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ON A NEW GENERALIZATION OF ALZER'S INEQUALITY

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ABSTRACT. Let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence of positive real numbers. Under certain conditions of this sequence we use the mathematical induction and the Cauchy mean-value theorem to prove the following inequality:

$$\frac{a_n}{a_{n+m}} \le \left(\frac{(1/n)\sum_{i=1}^{n} a_i^r}{(1/(n+m))\sum_{i=1}^{n+m} a_i^r}\right)^{1/r},$$

where n and m are natural numbers and r is a positive number. The lower bound is best possible. This inequality generalizes the Alzer's inequality (1993) in a new direction. It is shown that the above inequality holds for a large class of positive, increasing and logarithmically concave sequences.

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1. Introduction. Several authors including Alzer [1], Sandor [8], and Ume [10] proved the following inequality:

$$\frac{n}{n+1} < \left(\frac{(1/n)\sum_{i=1}^{n} i^r}{(1/(n+1))\sum_{i=1}^{n+1} i^r}\right)^{1/r}, \tag{1.1}$$

where $r > 0$ and $n \in \mathbb{N}$. The proof of this inequality involves the principle of the mathematical induction and other analytical methods.

Based on the mathematical induction, Elezović and Pečarić [2] generalized (1.1) and proved the following theorem.

THEOREM 1.1. *If the sequence $\{a_n\}_{n=1}^{\infty}$ of positive real numbers satisfies the inequality*

$$1 \le \left(\frac{a_{n+2}}{a_{n+1}}\right)^r \left[\frac{a_{n+2}}{a_{n+1}} - 1 + \left(\frac{a_n}{a_{n+1}}\right)^{r+1}\right], \quad n \ge 0,\ a_0 = 0, \tag{1.2}$$

then the following inequality holds:

$$\frac{a_n}{a_{n+1}} \le \left(\frac{(1/a_n)\sum_{i=1}^{n} a_i^r}{(1/a_{n+1})\sum_{i=1}^{n+1} a_i^r}\right)^{1/r}. \tag{1.3}$$

Recently, Qi [4] proved a generalized version of (1.1). The reader is referred to [4, Corollary 2].

The main purpose of this paper is to further generalize inequalities (1.1) and (1.3).

2. Main Results

Theorem 2.1. *Let n and m be natural numbers. Suppose $\{a_1, a_2, \dots\}$ is a positive and increasing sequence satisfying*

$$\frac{(k+2)a_{k+2}^r - (k+1)a_{k+1}^r}{(k+1)a_{k+1}^r - k a_k^r} \ge \left(\frac{a_{k+2}}{a_{k+1}}\right)^r \tag{2.1}$$

for any given positive real number r and $k \in \mathbb{N}$, then we have the inequality

$$\frac{a_n}{a_{n+m}} \le \left(\frac{(1/n)\sum_{i=1}^n a_i^r}{(1/(n+m))\sum_{i=1}^{n+m} a_i^r}\right)^{1/r}. \tag{2.2}$$

The lower bound of (2.2) is best possible.

Proof. The inequality (2.2) is equivalent to

$$\frac{a_n^r}{a_{n+m}^r} \le \frac{(1/n)\sum_{i=1}^n a_i^r}{(1/(n+m))\sum_{i=1}^{n+m} a_i^r}, \tag{2.3}$$

that is,

$$\frac{1}{n a_n^r}\sum_{i=1}^n a_i^r \ge \frac{1}{(n+m)a_{n+m}^r}\sum_{i=1}^{n+m} a_i^r. \tag{2.4}$$

This is also equivalent to

$$\frac{1}{n a_n^r}\sum_{i=1}^n a_i^r \ge \frac{1}{(n+1)a_{n+1}^r}\sum_{i=1}^{n+1} a_i^r. \tag{2.5}$$

Since

$$\sum_{i=1}^{n+1} a_i^r = \sum_{i=1}^n a_i^r + a_{n+1}^r, \tag{2.6}$$

inequality (2.5) reduces to

$$\sum_{i=1}^n a_i^r \ge \frac{n a_n^r a_{n+1}^r}{(n+1)a_{n+1}^r - n a_n^r}. \tag{2.7}$$

It is easy to see that inequality (2.7) holds for $n = 1$.

Assume that inequality (2.7) holds for $n > 1$. Using the principle of induction, it is easy to show that (2.7) holds for $n+1$. Using equality (2.6), the induction can be written as (2.1) for $k = n$. Thus, inequality (2.7) holds.

