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RECAPTURING SEMIGROUP COMPACTIFICATIONS OF A GROUP FROM THOSE OF ITS CLOSED NORMAL SUBGROUPS

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ABSTRACT. We know that if S is a subsemigroup of a semitopological semigroup T, and $\mathcal{F}$ stands for one of the spaces $\mathcal{AP}, \mathcal{WAP}, \mathcal{SAP}, \mathcal{D}$ or $\mathcal{LC}$, and $(\epsilon, T^{\mathcal{F}})$ denotes the canonical $\mathcal{F}$-compactification of T, where T has the property that $\mathcal{F}(S) = \mathcal{F}(T)_{|_S}$, then $(\epsilon_{|_S}, \overline{\epsilon(S)})$ is an $\mathcal{F}$-compactification of S. In this paper, we try to show the converse of this problem when T is a locally compact group and S is a closed normal subgroup of T. In this way we construct various semigroup compactifications of T from the same type compactifications of S.

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1. Introduction. For notation and terminology we follow Berglund et al. [2], as much as possible. Thus a topological semigroup is a semigroup S that is a Hausdorff topological space, the multiplication $(s,t) \to st : S \times S \to S$ being continuous. S is called a semitopological semigroup if the multiplication is separately continuous, i.e., the maps $\lambda_s : t \to st$ and $\rho_s : t \to ts$ from S into S are continuous for each $s \in S$. For S to be right topological only, the maps ρ_s are required to be continuous. Let G denote a locally compact group, and N is a closed normal subgroup of G. A semigroup compactification of G is a pair (φ, X), where X is a compact right topological semigroup with identity 1, and $\varphi : G \to X$ is a continuous homomorphism with $\overline{\varphi(G)} = X$, and $\varphi(G) \subset \Lambda(X) = \{x \in X \mid \lambda_x : X \to X \text{ is continuous}\}$; $\Lambda(X)$ is called the topological center of X. When there is no risk of confusion we often refer to (φ, X), or even to X, as a compactification of G.

A homomorphism from a compactification (ψ, X) of S to a compactification (φ, Y) of S is a continuous function $\theta : X \to Y$ such that $\theta \circ \psi = \varphi$. Then, Y is called a factor of X, and X is an extension of Y. A compactification with a given property P (such as that of being a semitopological semigroup or a topological group) is called a P-compactification. A universal P-compactification of S is a P-compactification which is an extension of every P-compactification of S (see [1, 2, 3]).

The C^*-algebra of all bounded continuous complex-valued functions on G is denoted by $\mathcal{C}(G)$ with left and right translation operators, L_s and R_s, defined for all $s \in G$ by $L_s f = f \circ \lambda_s$ and $R_s f = f \circ \rho_s$. If $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{C}(G)$ containing the constant functions, we denote by $G^{\mathcal{A}}$ the spectrum of $\mathcal{A}$ furnished with Gelfand topology (i.e., the weak*-topology induced from $\mathcal{A}^*$); the natural map $\epsilon : G \to G^{\mathcal{A}}$ is defined by $\epsilon(s)f = f(s)$. When $\mathcal{A}$ is left translation invariant (i.e., $L_s f \in \mathcal{A}$ for all $s \in G$ and $f \in \mathcal{A}$) we can define an action of G on $G^{\mathcal{A}}$ by $(s, \nu) \to \epsilon(s)\nu$, where $(\epsilon(s)\nu)(f) = \nu(L_s f)$. Right translation invariance and $\nu\epsilon(s)$ are analogously defined (see [5, 7]).

A left translation invariant C^*-subalgebra of $\mathscr{C}(G)$ containing the constant functions is called left m-introverted if the function $s \to (\nu f)(s) = \nu(L_s f)$ is in $\mathscr{A}$ for all $f \in \mathscr{A}$ and $\nu \in G^{\mathscr{A}}$; in this situation the product of $\mu, \nu \in G^{\mathscr{A}}$ can be defined by $(\mu\nu)(f) = \mu(\nu f)$. This makes $(\epsilon, G^{\mathscr{A}})$ a semigroup compactification of G. The spaces of almost periodic, weakly almost periodic, left continuous and distal functions, which are denoted by $\mathscr{AP}, \mathscr{WAP}, \mathscr{LC}$, and $\mathscr{D}$, respectively, are left m-introverted. We refer the reader to [2, 5] for the one-to-one correspondence between compactifications of G and left m-introverted C^*-subalgebras of $\mathscr{C}(G)$, and also for a discussion of properties P of compactifications and associated universal mapping properties.

