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NOTES ON FRÉCHET SPACES

WOO CHORL HONG

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ABSTRACT. First, we introduce sequential convergence structures and characterize Fréchet spaces and continuous functions in Fréchet spaces using these structures. Second, we give sufficient conditions for the expansion of a topological space by the sequential closure operator to be a Fréchet space and also a sufficient condition for a simple expansion of a topological space to be Fréchet. Finally, we study on a sufficient condition that a sequential space be Fréchet, a weakly first countable space be first countable, and a symmetrizable space be semi-metrizable.

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1. Introduction. Let us recall the following definitions in a topological space (X,c) endowed with the topological closure operator c. We denote the set of all positive integers by N.

(a) *Fréchet* [3] (also called *Fréchet-Urysohn* [2]): for each subset A of X, $c(A)=\{x\in X \mid (x_n)$ converges to x for some sequence (x_n) of points in $A\}$.

(b) *sequential* [5]: for every subset A of X which is not closed in X, there exists a sequence (x_n) of points in A converging to a point of the set $c(A)-A$.

(c) *weakly first countable* [4] (also called g-first *countable* [1] and [13]): for each $x\in X$, there exists a family $\{B(x,n)\mid n\in N\}$ of subsets of X such that the following conditions are satisfied:

(i) $x\in B(x,n+1)\subset B(x,n)$ for all $n\in N$,

(ii) a subset U of X is open if and only if for every $x\in U$ there exists an $n\in N$ such that $B(x,n)\subset U$.

Such a family $\{B(x,n)\mid n\in N\}$ is called a *weak base* at x.

(d) *Symmetrizable* [8]: there exists a symmetric (= a metric except for the triangle inequality) d on X satisfying the following condition: a subset U of X is open if and only if for every $x\in U$ there is a positive real number r such that $B(x,r)\subset U$, where $B(x,r)$ denotes the set $\{y\in X\mid d(x,y)<r\}$.

(e) *Semi-metrizable* [7]: there exists a symmetric d on X such that for each $x\in X$, the family $\{B(x,r)\mid r>0\}$ forms a (not necessarily open) neighborhood base at x.

The basic relationships among these spaces are indicated in the following diagram (see [4, 5, 6, 7, 8, 13]).

$$\begin{array}{ccc} \text{Fréchet} & \Longrightarrow & \text{sequential} \\ \Uparrow & & \Uparrow \\ \text{first countable} & \Rightarrow & \text{weakly first countable} \\ \Uparrow & & \Uparrow \\ \text{semi-metrizable} & \Longrightarrow & \text{symmetrizable.} \end{array}$$

It is clear that a symmetrizable and first countable space is semi-metrizable and every symmetrizable space need not be Fréchet (see [6, Ex. 5.1] and [13, Ex. 2.1]).

Many topologists have studied on properties of spaces mentioned above and relationships among the spaces. In particular, S. P. Franklin in [6, Props. 7.2 and 7.3] gave sufficient conditions for sequential spaces to be Fréchet and the following theorems were obtained by G. Gruenhage in [7] and F. Siwiec in [13].

Theorem A [7, Thm. 9.6]. *The following statements are equivalent.*

(1) *X is semi-metrizable.*

(2) *X is symmetrizable and first countable.*

(3) *X is symmetrizable and Fréchet.*

Theorem B [13, Thm. 1.10]. *If a Hausdorff space X satisfies one of the properties in the second column of the above diagram and is also Fréchet, then it satisfies the corresponding property in the first column.*

The purpose of this paper is to consolidate and investigate properties of Fréchet spaces. First, we introduce sequential convergence structures and characterize Fréchet spaces and continuous functions in Fréchet spaces using these structures. Second, we give sufficient conditions for the expansion of a topological space by the sequential closure operator to be a Fréchet space and also a sufficient condition for a simple expansion of a topological space to be Fréchet. Finally, we study on a sufficient condition that a sequential space be Fréchet, a weakly first countable space be first countable, and a symmetrizable space be semi-metrizable. And then we obtain generalizations of Theorems A and B.

