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A NOTE ON M-IDEALS IN CERTAIN ALGEBRAS OF OPERATORS

CHONG-MAN CHO and WOO SUK ROH

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ABSTRACT. Let $X = (\sum_{n=1}^{\infty} \ell_1^n)_p,\ p > 1$. In this paper, we investigate M-ideals which are also ideals in $L(X)$, the algebra of all bounded linear operators on X. We show that $K(X)$, the ideal of compact operators on X is the only proper closed ideal in $L(X)$ which is both an ideal and an M-ideal in $L(X)$.

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1. Introduction. Since Alfsen and Effros [1, 2] introduced the notion of an M-ideal in a Banach space, many authors have studied M-ideals in algebras of operators. An interesting problem has been characterizing and finding those Banach spaces X for which $K(X)$, the space of all compact linear operators on X, is an M-ideal in $L(X)$, the space of all continuous linear operators on X [4, 8, 9, 11, 12].

It is known that if X is a Hilbert space, $\ell_p\,(1 < p < \infty)$ or c_0, then $K(X)$ is an M-ideal in $L(X)$ [6, 8, 12] while $K(\ell_1)$ and $K(\ell_\infty)$ are not M-ideals in the corresponding spaces of operators [12]. Smith and Ward [12] proved that M-ideals in a complex Banach algebra with identity are subalgebras and that they are two-sided algebraic ideals if the algebra is commutative. They also proved that M-ideals in a C*-algebra are exactly the two-sided ideals [12]. Later, Cho and Johnson [5] proved that if X is a uniformly convex Banach space, then every M-ideal in $L(X)$ is a left ideal, and if X^* is also uniformly convex, then every M-ideal in $L(X)$ is a two-sided ideal in $L(X)$.

Flinn [7], and Smith and Ward [13] proved that $K(\ell_p)$ is the only nontrivial M-ideal in $L(\ell_p)$ for $1 < p < \infty$. Kalton and Werner [10] proved that if $1 < p,\ q < \infty$, $X = (\sum_{n=1}^{\infty} \ell_q^n)_p$ with complex scalars, then $K(X)$ is the only nontrivial M-ideal in $L(X)$. In their proof of this fact, Kalton and Werner [13] used the uniform convexity of X and X^*. In this case, M-ideals in $L(X)$ are two-sided closed ideals in $L(X)$ [5].

In this paper, we investigate M-ideals which are also ideals in $L(X)$ for $X = (\sum_{n=1}^{\infty} \ell_1^n)_p$, $1 < p < \infty$. In our case, neither X nor X^* is uniformly convex. Therefore, no relationships between M-ideals and algebraic ideals in $L(X)$ seem to be known. But still we can use Kalton and Werner's proof in [10] without using uniform convexity of X and X^* to prove that $K(X)$ is the only nontrivial M-ideal in $L(X)$ which is also a closed ideal in $L(X)$ (Theorem 3.3). By duality we have the same conclusion for the space $(\sum_{n=1}^{\infty} \ell_\infty^n)_p$, $1 < p < \infty$.

2. Preliminaries. A closed subspace J of a Banach space X is said to be an L-summand (respectively, M-summand) if there exists a closed subspace J' of X

such that X is an algebraic direct sum $X = J \oplus J'$ and satisfies a norm condition $\|j + j'\| = \|j\| + \|j'\|$ (respectively, $\|j + j'\| = \max\{\|j\|, \|j'\|\}$) for all $j \in J$ and $j' \in J'$. In this case, we write $X = J \oplus_1 J'$ (respectively, $X = J \oplus_\infty J'$) and the projection P on X with rang J is called an L-projection (respectively, an M-projection). A closed subspace J of a Banach space X is an M-ideal in X if the annihilator $J^\perp$ of J is an L-summand in X^*.

Let A be a complex Banach algebra with identity e. The state space S of A is defined to be $\{\phi \in A^* : \phi(e) = \|\phi\| = 1\}$. An element $h \in A$ is said to be Hermitian if $\|e^{i\lambda h}\| = 1$ for all real number λ. Equivalently, h is Hermitian if and only if $\phi(h)$ is real for every $\phi \in S$ [3, page 46].

In what follows, Z always denote a complex Banach space $(\sum_{n=1}^\infty \ell_1^n)_p$, the ℓ_p-sum of ℓ_1^n's for $1 < p < \infty$. For each n, let $\{e_{nl}\}_{l=1}^n$ be the standard basis of ℓ_1^n. Then these bases string together to form the standard basis $\{e_n\}_{n=1}^\infty$ of Z and each $T \in L(Z)$ has a matrix representation with respect to $\{e_n\}_{n=1}^\infty$. If $T \in L(Z)$ has the matrix whose (i,j)-entry is t_{ij}, then we can write $T = \sum_{i,j\ge 1} t_{ij} e_j \otimes e_i$, where $e_j \otimes e_i$ is the rank 1 map sending e_j to e_i. Observe that $T(e_j)$ forms the jth column vector of the matrix of T and $\|Te_j\| \le \|T\|$ for all $j = 1, 2, \ldots$. If the matrix of T has at most one nonzero entry in each row and column, then $\|T\|$ is the l_∞-norm of the sequence of nonzero entries.