2. Main results. Let G be a locally compact group with a closed normal subgroup N, and let (φ, X) be a compactification of N. Let $\sim$ be the equivalence relation on $G \times X$ with equivalence classes $\{(sr^{-1}, \varphi(r)x) \mid r \in N\}$. Thus

$$(s,x) \sim (t,y) \text{ if and only if } t^{-1}s \in N \text{ and } \varphi(t^{-1}s)x = y. \tag{2.1}$$

$\pi : G \times X \to (G \times X)/\sim$ will denote the quotient map. Clearly π is one-to-one on $\{e\} \times X$, so we can identify $X \cong \{e\} \times X$ with $\pi(\{e\} \times X)$. It is important that $(G \times X)/\sim$ is locally compact and Hausdorff. In this connection we have the following lemmas, which are stated in [6].

Lemma 2.1. (i) *The graph of $\sim$ is closed.*
(ii) *$\pi : (G \times X) \to (G \times X)/\sim$ is an open mapping.*
(iii) *Let K be a compact subset of G and let $L = KN$, then $\pi(K \times X) = \pi(L \times X)$.*

This lemma has the following easy consequences.

Lemma 2.2. *The quotient space $(G \times X)/\sim$ is locally compact and Hausdorff.*

Lemma 2.3. *If $G = KN$ for some compact subset K of G, then $(G \times X)/\sim$ is compact.*

Let $\mu : G \to (G \times X)$ be defined by $\mu(s) = (s, 1)$, where 1 is the identity of X. Then, $\pi \circ \mu : G \to (G \times X)/\sim$ is continuous as a composition of two continuous functions, and $\pi \circ \mu(G) = \pi(G \times \varphi(N))$, since for each $(s, \varphi(r)) \in G \times \varphi(N)$, $(s, \varphi(r)) \sim (sr, 1)$, and $\pi \circ \mu(sr) = \pi(sr, 1) = \pi(s, \varphi(s))$. Furthermore, if φ is a homeomorphism of N into X, then $\pi \circ \mu$ is also a homeomorphism.

We now define $\sigma_s(r) = s^{-1}rs$ for $s \in G$ and $r \in N$, it is obvious that $\sigma_s : N \to N$ is a surjective homomorphism for each $s \in G$.

Definition 2.4. A $\mathscr{P}$-compactification (φ, X) of N is said to be a conjugation invariant $\mathscr{P}$-compactification of N if $(\varphi \circ \sigma_s, X)$ is a $\mathscr{P}$-compactification of N for each $s \in G$. When we write $\mathscr{P}$-compactification instead of P-compactification, this means that we want to emphasize its conjugation invariance, see Corollary 2.7.

Remark. The reader may have noticed that, the definition of $\mathscr{P}$-conjugation invariant compactification is different from the compatibility of a compactification in [6], because if $\mathscr{P}$ is a property of compactifications that is not invariant under homomorphism and (ψ, X) is a $\mathscr{P}$-compactification of N compatible with G, then $(\psi \circ \sigma_s, X)$ is a compactification of N which may not be a $\mathcal{P}$-compactification of N, thus (ψ,X) can fail to be a $\mathcal{P}$-conjugation invariant compactification of N. On the other hand, if (ψ,X) is a $\mathcal{P}$-conjugation invariant compactification of N, i.e., $(\psi\circ\sigma_s,X)$ is a $\mathcal{P}$-compactification of N for each $s\in G$, it is not always true that σ_s has an extension from X to X.

Lemma 2.5. *Let G be a locally compact group, N a closed normal subgroup, and (φ,X) a conjugation invariant universal $\mathcal{P}$-compactifications of N, then each σ_s can be extended continuously to a mapping from X to X.*

Proof. By conjugation invariance of (φ,X), $(\varphi\circ\sigma_s,X)$ is a $\mathcal{P}$-compactification of N, and by universality of (φ,X) there exists a continuous homomorphism $\nu: X\to X$ such that $\varphi\circ\sigma_s=\nu\circ\varphi$ for each $s\in N$. This ν is the continuous function extending σ_s. □

It is obvious that if (φ,X) is a conjugation invariant universal $\mathcal{P}$-compactification of N, then each σ_s determines a continuous transformation of X, for which we use the same notation σ_s.

