

LACUNARY STATISTICAL CONVERGENCE AND INCLUSION PROPERTIES BETWEEN LACUNARY METHODS

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ABSTRACT. A lacunary sequence is an increasing integer sequence $\theta = \{k_r\}$ such that $k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. A sequence x is called S_θ -convergent to L provided that for each $\varepsilon > 0$, $\lim_r (1/(k_r - k_{r-1})) \{\text{the number of } k_{r-1} < k \leq k_r : |x_k - L| \geq \varepsilon\} = 0$. In this paper, we study the general description of inclusion between two arbitrary lacunary sequences convergent.

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1. Introduction. An increasing integer sequence $\theta = \{k_r\}$ is called a lacunary sequence if it satisfies that $k_0 = 0$ and $h_r := k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. Throughout this paper, we follow the notation used in [7]. A sequence x is called S_θ -convergent to L provided that for each $\varepsilon > 0$. The intervals determined by θ is denoted by $I_r := (k_{r-1}, k_r]$ and the ratio k_r/k_{r-1} is abbreviated by q_r . For a finite set E , $|E|$ indicate the number of elements in E .

Let θ be a lacunary sequence, a complex number sequence x is said to be S_θ -convergent to L provided that for each $\varepsilon > 0$.

$$\lim_r \left(\frac{1}{h_r} \right) \{\text{the number of } k_{r-1} < k \leq k_r : |x_k - L| \geq \varepsilon\} = 0. \quad (1.1)$$

In this case we write $S_\theta - \lim x = L$ or $x_k \rightarrow L(S_\theta)$, and we define

$$S_\theta = \{x : S_\theta - \lim x = L, \text{ for some } L\}. \quad (1.2)$$

In [4] it is defined that the lacunary sequence $\beta = \{l_r\}$ is called a lacunary refinement of the lacunary sequence $\theta = \{k_r\}$ if $\{k_r\} \subseteq \{l_r\}$. In [7], after the inclusion relationship between S_θ and S_β is studied for a special case if $\beta = \{l_r\}$ is a lacunary refinement of the lacunary sequence $\theta = \{k_r\}$, an open problem is raised: what is the general description of the inclusion between two arbitrary lacunary methods. In this paper, we study the inclusion properties of different lacunary methods and solve this open problem.

2. Inclusion properties. If $\beta = \{l_r\}$ is a lacunary refinement of the lacunary sequence $\theta = \{k_r\}$, it is shown by [7, Thm. 7] that $x_k \rightarrow L(S_\beta)$ implies $x_k \rightarrow L(S_\theta)$, i.e., $S_\beta \subseteq S_\theta$. We begin this section by giving the converse of [7, Thm. 7].

THEOREM 1. Suppose $\beta = \{l_r\}$ is a lacunary refinement of the lacunary sequence $\theta = \{k_r\}$. Let $I_r := (k_{r-1}, k_r]$ and $J_r := (l_{r-1}, l_r], r = 1, 2, 3, \dots$. If there exists $\delta > 0$ such that

$$\frac{|J_j|}{|I_i|} \geq \delta \quad \text{for every } J_j \subseteq I_i. \tag{2.1}$$

Then $x_k \rightarrow L(S_\theta)$ implies $x_k \rightarrow L(S_\beta)$, i.e., $S_\theta \subseteq S_\beta$.

PROOF. For any $\varepsilon > 0$, and every J_j , we can find I_i such that $J_j \subseteq I_i$; then we have

$$\begin{aligned} \left(\frac{1}{|J_j|}\right) \left| \{k \in J_j : |x_k - L| \geq \varepsilon\} \right| &= \left(\frac{|I_i|}{|J_j|}\right) \left(\frac{1}{|I_i|}\right) \left| \{k \in J_j : |x_k - L| \geq \varepsilon\} \right| \\ &\leq \left(\frac{|I_i|}{|J_j|}\right) \left(\frac{1}{|I_i|}\right) \left| \{k \in I_i : |x_k - L| \geq \varepsilon\} \right| \\ &\leq \left(\frac{1}{\delta}\right) \left(\frac{1}{|I_i|}\right) \left| \{k \in I_i : |x_k - L| \geq \varepsilon\} \right|, \end{aligned} \tag{2.2}$$

and the proof completes immediately. □

THEOREM 2. Suppose $\beta = \{l_r\}, \theta = \{k_r\}$ are two lacunary sequences. Let $I_r := (k_{r-1}, k_r], J_r := (l_{r-1}, l_r], r = 1, 2, 3, \dots$, and $I_{ij} = I_i \cap J_j, i, j = 1, 2, 3, \dots$. If there exists $\delta > 0$, such that

$$\frac{|I_{ij}|}{|I_i|} \geq \delta \quad \text{for every } i, j = 1, 2, 3, \dots, \text{ provided } I_{ij} \neq \emptyset. \tag{2.3}$$

Then $x_k \rightarrow L(S_\theta)$ implies $x_k \rightarrow L(S_\beta)$, i.e., $S_\theta \subseteq S_\beta$.