A number of facts which hold in $L(\ell_p), 1 < p < \infty$, still hold in $L(Z)$. If the matrix of $T \in L(Z)$ is a diagonal matrix (t_{ij}) with real diagonal entries, then for each real λ the matrix of $e^{i\lambda T}$ is also a diagonal matrix with diagonal matrix entries $e^{i\lambda t_{ii}}$. Thus $T \in L(Z)$ is Hermitian if the matrix T is a diagonal matrix with real entries.

Flinn [7] proved that if M is an M-ideal in $L(\ell_p), 1 < p < \infty$ and h is a Hermitian element in $L(\ell_p)$ with $h^2 = I$, then $hM \subseteq M$ and $Mh \subseteq M$. From this he proved that if h is any diagonal matrix with real entries, then $hM \subseteq M$ and $Mh \subseteq M$. His proof is valid for Z in place of ℓ_p. Thus we have the following.

Lemma 2.1. *If M is an M-ideal in $L(Z)$ and $h \in L(Z)$ is a diagonal matrix with real entries, then $hM \subseteq M$ and $Mh \subseteq M$.*

The M-ideal structure of $L(X)$ for $X = (\sum_{n=1}^\infty \ell_q^n)_p, 1 < p, q < \infty$ was studied by Kalton and Werner [10]. Some of their proofs for X are still good for Z. One of them is the following.

Lemma 2.2. *There is a constant C such that, whenever (k_n) is a sequence of positive integers with* $\limsup k_n = \infty$, *then $(\sum_{n=1}^\infty \ell_1^{k_n})_p$ is C-isomorphic to $(\sum_{n=1}^\infty \ell_1^n)_p$.*

Proof. See proof of Lemma 3.1 of [10]. □

We recall that a Banach space X is C-isomorphic to a Banach space Y if there exists an isomorphism T form X onto Y such that

$$\frac{1}{C}\|x\| \le \|Tx\| \le C\|x\| \tag{2.1}$$

for every $x \in X$. We use the following lemma which is due to Kalton and Werner [10].

Lemma 2.3 [10]. *Let X be a Banach space, $\mathcal{T} \subset L(X)$ be a two-sided ideal, and P a projection onto a complemented subspace E of X which is isomorphic to X.*

(a) *If $P \in \mathcal{T}$, then $\mathcal{T} = L(X)$.*

(b) *If E is C-isomorphic with X and $\mathcal{T}$ contains an operator T with $\|T-P\| < (C\|P\|^{-1})$, then $\mathcal{T} = L(X)$.*

3. M-ideals in $L((\sum_{n=1}^{\infty} \ell_1^n)_p)$.

A matrix carpentry used by Flinn [7] to characterize the M-ideal structure in $L(\ell_p)$ can be used to some extent in our case $Z = (\sum_{n=1}^{\infty} \ell_1^n)_p$. The proof of the following lemma is really a minor modification of Flinn's proof in [7].

Lemma 3.1. *If M is a nontrivial M-ideal in $L(Z)$, then $K(Z) \subseteq M$.*

Sketch of the proof. Let us call two positive integers i and j are in the same block if $n(n+1)/2 < i,\ j \le (n+1)(n+2)/2$ for some n. Using Lemma 2.1, we can follow Flinn's proof of the second corollary to Lemma 1 in [7]. The only modification is the following: to prove $2^{1/q} < |t_{pl} + t_{kl}| \le 2^{1/q}$, we consider two cases. If p and k are in a different block, Flinn's proof just run through. If p and k are in the same block, then $2^{1/q} < |t_{pl} + t_{kl}| \le \|T(e_l)\| \le 2^{1/q}$. □

The proof of the following lemma is contained in the proof of Theorem 3.3 in [10].

Lemma 3.2. *If $\mathcal{T}$ is a closed ideal strictly containing $K(Z)$ then $\mathcal{T}$ contains all the operators which factor through ℓ_p.*

The proof of the following theorem is a modification of that of Kalton and Werner [10]. Here we can go around the use of uniform convexity.

Theorem 3.3. *If $\mathcal{T}$ is a closed ideal and also an M-ideal in $L(Z)$ strictly containing $K(Z)$, then $\mathcal{T} = L(Z)$.*

Proof. We recall that the standard basis $\{e_{nl}\}_{l=1}^{n}$ of ℓ_1^n string together to form the standard basis $\{e_n\}_{n=1}^{\infty}$ of Z. If $\{e'_n\}_{n=1}^{\infty}$ is the standard basis of ℓ_p, then the map $e_n \to e'_n$ gives a contraction from Z to ℓ_p. Since $E = \overline{\operatorname{span}}\{e_{nl}\}_{n=1}^{\infty}$ is isometric to ℓ_p, there exists a norm one operator A from Z to E carrying e_n to e_{n1} via e'_n. Thus A factors through ℓ_p. By Lemma 3.2, $A \in \mathcal{T}$.