Corollary 2.6. *Let N be contained in the center of G, then each compactification (φ,X) of N is conjugation invariant.*

Corollary 2.7. *Let $(\epsilon,N^{\mathcal{P}})$ denote a universal $\mathcal{P}$-compactification of N and let $\mathcal{P}$ be a purely algebraic property, then $(\epsilon,N^{\mathcal{P}})$ is a conjugation invariant $\mathcal{P}$-compactification of N.*

Notice our deviation from the usual notation.

Corollary 2.8. *Let (φ,X) be an $\mathcal{F}$-compactification of N, where $\mathcal{F}$ stands for either of the spaces $\mathcal{AP}$ and $\mathcal{WAP}$, then (φ,X) is a conjugation invariant universal $\mathcal{F}$-compactification of N.*

Lemma 2.9. *Let (φ,X) be a conjugation invariant $\mathcal{P}$-compactification of N, then for each $s\in G$, σ_s is a continuous automorphism of X.*

Proof. σ_s is a homeomorphism of X onto X (since $\sigma_s(N)=N$ and $\sigma_s\sigma_{s^{-1}}=I$, the identity mapping). Now, we show that σ_s is a homomorphism. Obviously,

$$\sigma_s(xy)=\sigma_s(x)\sigma_s(y)\quad\text{for each } x,y\in\varphi(N). \tag{2.2}$$

Since X is a right topological semigroup with $\varphi(N)\subset\Lambda(X)$, we conclude that (2.2) holds for each $x\in\varphi(N)$, $y\in X$. Then it follows that (2.2) holds for all $x,y\in X$, as required. □

If N is a closed subgroup of G, and X is a conjugation invariant $\mathcal{P}$-compactification of N, then we can define a semidirect product structure on $G\times X$ by $(s,x)(t,y)=(st,\sigma_t(x)y)$, where σ_t is the conjugation map.

Lemma 2.10. *Let G be a locally compact group with a closed normal subgroup N, and let (φ,X) be a conjugation invariant $\mathcal{P}$-compactification of N, then $G\times X$ is a right topological semigroup. Furthermore, the map*

$$((s,r),(t,y))\longrightarrow(st,\varphi(\sigma_t(r))y):(G\times N)\times(G\times X)\longrightarrow G\times X \tag{2.3}$$

is continuous, and the equivalence relation $\sim$ is a congruence on $G\times X$.

Proof. The continuity is an easy conclusion of Ellis theorem. Now, we show that $\sim$ is a congruence. Suppose $(s,x)\sim(t,y)$ and $(u,z)\in G\times X$, then $t^{-1}s\in N$ and $\varphi(t^{-1}s)x=y$, so $(s,x)(u,z)=(su,\sigma_u(x)z)$ and $(t,y)(u,z)=(tu,\sigma_u(y)z)$.

On the other hand, $(su,\sigma_u(x)z)\sim(tu,\sigma_u(y)z)$ since $(tu)^{-1}su=u^{-1}t^{-1}su\in N$ and

$$\varphi((tu)^{-1}su)\sigma_u(x)z=\sigma_u(y)z, \tag{2.4}$$

thus

$$(s,x)(u,z)\sim(t,y)(u,z). \tag{2.5}$$

Similarly

$$(u,z)(s,x)\sim(u,z)(t,y). \tag{2.6}$$

□

The following theorem is an easy consequence of the previous corollaries and lemmas.

Theorem 2.11. *Let G be a locally compact group with a closed normal subgroup N, and let (φ,X) be a conjugation invariant compactification of N. Then $(G\times X)/\sim$ is a locally compact right topological semigroup, and a compactification of G, provided that $G=KN$ for some compact subset K of G.*

Theorem 2.12. *The compactification $(\pi\circ\mu,(G\times X)/\sim)$ of G described in the previous theorem has the following universal property; let (φ,Y) be a semigroup compactification of G such that $\varphi|_N$ extends to a continuous homomorphism $\phi:X\to Y$ in such a way that for each $s\in G$ and $x\in X$,*

$$\phi(\sigma_s(x))=\varphi(s^{-1})\phi(x)\varphi(s), \tag{2.7}$$

then there exists a (unique) continuous homomorphism $\theta:(G\times X)/\sim\to Y$ such that $\theta\circ\pi\circ\mu=\varphi$.