PROOF. Let $\alpha = \beta \cup \theta$. Then α is a lacunary refinement of the lacunary sequence β , also θ . The interval sequence of α is $\{I_{ij} = I_i \cap J_j : I_{ij} \neq \emptyset\}$. From Theorem 1, the condition in Theorem 2: $|I_{ij}|/|I_i| \geq \delta$, for every $i, j = 1, 2, 3, \dots$, provided $I_{ij} \neq \emptyset$ yields that $x_k \rightarrow L(S_\theta)$ implies $x_k \rightarrow L(S_\alpha)$. Since α is also a lacunary refinement of the lacunary sequence β , from [4, Thm. 7] we have that $x_k \rightarrow L(S_\alpha)$ implies $x_k \rightarrow L(S_\beta)$. The proof follows immediately. □

In Theorem 2, if the condition is replaced by $|I_{ij}|/|J_j| \geq \delta$, for every $i, j = 1, 2, 3, \dots$, provided $I_{ij} \neq \emptyset$, it can be seen that $x_k \rightarrow L(S_\beta)$ implies $x_k \rightarrow L(S_\theta)$. Combining this remark and Theorem 2 we get the following theorem.

THEOREM 3. Suppose $\beta = \{l_r\}, \theta = \{k_r\}$ are two lacunary sequences. Let $I_r := (k_{r-1}, k_r], J_r := (l_{r-1}, l_r], r = 1, 2, 3, \dots$, and $I_{ij} = I_i \cap J_j, i, j = 1, 2, 3, \dots$. If there exists $\delta > 0$, such that

$$\frac{|I_{ij}|}{(|I_i| + |J_j|)} \geq \delta, \quad \text{for every } i, j = 1, 2, 3, \dots, \text{ provided } I_{ij} \neq \emptyset. \tag{2.4}$$

Then $x_k \rightarrow L(S_\theta)$ if and only if $x_k \rightarrow L(S_\beta)$, i.e., $S_\theta = S_\beta$.

Theorem 2 provides a sufficient condition for lacunary sequence $\beta = \{l_r\}, \theta = \{k_r\}$ to yield the inclusion relation $S_\theta \subseteq S_\beta$. In Theorem 4 we give some necessary and sufficient conditions for two lacunary sequences to have the relationship.

THEOREM 4. Suppose $\beta = \{l_r\}, \theta = \{k_r\}$ are two lacunary sequences. Let $J_r := (l_{r-1}, l_r], I_r := (k_{r-1}, k_r], r = 1, 2, 3, \dots$. There exists a sequence x and a number L such that $S_\beta - \lim x = L$ and $S_\theta - \lim x \neq L$, if and only if there exist $\{S_i\}, \{t_i\} \subseteq N$ and $\delta > 0$ which satisfies the following conditions:

- (i) $J_{s_i} \cap I_{t_i} \neq \emptyset$;
- (ii) $\lim_i |J_{s_i}|/|I_{t_i}| = \infty$;
- (iii) $|J_{s_i} \cap I_{t_i}|/|I_{t_i}| \geq \delta, i = 1, 2, 3, \dots$

PROOF

(I) NECESSITY. If there exists a sequence $x = \{x_k\}$ and a number L such that $S_\beta - \lim x = L$ and $S_\theta - \lim x \neq L$, then there exist a subsequence $\{t_i\} \subseteq N, \varepsilon > 0$, and $\delta > 0$, such that

$$\left(\frac{1}{|I_{t_i}|}\right) |\{k \in I_{t_i} : |x_k - L| \geq \varepsilon\}| \geq \delta, \quad i = 1, 2, 3, \dots \tag{2.5}$$

For each t_i , there exist a positive integer s_i and a whole number r_i such that

$$J_{s_i+j} \cap I_{t_i} \neq \emptyset \quad \text{for every } j = 0, 1, 2, \dots, i = 1, 2, 3, \dots \tag{2.6}$$