Since $\mathcal{T}$ is also an M-ideal, by Proposition 2.3 in [14], there exists a net $(H_\alpha) \subseteq \mathcal{T}$ such that

$$\limsup \|\pm A + (\mathrm{Id} - H_\alpha)\| = 1. \tag{3.1}$$

To simplify subsequent calculations, let us write the standard basis of Z as $\{e_{nl} : n \in \mathbb{N},\ 1 \le l \le n\}$ and let $\{e_{nl}^* : n \in \mathbb{N},\ 1 \le l \le n\}$ be the corresponding biorthogonal functionals. Then $Ae_{nl} = e_{ml}$, where $m = (n-1)n/2 + l$.

Given $0 < \varepsilon < 1$,

$$\max_{\pm} \|\pm A + (\mathrm{Id} - H_\alpha)\| < 1 + \varepsilon \tag{3.2}$$

for infinitely many α's. For such an α and every e_{nl},

$$\max_{\pm} \|\pm Ae_{nl} - (\mathrm{Id} - H_\alpha)e_{nl}\| < 1 + \varepsilon. \tag{3.3}$$

Put $\alpha_{kj} = e_{kj}^*(\mathrm{Id}-H_\alpha)e_{nl}$. Then,

$$\begin{aligned}\max_{\pm} &\|\pm Ae_{nl}+(\mathrm{Id}-H_\alpha)e_{nl}\|^p\\ &= \max_{\pm}\|\pm e_{m1}-(\mathrm{Id}-H_\alpha)e_{nl}\|^p\\ &= \Big(\max_{\pm}|\alpha_{m1}\pm 1|+|\alpha_{m2}|+\cdots+|\alpha_{mm}|\Big)^p+\sum_{k\neq m}\left(\sum_{j=1}^{k}|\alpha_{kj}|\right)^p\\ &< (1+\varepsilon)^p.\end{aligned} \tag{3.4}$$

Since $\max_\pm|\alpha_{m1}\pm 1|\geq 1$, it follows that $\sum_{k\neq m}(\sum_{j=1}^{k}|\alpha_{kj}|)^p<(1+\varepsilon)^p-1$ and $|\alpha_{m2}|+\cdots+|\alpha_{mm}|<\varepsilon$. Since $\sqrt{1+|\alpha_{m1}|^2}\leq\max_\pm|\alpha_{m1}\pm 1|<1+\varepsilon, |\alpha_{m1}|<\sqrt{2\varepsilon+\varepsilon^2}<2\sqrt{\varepsilon}$. Thus $\|(\mathrm{Id}-H_\alpha)e_{nl}\|<((3\sqrt{\varepsilon})^p+(1+\varepsilon)^p-1)^{1/p}\to 0$ as $\varepsilon\to 0$ uniformly in n and l. It follows that, for any n,

$$\|P_n(\mathrm{Id}-H_\alpha)P_n\|\leq\|P_n(\mathrm{Id}-H_\alpha)j_n\|\leq\big((3\sqrt{\varepsilon})^p+(1+\varepsilon)^p-1\big)^{1/p}, \tag{3.5}$$

where P_n is the projection on Z with range $\ell_1^n\subseteq Z$ and j_n is the canonical injection of ℓ_1^n into Z.

By Lemma 3.2 in [10], there exists a sequence (k_n) such that, for the canonical projection P from Z onto $(\sum_{n=1}^{\infty}\ell_1^{k_n})_p$,

$$\|P-PH_\alpha P\|=\|P(\mathrm{Id}-H_\alpha)P\|<3\big((3\sqrt{\varepsilon})^p+(1+\varepsilon)^p-1\big)^{1/p}. \tag{3.6}$$

Since $PH_\alpha P\in\mathcal{T}$ and $\varepsilon>0$ is arbitrary small, by Lemmas 2.2 and 2.3, $\mathcal{T}=L(Z)$. □

From Lemma 3.1 and Theorem 3.3, we have the following.

Corollary 3.4. *If $\mathcal{T}$ is a proper ideal and also an M-ideal in $L(Z)$, then $\mathcal{T}=K(Z)$.*

Remark. By duality, all the lemmas, Theorem 3.3 and Corollary 3.4 hold with $Z^*=(\sum_{n=1}^{\infty}\ell_\infty^n)_p$, $1<p<\infty$, in place of Z.

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Cho: Department of Mathematics, Hanyang University, Seoul 133-791, Korea
E-mail address: cmcho@email.hanyang.ac.kr

Roh: Department of Mathematics, Hanyang University, Seoul 133-791, Korea