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CONVEX ISOMETRIC FOLDING

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ABSTRACT. We introduce a new type of isometric folding called "convex isometric folding." We prove that the infimum of the ratio $\mathrm{Vol}\,N/\mathrm{Vol}\,\varphi(N)$ over all convex isometric foldings $\varphi : N \to N$, where N is a compact 2-manifold (orientable or not), is $1/4$.

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1. Introduction. A map $\varphi : M \to N$, where M and N are C^∞ Riemannian manifolds of dimensions m and n, respectively, is said to be an isometric folding of M into N if and only if for any piecewise geodesic path $\gamma : J \to M$, the induced path $\varphi \circ \gamma : J \to N$ is a piecewise geodesic and of the same length. The definition is given by Robertson [4]. The set of all isometric foldings $\varphi : M \to N$ is denoted by $\mathcal{F}(M,N)$.

Let $p : M \to N$ be a regular locally isometric covering and let G be the group of covering transformations of p. An isometric folding $\phi \in \mathcal{F}(M)$ is said to be p-invariant if and only if for all $g \in G$ and all $x \in X$, $p(\varphi(x)) = p(\varphi(g,x))$. See Robertson and Elkholy [5]. The set of p-invariant isometric foldings is denoted by $\mathcal{F}_i(M,p)$.

DEFINITION 1.1. Let $\varphi \in \mathcal{F}(M,N)$, where M and N are C^∞ Riemannian manifolds of dimensions m and n, respectively. We say that φ is a convex isometric folding if and only if $\varphi(M)$ can be embedded as a convex set in $\mathbb{R}^n$.

We denote the set of all convex isometric foldings of M into N by $C(M,N)$, and if $C(M,N) \neq \emptyset$, then it forms a subsemigroup of $\mathcal{F}(M,N)$.

DEFINITION 1.2. We say that $\varphi \in \mathcal{F}_i(M,p)$ is a p-invariant convex isometric folding if and only if $\varphi(M)$ can be embedded as a convex set in $\mathbb{R}^m$.

We denote the set of p-invariant convex isometric foldings of M by $C_i(M,p)$. If $C_i(M,p) \neq \emptyset$, then for any covering map, $p : M \to N$, $C_i(M,p)$ is a subsemigroup of $C(M)$.

To solve our main problem we need the following:

(1) Robertson and Elkholy [5] proved that if N is an n-smooth Riemannian manifold, $p : M \to N$ is its universal covering, and G is the group of covering transformations of p, then $\mathcal{F}(N)$ is isomorphic as a semigroup to $\mathcal{F}_i(M,p)/G$.

(2) Elkholy [1] proved that if N is an n-smooth Riemannian manifold, $p : M \to N$ is its universal covering, and $\varphi \in \mathcal{F}(N)$ such that $\varphi_* : \pi_1(N) \to \pi_1(N)$ is trivial, then the

corresponding folding $\psi \in \mathcal{J}_i(M,p)$ maps each fiber of p to a single point.

(3) Elkholy and Al-Ahmady [3] proved that under the same conditions of (2), if N is a compact 2-manifold, then

$$\frac{\operatorname{Vol} N}{\operatorname{Vol} \varphi(N)} = \frac{\operatorname{Vol} F}{\operatorname{Vol} \psi(F)}, \tag{1.1}$$

where F is a fundamental region of G in M.

2. Convex isometric folding and covering spaces.

The next theorem establishes the relation between the set of convex isometric folding of a manifold, $C(N)$, and the set of p-invariant convex isometric folding of its universal covering space, $C_i(M,p)$.