Proof. We define $\theta_0:G\times X\to Y$ by $\theta_0(s,x)=\varphi(s)\phi(x)$, then θ_0 is a continuous homomorphism which is constant on $\sim$-classes of $G\times X$. Now we take $\theta=\theta_0\circ\pi$. □

Theorem 2.13. *Let N be a closed normal subgroup of G with $G=KN$ for some compact subset K of G. Suppose that $\mathcal{P}$ is a property of compactifications such that $(\varphi|_N,\overline{\varphi(N)})$ is a $\mathcal{P}$-compactification of N whenever $(\varphi,\overline{\varphi(G)})$ is a $\mathcal{P}$-compactification of G. Suppose that $(\epsilon,N^{\mathcal{P}})$ is a conjugation invariant universal $\mathcal{P}$-compactification of N. If $(G\times N^{\mathcal{P}})/\sim$ has the property $\mathcal{P}$, then $(G\times N^{\mathcal{P}})/\sim$ is the universal $\mathcal{P}$-compactification of G.*

Proof. We show that $(G\times N^{\mathcal{P}})/\sim$ is the universal $\mathcal{P}$-compactification of G. Let (φ,X) be a $\mathcal{P}$-compactification of G such that $(\varphi|_N,\overline{\varphi(N)})$ is a $\mathcal{P}$-compactification of N, by the universal property of $N^{\mathcal{P}}$ there exists a continuous homomorphism $\phi:N^{\mathcal{P}}\to X$ such that $\phi\circ\epsilon=\varphi|_s$, and we have $\phi(\sigma_s(x))=\varphi(s^{-1})\phi(x)\varphi(s)$ for all $s\in G$ and $x\in N^{\mathcal{P}}$. Notice that we use two different scripts of the same letter to emphasize their connection. Indeed, for fixed $s \in G$, both sides represent homomorphisms of $N^{\mathcal{P}}$ into X, both sides are continuous in x, and coincide on the dense subspace N. Now the map $\varphi\times\phi : (G\times N^{\mathcal{P}}) \to X$ defined by $(\varphi\times\phi)(s,x) = \varphi(s)\phi(x)$ is continuous and a homomorphism, since

$$\begin{aligned}(\varphi\times\phi)\big((s,x)(t,y)\big) &= (\varphi\times\phi)(st,\sigma_t(x)y) = \varphi(st)\phi(\sigma_t(x)y)\\ &= \varphi(s)\varphi(t)\phi(\sigma_t(x))\phi(y) = \varphi(s)\phi(x)\varphi(t)\phi(y)\\ &= \varphi\times\phi(s,x)\varphi\times\phi(t,y).\end{aligned} \tag{2.8}$$

Also $\varphi\times\phi$ is constant on $\sim$-classes, thus the quotient of $\varphi\times\phi$ gives a continuous homomorphism from $(G\times N^{\mathcal{P}})/\sim$ to X. □

Corollary 2.14. *Let N be a closed normal subgroup of G with $G = KN$ for some compact subset K of G, then*
(i) *$(G\times N^{\mathcal{LC}})/\sim$ is the universal $\mathcal{LC}$-compactification of G.*
(ii) *$(G\times N^{\mathcal{D}})/\sim$ is the universal $\mathcal{D}$-compactification of G.*

Proof. (i) Since $(G\times N^{\mathcal{LC}})/\sim$ is a compactification of G, by Theorem 2.13, $(G\times N^{\mathcal{LC}})/\sim$ is the universal $\mathcal{LC}$-compactification of G.
(ii) Since $N^{\mathcal{D}}$ is a group, $(G\times N^{\mathcal{D}})/\sim$, the quotient by a congruence of a semidirect product of groups is also a group, thus by Theorem 2.13 $(G\times N^{\mathcal{D}})/\sim$ is the universal $\mathcal{D}$-compactification of G. □

In some situations, we want to be able to conclude that the right topological semigroup $(G\times X)/\sim$ of Theorem 2.13 is also left topological. The following lemma can be helpful in this connection.