Then, we can write

$$\begin{aligned} \delta &\leq \left(\frac{1}{|I_{t_i}|}\right) |\{k \in I_{t_i} : |x_k - L| \geq \varepsilon\}| \\ &= \left(\frac{1}{|I_{t_i}|}\right) \left| \left\{ k \in \left(\bigcup_{j=0}^{r_i} J_{s_i+j} \right) \cap I_{t_i} : |x_k - L| \geq \varepsilon \right\} \right| \\ &= \left(\frac{1}{|I_{t_i}|}\right) \sum_{j=0}^{r_i} |\{k \in J_{s_i+j} \cap I_{t_i} : |x_k - L| \geq \varepsilon\}| \\ &= \left(\frac{1}{|I_{t_i}|}\right) \sum_{j=0, r_i} |\{k \in J_{s_i+j} \cap I_{t_i} : |x_k - L| \geq \varepsilon\}| \\ &\quad + \left(\frac{1}{|I_{t_i}|}\right) \sum_{0 < j < r_i} |\{k \in J_{s_i+j} \cap I_{t_i} : |x_k - L| \geq \varepsilon\}| \\ &= \left(\frac{1}{|I_{t_i}|}\right) \sum_{j=0, r_i} |\{k \in J_{s_i+j} \cap I_{t_i} : |x_k - L| \geq \varepsilon\}| \\ &\quad + \left(\frac{1}{|I_{t_i}|}\right) \sum_{0 < j < r_i} |\{k \in J_{s_i+j} : |x_k - L| \geq \varepsilon\}| \\ &= \left(\frac{1}{|I_{t_i}|}\right) \sum_{j=0, r_i} |\{k \in J_{s_i+j} \cap I_{t_i} : |x_k - L| \geq \varepsilon\}| \\ &\quad + \sum_{0 < j < r_i} \left(\frac{|J_{s_i+j}|}{|I_{t_i}|}\right) \left(\frac{1}{|J_{s_i+j}|}\right) |\{k \in J_{s_i+j} : |x_k - L| \geq \varepsilon\}|. \end{aligned} \tag{2.7}$$

Since $S_\beta - \lim x = L$, if $s_i + j$ is sufficiently large, we have

$$\left(\frac{1}{|J_{s_i+j}|}\right) \sum_{0 < j < r_i} |\{k \in J_{s_i+j} : |x_k - L| \geq \varepsilon\}| < \frac{\delta}{2}. \tag{2.8}$$

It can be seen that $\sum_{0 < j < r_i} (|J_{s_i+j}|/|I_{t_i}|) \leq 1$. Hence, we get

$$\begin{aligned} \delta &\leq \left(\frac{1}{|I_{t_i}|} \right) |\{k \in I_{t_i} : |x_k - L| \geq \varepsilon\}| \\ &\leq \left(\frac{1}{|I_{t_i}|} \right) \sum_{j=0, r_i} |\{k \in J_{s_i+j} \cap I_{t_i} : |x_k - L| \geq \varepsilon\}| + \frac{\delta}{2}. \end{aligned} \quad (2.9)$$

This implies that $(1/|I_{t_i}|) \sum_{j=0, r_i} |\{k \in J_{s_i+j} \cap I_{t_i} : |x_k - L| \geq \varepsilon\}| \geq \delta/2$, which ensures that at least one of the following inequalities hold:

$$\left(\frac{1}{|I_{t_i}|} \right) |\{k \in J_{s_i} \cap I_{t_i} : |x_k - L| \geq \varepsilon\}| \geq \frac{\delta}{4}, \quad (2.10)$$

or

$$\left(\frac{1}{|I_{t_i}|} \right) |\{k \in J_{s_i+r_i} \cap I_{t_i} : |x_k - L| \geq \varepsilon\}| \geq \frac{\delta}{4}. \quad (2.11)$$

Suppose the first one holds, i.e., $(1/|I_{t_i}|) |\{k \in J_{s_i} \cap I_{t_i} : |x_k - L| \geq \varepsilon\}| \geq \delta/4$. From this inequality and since $|\{k \in J_{s_i} \cap I_{t_i} : |x_k - L| \geq \varepsilon\}| \leq |J_{s_i} \cap I_{t_i}|$, we conclude that $\delta/4 \leq |J_{s_i} \cap I_{t_i}|/|I_{t_i}|$ which proves (iii).