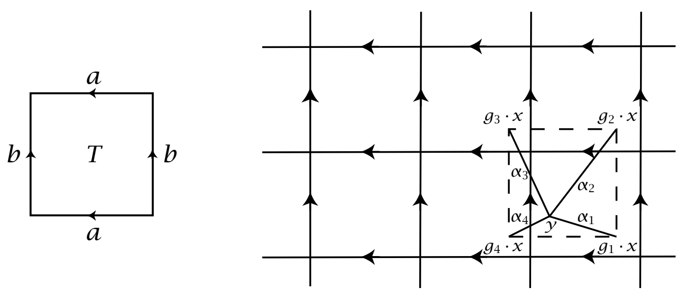
Theorem 2.1. *Let N be a manifold and $p : M \to N$ its universal covering. Let G be the group of covering transformations of p. If $C(N) \neq \emptyset$, then $C(N)$ is isometric as a semigroup to $C_i(M,p)/G$.*

Proof. Let $C(N) \neq \varphi$. Then by using (1), there exists an isomorphism f from $\mathcal{J}_i(M,p)/G$ into $\mathcal{J}(N)$. Since $C_i(M,p)$ is a subsemigroup of $\mathcal{J}_i(M,p)$, $C_i(M,p)/G$ is a subsemigroup of $\mathcal{J}_i(M,p)/G$.

Let $h = f \mid (C_i(M,P)/G)$. Since $C_i(M,p)/G$ is a semigroup, h is a homeomorphism and also it is one-one. To show that h is an onto map, we suppose that $\varphi \in C(N)$. Hence, $\varphi \in \mathcal{J}(N)$ and, consequently, there exists $\psi \in \mathcal{J}_i(M,p)/G$. Since $\varphi \in C(N)$, φ_* is trivial and hence for all $x \in M$, $\psi(G,x) = \psi(x)$, and therefore $\psi \in C_i(M,p)/G$. □

Theorem 2.2. *Let N be a compact orientable 2-manifold and consider the universal covering space $(\mathbb{R}^2,P)$ of N. Let $\varphi \in C(N)$ and $\psi \in C_i(\mathbb{R}^2,p)$. Then for all x, $y \in \mathbb{R}^2$, $d(\psi(x),\psi(y)) \leq \Delta$, where Δ is the radius of a fundamental region for the covering space.*

Proof. Elkholy [1] proved the truth of the theorem for $N = S^2$. So, we have to prove it for the connected sum of n-tori. First, let $N = T$ be a torus homomorphic to the quotient space obtained by identifying opposite sides of a square of length "a" as shown in Figure 1(a)



(a) (b)

Figure 1.

Suppose that $\varphi : T \to T$ is a convex isometric folding. Then $\varphi_*(\pi_1(T))$ is trivial. By Theorem 2.1, there exists a convex isometric folding $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ such that for all x, $y \in \mathbb{R}^2$ and for all $g \in G$, $p(\psi(x)) = p(\psi(g,x))$. Equivalently, for all $(P,Q) \in \mathbb{R}^2$ and for all $g \in \mathbb{Z}\times\mathbb{Z}$, there exists a unique $h \in \mathbb{Z}\times\mathbb{Z}$ such that $h \circ \psi(P,Q) = \psi(g(P,Q))$, i.e.,

$$\psi(P,Q) + \left(\sqrt{2}\Delta m, \sqrt{2}\Delta n'\right) = \psi\left(P + \sqrt{2}\Delta m, Q + \sqrt{2}\Delta n\right), \quad \text{where } m,n,m',n' \in \mathbb{Z}. \tag{2.1}$$

Consider any fundamental region F of the covering space $(\mathbb{R}^2, p)$ of T, i.e., a closed square of length "a" with sides identified as shown in Figure 1(b). Since φ_* is trivial, by (2), for all $x \in \mathbb{R}^2$, $\psi(G,x) = \psi(x)$. Now, let x and y be distinct points of $\mathbb{R}^2$ such that $x = g \cdot y$ for all $g \in G$ and let $d(x,y) = \alpha_1$. Then there exists a point $x^* = g \cdot x$ such that

$$d(y,x^*) = \min(\alpha_i), \qquad \alpha_i = d(y,g_i,x), \quad i = 1,\dots,4. \tag{2.2}$$

Thus, there are always four equivalent points $g_i \cdot x$, $i = 1,\dots,4$ which form the vertices of a square of length "a" and such that $d(g_i \cdot x, y) \le 2\Delta$. From Figure 1(b), it is clear that $\max d(x^*,y) \le \Delta$ and since ψ is an isometric folding, by Robertson [4], $d(\psi(x),\psi(y)) \le d(x,y)$, i.e.,

$$d\left(\psi(x),\psi(y)\right) = d\left(\psi(g_i \cdot x),\psi(y)\right) \le d(g_i \cdot x, y) = d(x,y) \le \Delta, \tag{2.3}$$

and this proves the theorem for $N = T$.