Standard notations, not explained below, is the same as in [3, 14].

2. Results

2.1. Sequential convergence structures. Let X be a nonempty set and let $S(X)$ denote the set of all sequences of points in X. Recall that a nonempty subfamily L of the cartesian product $S(X)\times X$ is called *a sequential convergence structure on X* [9] if it satisfies the following conditions:

(SC 1): For each $x \in X, ((x),x) \in L$, where (x) is the constant sequence whose nth term is x for all indices $n \in N$.

(SC 2): If $((x_n),x) \in L$, then $((x_{\phi(n)}),x) \in L$ for each subsequence $(x_{\phi(n)})$ of (x_n).

(SC 3): Let $x \in X$ and $A \subset X$. If $((y_n),x) \in L$ for some $(y_n) \in S(\{y \in X \mid ((x_n),y) \in L$ for some $(x_n) \in S(A)\})$, then $((x_n),x) \in L$ for some $(x_n) \in S(A)$.

Let $\mathrm{SC}[X]$ denote the set of all sequential convergence structures on X.

Theorem 2.1 [9]. *For $L \in \mathrm{SC}[X]$, define a function $c_L : P(X) \to P(X)$ as follows: for* *each subset A of X, $c_L(A) = \{x \in X \mid ((x_n),x) \in L$ for some $(x_n) \in S(A)\}$. Then, (X,c_L) is a Fréchet space endowed with the topological closure operator c_L.*

Let $\mathcal{L}(c)$ denote the set of all pairs $((x_n),x) \in S(X)\times X$ such that (x_n) converges to x in a topological space (X,c) endowed with the topological closure operator c.

Theorem 2.2 [9]. *For each $L \in \mathrm{SC}[X]$, we have*
(1) $L \subset \mathcal{L}(c_L) \in \mathrm{SC}[X]$,
(2) $c_L = c_{\mathcal{L}(c_L)}$, *and*
(3) $\bigcup\{L' \in \mathrm{SC}[X] \mid c_L = c_{L'}\} = \mathcal{L}(c_L)$.

Note that there is an $L \in \mathrm{SC}[X]$ such that $L \neq \mathcal{L}(c_L)$. Let Q be the rational number space with the usual topology and let $L_Q = \{((x_n),x) \in S(Q)\times Q \mid (x_n)$ converges to x in $Q\}$ and $L = \{((x),x) \mid x \in Q\} \cup \{((x_n),x) \in S(Q)\times Q \mid (x_n)$ converges to x in Q and (x_n) is either strictly increasing or strictly decreasing$\}$. Then we have $L \subsetneq L_Q = \mathcal{L}(c_{L_Q}) = \mathcal{L}(c_L)$.

According to Theorem 2.1 and Theorem 2.2, we immediately obtain the following theorem and hence we omit the proof.

Theorem 2.3. *A topological space (X,c) is Fréchet if and only if $c = c_L$ for some $L \in \mathrm{SC}[X]$.*

Now, we characterize continuous functions in Fréchet spaces using sequential convergence structures.

Theorem 2.4. *Let (X,c_X) and (Y,c_Y) be two Fréchet spaces endowed with the topological closure operators c_X and c_Y, respectively, and let $L_X \in \mathrm{SC}[X]$ with $c_X = c_{L_X}$. Then, a function $f : (X,c_X) \to (Y,c_Y)$ is continuous if and only if for each $((x_n),x) \in L_X$, $((f(x_n)),f(x)) \in \mathcal{L}(c_Y)$.*

Proof. Let $((x_n),x) \in L_X$. Then, by Theorem 2.2(1) and hypothesis, $((x_n),x) \in \mathcal{L}(c_X)$; equivalently, (x_n) converges to x in (X,c_X). Since f is continuous, it is clear that $((f(x_n)),f(x)) \in \mathcal{L}(c_Y)$.