For such s_i, t_i chosen in the proof of (iii), from (2.10), we have

$$\begin{aligned} \frac{\delta}{4} &\leq \left(\frac{1}{|I_{t_i}|} \right) |\{k \in J_{s_i} \cap I_{t_i} : |x_k - L| \geq \varepsilon\}| \\ &\leq \left(\frac{|J_{s_i}|}{|I_{t_i}|} \right) \left(\frac{1}{|J_{s_i}|} \right) |\{k \in J_{s_i} \cap I_{t_i} : |x_k - L| \geq \varepsilon\}|. \end{aligned} \quad (2.12)$$

Since $(1/|J_{s_i}|) |\{k \in J_{s_i} \cap I_{t_i} : |x_k - L| \geq \varepsilon\}| \rightarrow 0$, as $i \rightarrow \infty$, (2.12) implies $|J_{s_i}|/|I_{t_i}| \rightarrow \infty$, as $i \rightarrow \infty$, which proves (ii). It is clear that the intervals J_{s_i}, I_{t_i} chosen in the proof of (iii) satisfy (i).

(II) SUFFICIENCY. Suppose that for the two lacunary sequences $\beta = \{l_r\}, \theta = \{k_r\}$ there exist sequences $\{s_i\}, \{t_i\} \subseteq N$ and $\delta > 0$ which satisfy the conditions (i), (ii), and (iii) in the theorem. Define

$$x_k = \begin{cases} 1, & \text{if } k \in J_{s_i} \cap I_{t_i}, \\ 0, & \text{otherwise.} \end{cases} \quad (2.13)$$

For any $1 < \varepsilon < 0$, if $j \neq s_i$ for any $i = 1, 2, 3, \dots$, $(1/|J_j|) |\{k \in J_j : |x_k - 0| \geq \varepsilon\}| = |\emptyset|/|J_j| = 0$; if $j = s_i$, for some i , $(1/|J_{s_i}|) |\{k \in J_{s_i} \cap I_{t_i} : |x_k - 0| \geq \varepsilon\}| = |J_{s_i} \cap I_{t_i}|/|J_{s_i}| \leq |I_{t_i}|/|J_{s_i}| \rightarrow 0$. Hence $S_\beta - \lim x = 0$. But $(1/|I_{t_i}|) |\{k \in I_{t_i} : |x_k - 0| \geq \varepsilon\}| = |J_{s_i} \cap I_{t_i}|/|I_{t_i}| \geq \delta$, for $i = 1, 2, 3, \dots$. This implies that $S_\theta - \lim x \neq 0$. \square

COROLLARY 1. Suppose $\beta = \{l_r\}, \theta = \{k_r\}$ are two lacunary sequences. Let $J_r := (l_{r-1}, l_r], I_r := (k_{r-1}, k_r], r = 1, 2, 3, \dots$. If there exist $\{s_i\}, \{t_i\} \subseteq N$ such that $J_{s_i} \supseteq I_{t_i}, i = 1, 2, 3, \dots$, and $\lim_i |J_{s_i}|/|I_{t_i}| = \infty$, then there exists a sequence x and a number L such that $S_\beta - \lim x = L$ and $S_\theta - \lim x \neq L$.

PROOF. The conditions of the Corollary 1 imply all the conditions of the Theorem 4. So the corollary follow Theorem 4 immediately. \square

3. The Space X_θ . Let $\theta = \{k_r\}$ be a lacunary sequence. A S_θ -convergent sequence x is not necessary to be bounded. The superior norm in l^∞ does not apply to the set S_θ , where l^∞ is the set of bounded complex numbers. Hence we need to define a new norm on a suitable space. If we apply the superior norm in the subset of l^∞ , $S_\theta \cap l^\infty$, we get a closed subset of l^∞ .

THEOREM 5. $S_\theta \cap l^\infty$ is a closed subset of l^∞ with the superior norm.