It can easily be shown that

$$\lim_{r \to +\infty}\left(\frac{(1/n)\sum_{i=1}^n a_i^r}{(1/(n+m))\sum_{i=1}^{n+m} a_i^r}\right)^{1/r} = \frac{a_n}{a_{n+m}}. \tag{2.8}$$

Hence, the lower bound of (2.2) is best possible. The proof is complete. □

Corollary 2.2. *Let n and m be natural numbers. Suppose $a = \{a_1, a_2, \dots\}$ is a positive and increasing sequence satisfying*

$$a_{k+1}^2 \geq a_k a_{k+2}, \tag{2.9}$$

$$\frac{a_{k+1} - a_k}{a_{k+1}^2 - a_k a_{k+2}} \geq \max\left\{\frac{k+1}{a_{k+1}}, \frac{k+2}{a_{k+2}}\right\}, \quad k \in \mathbb{N}. \tag{2.10}$$

Then, for any given positive real number r, we have the inequality (2.2). The lower bound of (2.2) is best possible.

Proof. For $x \in [n, n+1]$, let

$$f(x) = (n+1-x)a_{n+1} + (x-n)a_{n+2}, \tag{2.11}$$

$$g(x) = (n+1-x)a_n + (x-n)a_{n+1}. \tag{2.12}$$

Further, we define

$$F(x) = (x+1)f^r(x), \quad G(x) = xg^r(x), \quad x \in [n, n+1]. \tag{2.13}$$

Direct calculation yields

$$F(n) = (n+1)a_{n+1}^r, \qquad F(n+1) = (n+2)a_{n+2}^r; \tag{2.14}$$

$$G(n) = na_n^r, \qquad G(n+1) = (n+1)a_{n+1}^r; \tag{2.15}$$

$$F'(x) = f^{r-1}(x)\left[f(x) + r(x+1)(a_{n+2} - a_{n+1})\right]; \tag{2.16}$$

$$G'(x) = g^{r-1}(x)\left[g(x) + rx(a_{n+1} - a_n)\right]. \tag{2.17}$$

Therefore, using the inequality (2.10) and standard arguments gives

$$\begin{aligned} \frac{F'(x)}{G'(x)} &= \left(\frac{(n+1-x)a_{n+1} + (x-n)a_{n+2}}{(n+1-x)a_n + (x-n)a_{n+1}}\right)^r \\ &\quad\times \frac{1 + r(x+1)(a_{n+2} - a_{n+1})/[(n+1-x)a_{n+1} + (x-n)a_{n+2}]}{1 + rx(a_{n+1} - a_n)/[(n+1-x)a_n + (x-n)a_{n+1}]} \\ &\geq \left(\frac{(n+1-x)a_{n+1} + (x-n)a_{n+2}}{(n+1-x)a_n + (x-n)a_{n+1}}\right)^r. \end{aligned} \tag{2.18}$$

Applying the Cauchy's mean-value theorem to the left side of inequality (2.1), it turns out that there exists one point $\zeta \in (n, n+1)$ such that

$$\begin{aligned} &\frac{(n+2)a_{n+2}^r - (n+1)a_{n+1}^r}{(n+1)a_{n+1}^r - na_n^r} \\ &\quad = \frac{F'(\zeta)}{G'(\zeta)} \geq \left(\frac{(n+1-\zeta)a_{n+1} + (\zeta-n)a_{n+2}}{(n+1-\zeta)a_n + (\zeta-n)a_{n+1}}\right)^r \geq \left(\frac{a_{n+2}}{a_{n+1}}\right)^r, \end{aligned} \tag{2.19}$$

in which the logarithmic convexity of the sequence $\{a_n\}_{n=1}^{\infty}$ is used. Thus, the inequality (2.1) is proved. $\square$

Corollary 2.3 [4]. *Let n and m be natural numbers and k a nonnegative integer. Then*

$$\frac{n+k}{n+m+k} < \left(\frac{(1/n)\sum_{i=k+1}^{n+k} i^r}{(1/(n+m))\sum_{i=k+1}^{n+m+k} i^r}\right)^{1/r}, \tag{2.20}$$

where r is any given positive real number. The lower bound is best possible.

Proof. This follows from Corollary 2.2 applied to $a = (k+1, k+2, \dots)$. □

Note. When $k = 0$ and $m = 1$, inequality (2.20) reduces to (1.1).

Note. Recently, some inequalities related to Alzer's inequality and the sum of powers of positive integers or sequences have been proved. For details, see Qi [6, 5, 3], Sándor [9], and Qi and Luo [7].

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