Conversely, suppose that there is a closed subset F of Y such that $f^{-1}(F)$ is not closed in X. Since $c_X = c_{L_X}$, it is clear that $f^{-1}(F) \subsetneq c_{L_X}(f^{-1}(F))$. Let $x \in c_{L_X}(f^{-1}(F)) - f^{-1}(F)$. Then, by the definition of c_{L_X} (see Theorem 2.1), $((x_n),x) \in L_X$ for some $(x_n) \in S(f^{-1}(F))$. By hypothesis, $((f(x_n)),f(x)) \in \mathcal{L}(c_Y)$, and hence we have that $f(x) \in F$ because F is closed. Thus, $x \in f^{-1}(F)$, which is a contradiction. □

By Theorem 2.4, we have the following well-known fact.

Corollary 2.5. *Let (X,c_X) and (Y,c_Y) be two Fréchet spaces. Then, a function $f : (X,c_X) \to (Y,c_Y)$ is continuous if and only if f is sequentially continuous.*

Proof. Since (X,c_X) is a Fréchet space, $c_x = c_{\mathcal{L}(c_X)}$, and hence we can replace L_X by $\mathcal{L}(c_X)$ in Theorem 2.4 above. □

Remark. Note that we thus obtain by Theorem 2.4 a convenient method to check whether a function in Fréchet spaces is continuous or not. For example, let f be a real-valued function defined on a subspace X of the real line R with the usual topology and let $L_X = \{((x),x) \mid x \in X\} \cup \{((x_n),x) \in S(X)\times X \mid (x_n)$ converges to x in X and (x_n)

is either strictly increasing or strictly decreasing}. Then, it is obvious that $L_X \in \mathrm{SC}[X]$ and moreover (X, c_{L_X}) is precisely equal to the space X itself. By Theorem 2.4, we see that f is continuous if and only if for each $((x_n), x) \in L_X, (f(x_n))$ converges to $f(x)$ in R.

2.2. Expansions. Let (X, c) be a topological space endowed with the topological closure operator c. A function $c_* : P(X) \to P(X)$ defined by for each $A \subset X$, $c_*(A) = \{x \in X \mid (x_n)$ converges to x in the space X for some $(x_n) \in S(A)\}$ is called *the sequential closure operator* [3] on the topological space (X, c). It is obvious that a topological space (X, c) is Fréchet (sequential) if and only if for each $A \subset X$, $c(A) = c_*(A)$ (resp. if $A = c_*(A)$ then $A = c(A)$). It is known that the sequential closure operator c_* on a topological space (X, c) need not be a topological closure operator on the set X and if c_* is idempotent, then (X, c_*) is a Fréchet expansion of (X, c).

Consider the following two properties in a topological space (X, c):

$(*)$ *For each countable subset A of X, $c(A) \subset c_*(A)$.*

$(**)$ *For each double sequence (or called bi-sequence) $(x_{nm} \mid n, m \in N)$ of points in X, if for each $n \in N, ((x_{nm} \mid m \in N), y_n) \in \mathcal{L}(c)$ and $((y_n), z) \in \mathcal{L}(c)$, then there exists $(p_n) \in S(\{x_{nm} \mid n, m \in N\})$ such that $((p_n), z) \in \mathcal{L}(c)$.*

We then have easily that the following implications hold: Fréchet $\Rightarrow (*) \Rightarrow (**)$. But, according to [10, Ex. 2.4], we see that the converses are not true. Note that $(**)$ is equivalent to (SC 3) of $\mathcal{L}(c)$.

Theorem 2.6 [10]. *Let (X, c) be a topological space. If X satisfies the property $(**)$, then (X, c_*) is Fréchet and moreover, $\mathcal{L}(c) = \mathcal{L}(c_*)$.*

Immediately we have that Theorem 2.6 holds whenever we replace $(**)$ with $(*)$.

