(1/2)$, we know that the radius of convexity is at least this large. The following consequence of Theorem 4 is that functions in $G_1$ are convex in the disk $|z| < \sqrt{2}/2$.

**Corollary.** *If $f \in G_b$, $\sqrt{2}/2 \le b \le 1$, then $f$ is convex in the disk $|z| < \sqrt{2}/2b$.*

**Proof.** If $|z\omega'(z)| < 1$ for $z \in \Delta$, then $|z\omega'(z)| < t$ for $|z| < t < 1$. If $f \in G_b$, then $|z\omega'(z)| < b$ for $z \in \Delta$. Hence, $|z\omega'(z)| < \sqrt{2}/2$ when $|z| < \sqrt{2}/2b$. □

**5. Examples.** Theorem 1 gives a sharp order of starlikeness for $G_b$ when $0 < b \le 1$, with $G_1 \subset S^*(1/2)$. Our methods do not extend to $b > 1$, but we expect the order of starlikeness to decrease from $1/2$ to 0 as $b$ increases from 1 to some value $b_0$ after which functions in $G_b$ need not be starlike. We do not have a sharp result for $b > 1$, but our next example shows that the univalent functions in $G_b$ are not necessarily starlike for $b \ge 11.66$.

The function $h(z) = z(1-iz)^{i-1}$ is spiral-like [11] and, hence, in $S$ because

$$\operatorname{Re}\left\{e^{\pi i/4}\frac{zh'(z)}{h(z)}\right\} = \frac{1}{\sqrt{2}}\left(\frac{1-|z|^2}{|1-iz|^2}\right) > 0, \quad z \in \Delta. \tag{5.1}$$

Since $zh'(z)/h(z) = (1+z)/(1-iz)$, we see that $h$ is not starlike for $|z| < a$, $\sqrt{2}/2 < a < 1$. Thus, $f(z) = f_a(z) = h(az)/a$ is not starlike for $z \in \Delta$. Setting $p(z) = zf'(z)/f(z) = (1+az)/(1-aiz)$, we have

$$\left|\frac{zp'(z)}{(p(z))^2}\right| = \left|\frac{(1+i)az}{(1+az)^2}\right| \le \frac{\sqrt{2}a}{(1-a)^2} < 11.66 \tag{5.2}$$

for $a$ sufficiently close to $\sqrt{2}/2$. Hence, $f \in G_b - S^*(0)$ for $b = 11.66$.

Finally, we show that the functions in $G_b$ need not be univalent. In [5], it is shown for $h(z) = z(1-iz)^{i-1}$ that $g(z) = \int_0^z h(t)/t\,dt = (1-iz)^i - 1$ is not in $S$ because $g(z_0) = g(-z_0)$ for $z_0 = i(e^{2\pi}-1)/(e^{2\pi}+1), |z_0| = 0.996\ldots$. We, thus, conclude that for $f(z) = g(cz)/c$, $c = 0.997$, $f \in G_b - S$ for $b$ sufficiently large.

**Acknowledgement.** This paper was completed while the author was on a sabbatical leave as a visiting scholar at the University of California at San Diego. I would like to express my deep appreciation to Professor Carl FitzGerald for enlightening discussions, especially for his insight and guidance on the example in Theorem 3.

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