Lemma 2.15. *Let G be a locally compact group with a closed normal subgroup N and let X be a universal conjugation invariant compactification of N. Suppose that $G = KN$ for some compact subset K of G and $s \to \sigma_s(x) : G\times X \to X$ is continuous for all $x \in X$. Then $(G\times X)/\sim$ is semitopological.*

Proof. Since $(s,x) \to \sigma_s(x) : G\times X \to X$ is a group action, it is continuous by Ellis theorem, thus $G\times X$ is semitopological semigroup and also $(G\times X)/\sim = \pi(G\times X)$. □

Corollary 2.16. *Let G be a locally compact group with a closed normal subgroup N, $G = KN$ for some compact subset K of G and suppose that $s \to \sigma_s(x) : G \to N^{\mathcal{WAP}}$ is continuous for all $x \in N^{\mathcal{WAP}}$, then $(G\times N^{\mathcal{WAP}})/\sim$ is the universal semitopological semigroup compactification of G.*

Proof. Since $N^{\mathcal{WAP}}$ is a semitopological semigroup, by Lemma 2.15, $(G\times N^{\mathcal{WAP}})/\sim$ is semitopological semigroup. Thus by Theorem 2.13, $(G\times N^{\mathcal{WAP}})/\sim$ is the universal semitopological semigroup compactification of G. □

A similar argument yields the following corollary.

Corollary 2.17. *Let G be a locally compact group with a closed normal subgroup N, $G = KN$ for some compact subset K of G and suppose that $s \to \sigma_s(x) : G \to N^{\mathcal{AP}}$ is* *continuous for all $x \in N^{\mathcal{AP}}$, then $(G\times N^{\mathcal{AP}})/\sim$ is the universal topological semigroup compactification of G.*

Lemma 2.18. *Let N be a closed normal subgroup of G with $G = KN$ for some compact subset K of G. Let $\mathcal{F}$ and $\mathcal{G}$ be left m-introverted subalgebras of $\mathcal{LC}(N)$ and $\mathcal{LC}(G)$, respectively. Then $N^{\mathcal{F}}$ is a conjugation invariant $\mathcal{F}$-compactification of N if and only if $\mathcal{G}|_N = \mathcal{F}$ and $(G\times N^{\mathcal{F}})/\sim$ is the $\mathcal{G}$-compactification of G.*

Proof. Let $\mathcal{G}|_N = \mathcal{F}$, we define $\sigma_s(x)(f)$ for $s \in G$, $x \in N^{\mathcal{F}}$ and $f \in \mathcal{F}$ by $\sigma_s(x)(f) = x(g\circ\sigma_s|_N)$, where $g \in \mathcal{G}, g|_N = f$. Since every such extension g yields a $g\circ\sigma_s$ agreeing with $f\circ\sigma_s$ on N, $\sigma_s(x)$ is well defined. So $N^{\mathcal{F}}$ is a conjugation invariant $\mathcal{F}$- compactification of N.

Conversely, since the quotient map $\pi : G\times N^{\mathcal{F}} \to (G\times N^{\mathcal{F}})/\sim$ is injective on the compact set $N^{\mathcal{F}} \cong \{e\}\times N^{\mathcal{F}}$, it gives a topological isomorphism of $N^{\mathcal{F}}$ into $G^{\mathcal{G}} \cong (G\times N^{\mathcal{F}})/\sim$. □

Corollary 2.19. *Let G be a compact group with a closed normal subgroup N, then*
(i) $(G\times N^{\mathcal{WAP}})/\sim\,\cong G^{\mathcal{WAP}}$.
(ii) $(G\times N^{\mathcal{AP}})/\sim\,\cong G^{\mathcal{AP}}$.

Corollary 2.20. *Let N be a closed normal subgroup of a locally compact group G contained in the center of G, then*

$$\left(G\times N^{\mathcal{WAP}}\right)\Big/\sim\,\cong G^{\mathcal{WAP}}. \tag{2.9}$$

The next example shows that the continuity of $s \to \sigma_s(x)$ in Corollary 2.14 and Lemma 2.15 is an essential condition.

Example 2.21. Let $G = \mathbb{C}\times\mathbb{T}$ be the Euclidean group of the plane with $(z,w)(z_1,w_1) = (z+wz_1, ww_1)$ and $N = \mathbb{C}\times\{1\}$, then N is a closed normal subgroup of G and $\mathcal{AP}(G)|_N$ is a proper subset of $\mathcal{AP}(N)$ [4, 8], so by Lemma 2.15 $(G\times\mathbb{C}^{\mathcal{AP}})/\sim$ is not the universal $\mathcal{AP}$ compactification of G. $\mathbb{C}^{\mathcal{AP}}$ is a conjugation invariant compactification of N, so the continuity of $s \to \sigma_s$ must fail to hold Lemma 2.15. From [4, 8], we can similarly conclude that $(G\times\mathbb{C}^{\mathcal{WAP}})/\sim$ is not the universal $\mathcal{WAP}$-compactification of G and that the continuity of $s\to\sigma_s$, as required by Corollary 2.14, also fails to hold.

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