Remark. It is an interesting and important fact that $\mathcal{L}(c) = \mathcal{L}(c_*)$, even though $c \neq c_*$. From this fact, we have naturally the following:

(1) The properties $(*)$ and $(**)$ are sufficient conditions for a non-Fréchet space (X, c) to have the Fréchet expansion (X, c_*).

(2) There are close corelations between some topological properties of the two spaces (X, c) and (X, c_*). For example,

- (i) the separation properties of (X, c) transfer to the space (X, c_*),
- (ii) if (X, c_*) is compact (connected or separable), then (X, c) is compact (resp. connected or separable), and
- (iii) (X, c) is sequentially compact if and only if (X, c_*) is sequentially compact, etc.

Let $(X, \mathcal{T})$ be a topological space and $\mathcal{A}$ a family of subsets of X. The expansion of $\mathcal{T}$ by $\mathcal{A}$ denoted by $\mathcal{T}(\mathcal{A})$ is the topology on X with $\mathcal{T} \cup \mathcal{A}$ as sub-base. In case $\mathcal{A} = \{A\}$ the expansion by $\mathcal{A}$ is a simple expansion, denoted by $\mathcal{T}(A)$. Simple extensions (expansions) were introduced by N. Levine in [11]. This and subsequent works have been concerned with the preservation of topological properties under these expansions. In particular, J. A. Narvarte and J. A. Guthrie [12] investigated the preservation of Fréchet spaces, sequential spaces and k-spaces under simple expansions.

We recall that a family $\mathcal{A}$ of subsets of X *is point finite* [14] if and only if each $x \in X$ belongs to only finitely many $A \in \mathcal{A}$.

Theorem 2.7. *Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{A} = \{c(B) - c_*(B) \mid B \subset$*

X and $c(B)-c_*(B)\neq\emptyset\}$, where $c(c_*)$ is the topological closure operator (resp. the sequential closure operator) on $(X,\mathcal{T})$. If $\mathcal{A}$ is point finite, then $(X,\mathcal{T}(\mathcal{A}))$ is a Fréchet space.

Proof. If $(X,\mathcal{T})$ is a Fréchet space, then clearly $\mathcal{A}=\emptyset$, and so $\mathcal{T}(\mathcal{A})=\mathcal{T}$. Hence, it remains to prove the case that $(X,\mathcal{T})$ is not Fréchet. Let $Y\subset X$ and $p\in c_{\mathcal{T}(\mathcal{A})}(Y)-Y$, where $c_{\mathcal{T}(\mathcal{A})}$ is the topological closure operator on $(X,\mathcal{T}(\mathcal{A}))$. Since $\mathcal{A}$ is point finite, $\{A\in\mathcal{A}\mid p\in A\}$ is finite, say $\{K_1,K_2,\dots,K_n\}$. Let $M=\cap\{K_i\mid i=1,2,\dots,n\}$. Then, clearly, $Y\cap M\neq\emptyset$ since M is a basic open set in $(X,\mathcal{T}(\mathcal{A}))$ containing p. We first show that $p\in c_{\mathcal{T}(\mathcal{A})}(Y\cap M)$. Since $\mathcal{T}\cup\mathcal{A}$ is a sub-base for $\mathcal{T}(\mathcal{A})$, by the definition of M, we have that for each basic open set U in $(X,\mathcal{T}(\mathcal{A}))$ containing $p,(\cap\{V_j\mid i\in J\})\cap M\subset U$ for some finite family $\{V_j\mid j\in J$ and J is finite$\}$ of open sets V_j in $(X,\mathcal{T})$ containing p, and so $V\cap M\subset U$ for some open set V in $(X,\mathcal{T})$ containing p. Hence, it is sufficient to show that for each open set V in $(X,\mathcal{T})$ containing p, $(Y\cap M)\cap V\neq\emptyset$. Suppose on the contrary that there exists an open set V in $(X,\mathcal{T})$ containing p such that $(Y\cap M)\cap V=\emptyset$. Then, since $M\cap V$ is a basic open set in $(X,\mathcal{T}(\mathcal{A}))$ containing p and since $p\in c_{\mathcal{T}(\mathcal{A})}(Y),Y\cap(M\cap V)\neq\emptyset$, which is a contradiction. It is easy to see that for each subset Z of X, $c_{\mathcal{T}(\mathcal{A})}(Z)\subset c_*(Z)\subset c(Z)$. Hence, we have that there exists $(x_n)\in S(Y\cap M)$ such that (x_n) converges to p in $(X,\mathcal{T})$. To end the proof, we claim that (x_n) converges to p in $(X,\mathcal{T}(\mathcal{A}))$. Suppose that it is not. Then there exists a basic open set U in $(X,\mathcal{T}(\mathcal{A}))$ containing p such that (x_n) is not eventually in U. We have already known that $V\cap M\subset U$ for some open set V in $(X,\mathcal{T})$ containing p. It follows that there is an open set V in $(X,\mathcal{T})$ containing p such that (x_n) is not eventually in $V\cap M$, and hence (x_n) is also not eventually in V because $(x_n)\in S(M)$, which is a contradiction. □