PROOF. Suppose $\{x^n\} \subseteq S_\theta \cap l^\infty$ is a convergent sequence and it converges to $x \in l^\infty$. We need to prove that $x \in S_\theta \cap l$. Assume $x^n \rightarrow L_n(S_\theta), n = 1, 2, 3, \dots$. Take a positive decreasing convergent sequence $\{\varepsilon_n\}$, then for every $n = 1, 2, 3, \dots$, there is a positive integer m_n , such that if $n \geq m_n$, then $\|x - x^n\|_\infty \leq \varepsilon_n/4$. Without loss of generality, we just assume $m_n = n$. For the fixed n , there is k_{r_n} such that

$$\begin{aligned} \left(\frac{1}{h_{r_n}}\right) \left| \left\{ k \in I_{r_n} : |x_k^n - L_n| \geq \frac{\varepsilon_n}{4} \right\} \right| &< \frac{1}{4}, \\ \left(\frac{1}{h_{r_n}}\right) \left| \left\{ k \in I_{r_n} : |x_k^{n+1} - L_{n+1}| \geq \frac{\varepsilon_{n+1}}{4} \right\} \right| &< \frac{1}{4}. \end{aligned} \tag{3.1}$$

Then

$$\left\{ k \in I_{r_n} : |x_k^n - L_n| < \frac{\varepsilon_n}{4} \right\} \cap \left\{ k \in I_{r_n} : |x_k^{n+1} - L_{n+1}| < \frac{\varepsilon_{n+1}}{4} \right\} \neq \emptyset. \tag{3.2}$$

Pick a member k in this intersection, we can write

$$\begin{aligned} |L_n - L_{n+1}| &\leq |x_k^n - L_n| + |x_k^{n+1} - L_{n+1}| + |x_k^n - x_k^{n+1}| \\ &\leq |x_k^n - L_n| + |x_k^{n+1} - L_{n+1}| + \|x - x^n\|_\infty + \|x - x^{n+1}\|_\infty \\ &\leq \frac{\varepsilon_n}{4} + \frac{\varepsilon_{n+1}}{4} + \frac{\varepsilon_n}{4} + \frac{\varepsilon_{n+1}}{4} \leq \varepsilon_n. \end{aligned} \tag{3.3}$$

This implies that $\{L_n\}$ is a Cauchy sequence in C , and there is a complex number L such that $L_n \rightarrow L$, as $n \rightarrow \infty$. We need to prove that $x \rightarrow L(S_\theta)$. For any $\varepsilon > 0$, there is n , such that $\varepsilon_n < \varepsilon/4, \|x - x^n\|_\infty < \varepsilon/4, |L_n - L| < \varepsilon/4$, then

$$\begin{aligned} \left(\frac{1}{h_r}\right) \left| \left\{ k \in I_r : |x - L| \geq \varepsilon \right\} \right| &\leq \left(\frac{1}{h_r}\right) \left| \left\{ k \in I_r : |x^n - L_n| + \|x - x^n\|_\infty + |L_n - L| \geq \varepsilon \right\} \right| \\ &\leq \left(\frac{1}{h_r}\right) \left| \left\{ k \in I_r : |x^n - L_n| + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \geq \varepsilon \right\} \right| \\ &= \left(\frac{1}{h_r}\right) \left| \left\{ k \in I_r : |x^n - L_n| \geq \frac{\varepsilon}{2} \right\} \right| \rightarrow 0, \text{ as } r \rightarrow \infty. \end{aligned} \tag{3.4}$$

This gives that $x \rightarrow L(S_\theta)$, which completes the proof. □

Let $\theta = \{k_r\}$ be a lacunary sequence. for every $L \in C$, define

$$X_L := \{x \in S_\theta : x \rightarrow L(S_\theta)\} \quad \text{and} \quad X_\theta := \{X_L : L \in C\}. \tag{3.5}$$

If $x \rightarrow L(S_\theta), y \rightarrow K(S_\theta)$, and $a \in C$, it can be seen that

$$x + y \rightarrow (L + K)(S_\theta) \quad \text{and} \quad ax \rightarrow aL(S_\theta), \tag{3.6}$$

where $x + y = \{x_n + y_n\}$ and $ax = \{ax_n\}$. Then the following definitions on X_θ well defined

$$X_L + X_K := X_{L+K} \quad \text{and} \quad aX_L = X_{aL}. \quad (3.7)$$

For every $X_L \in X_\theta$, we define a norm of it by $\|X_L\| := |L|$.

THEOREM 6. *With the addition, scalar multiplication and norm defined on X_θ as above, X_θ is a Banach space.*

PROOF. To prove that X_θ is complete norm space, we just notice that $\{X_{L_n}\}$ is a Cauchy sequence of X_θ , if and only if $\{L_n\}$ is a Cauchy sequence of C , and for any complex number L , the sequence with constant terms L should converge to $L(S_\theta)$. Rest of the proof is clear. \square

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