2.3. Fréchet and related spaces. We start with the following property in a topological space (X,c) endowed with the topological closure operator c:

$(***)$ *For each $A\subset X$ and $x\in X$, if $x\in c(A)-A$ then there exists $B\subset A$ such that $x\in c(B)$ and $c(B)-\{x\}\subset A$.*

Obviously, every Hausdorff Fréchet space satisfies the property $(***)$. On the other hand, there are Hausdorff compact spaces satisfying the property $(***)$ which are not sequential and hence are not Fréchet.

Example 2.8. The ordinal space $X=[0,\omega_1]$, where ω_1 is the first uncountable ordinal, is a Hausdorff compact space which is not sequential (see [10, Ex. 2.4(2)]). Now we show that X satisfies the property $(***)$. Since $[0,\omega_1)$ is discrete, to show this it is sufficient to prove that for each $A\subset[0,\omega_1)$, if $\omega_1\in c(A)$ then there exists a subset B of A such that $\omega_1\in c(B)$ and $c(B)=B\cup\{\omega_1\}$. If there exists $\alpha\in[0,\omega_1)$ such that $[\alpha,\omega_1)\subset A$, then $[\alpha,\omega_1)$ is a desired subset of A. Hence, it remains to prove the case where for each $\alpha\in[0,\omega_1)$, $(\alpha,\omega_1)\not\subseteq A$. We first recall a definition that a subset C of $[0,\omega_1]$ is an interval if and only if for each $\alpha,\beta\in C$ with $\alpha\leqq\beta$, $[\alpha,\beta]\subset C$. Then, it is clear that A is the union of infinitely many intervals: $A=\cup\{C_i\mid i\in I\}$, where C_i is an interval for each $i\in I$ and $C_i\cap C_j=\emptyset$ if $i\neq j$. Let α_i denote the first element of C_i for each $i\in I$. Note that I must be uncountable. Suppose on the contrary that I is countable. Then, let 1_0 be the first element of I and assume $\alpha_{1_0}\neq 0,(\alpha_i-1\mid i\in I)\in S([0,\omega_1)-A)$, and hence $\sup\{\alpha_i-1\mid i\in I\}\lneqq\omega_1$ since there does not exist any sequence in $[0,\omega_1)$ converging to ω_1. It follows that for each α with $\sup\{\alpha_i - 1 \mid i \in I\} \lneqq \alpha \lneqq \omega_1$, $[\alpha,\omega_1) \subset A$. It is impossible in this case. Thus we have immediately that $(\alpha_i \mid i \in I)$ is a net in A converging to ω_1 and there isn't any interval in the range $\{\alpha_i \mid i \in I\}$ of the net (α_i) which contains more than one point. From this, we see that $c(\{\alpha_i \mid i \in I\}) = \{\alpha_i \mid i \in I\} \cup \{\omega_1\}$, and so it is a desired subset of A. The proof is complete.

From the above facts and the following two examples:

(1) the Sierpiński space, $X = \{a,b\}$ with the topology $\{\emptyset, X, \{a\}\}$, is a Fréchet space satisfying the property $(***)$, but not Hausdorff, and

(2) the space $X = \{a,b,c\}$ with the topology $\{\emptyset, X, \{a\}\}$ is Fréchet, but doesn't satisfy the property $(***)$ and not Hausdorff, we have that the three properties $(***)$, Fréchet and Hausdorff are independent.

Theorem 2.9. *Every sequential space satisfying the property $(***)$ is Fréchet.*

Proof. Suppose that there exists a sequential space (X,c) satisfying the property $(***)$ which is not Fréchet. Then, there exists a subset A of X such that $c(A) \neq c_*(A)$, let $x \in (c(A) - c_*(A))$. By $(***)$, there is a subset B of A such that $x \in c(B)$ and $c(B) - \{x\} \subset A$. Let $C = c(B) - \{x\}$. Clearly, $c(C) = c(B)$ and $c(C) = C \cup \{x\}$. Hence, by sequentiality of X, we have that there exists $(x_n) \in S(C)$ such that $((x_n),x) \in \mathcal{L}(c)$, and so $x \in c_*(C)$, which is a contradiction. □

Theorem 2.10. *Every weakly first countable space satisfying the property $(***)$ is first countable.*

Proof. Let (X,c) be a weakly first countable space satisfying the property $(***)$. Then, for each $x \in X$, there is a weak base $\{B(x,n) \mid n \in N\}$ at x. To prove this, it suffices to show that for each $x \in X$ and each $n \in N$, $B(x,n)$ is a neighborhood of x in the space X. Suppose on the contrary that there are $x \in X$ and $n \in N$ such that $x \notin \text{int}(B(x,n))$, where $\text{int}(B(x,n))$ denotes the interior of $B(x,n)$ in X. Then, clearly, $x \in c(X - B(x,n))$. Since X satisfies the property $(***)$, there exists a subset Y of $X - B(x,n)$ such that $x \in c(Y)$ and $c(Y) - \{x\} \subset X - B(x,n)$. Since $X - c(Y)$ is open in X and $B(x,n) \subset ((X - c(Y)) \cup \{x\})$, by condition (ii) of the definition of weak first countability, we have that the set $(X - c(Y)) \cup \{x\}$ is open in X and $Y \cap ((X - c(Y)) \cup \{x\}) = \emptyset$, which is a contradiction. □

The following corollaries follow directly from Theorem 2.10.

Corollary 2.11. *Every weakly first countable, Hausdorff and Fréchet space is first countable.*

Corollary 2.12. *Every symmetrizable space satisfying the property $(***)$ is first countable and hence semi-metrizable.*

Remark. (1) Notice that Corollary 2.11 was stated by A. V. Arhangel'skii [1] without proof and a proof can be found in F. Siwiec [13, Thm. 1.10].

(2) One can easily observe that Corollary 2.12 is in part a generalization of Theorem A [7, Thm. 9.6]. Note that in a Hausdorff space X, X is a symmetrizable space satisfying the property $(***)$ if and only if X is symmetrizable and first countable.

(3) From Theorems 2.9, 2.10 and Corollary 2.12, we know that the property $(***)$ is a sufficient condition that a topological space satisfying one of the properties in the second column of the above diagram given in the introduction satisfy the corresponding property in the first column. This is a generalization of Theorem B [13, Thm. 1.10].

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Hong: Department of Mathematics Education, Pusan National University, Pusan 609-735, Korea

E-mail address: wchong@hyowon.pusan.ac.kr