Now, consider the connected sum of two tori, obtained as a quotient space of an octagon with sides identified as shown in Figure 2(a). The group of covering transformations G is isometric to $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$. Using the same previous technique, we can

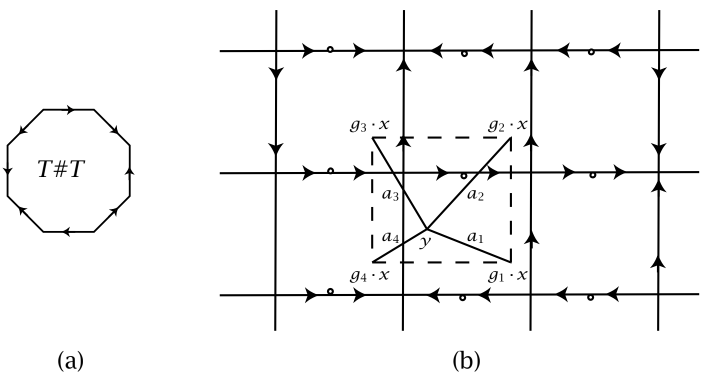


FIGURE 2.

obtain four equivalent points as the vertices of a square of diameter 2Δ such that $\max d(y,x^*) \le \Delta$, and the result follows. This theorem, by using the above method, is true for the connected sum of n-tori. $\square$

Theorem 2.3. *Let N be a compact nonorientable 2-manifold and consider the universal covering space (M,p) of N. Let $\phi \in C(N)$ and $\psi \in C_i(M,p)$. Then for all $x,y \in M$, $d(\psi(x),\psi(y)) \le \Delta$, where Δ is the radius of a fundamental region for the covering space.*

Proof. By Elkholy [2], the theorem is true for $N = p^2$ and $M = S^2$. Now, consider the connected sum of two projective planes, the Klein bottle K, homeomorphic to the quotient space obtained by identifying the opposite sides of a square as shown in Figure 3(a).

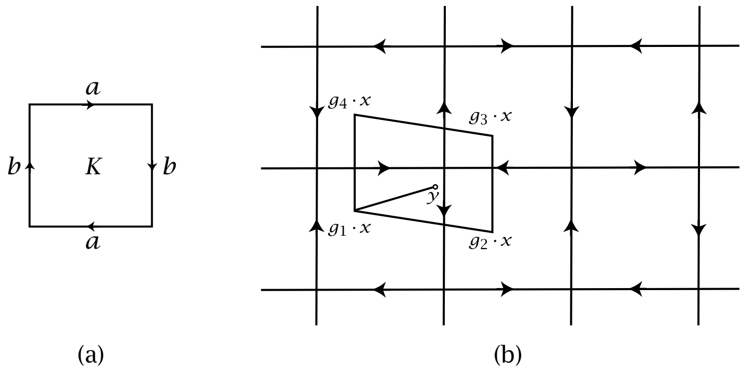


Figure 3.

Suppose that $\varphi : K \to K$ is a convex isometric folding. Then there exists a convex isometric folding $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ such that for all $x \in \mathbb{R}^2$ and $g \in G$, $p(\psi(x)) = p(\psi(g \cdot x))$. Equivalently, for all $(P,Q) \in \mathbb{R}^2$ and for all $g \in \mathbb{Z} \times \mathbb{Z}_2$, there exists a unique $h \in \mathbb{Z} \times \mathbb{Z}_2$ such that $h \circ \psi(P,Q) = \psi(g(P,Q))$, i.e.,

$$\begin{aligned} &\psi(P,Q) + \left(\sqrt{2}\Delta m', \sqrt{2}\Delta n'\right) \\ &\qquad = \psi\left(P + \sqrt{2}\Delta m, \sqrt{2}\Delta n + (-)^m Q\right), \quad \text{where } m,n,m',n' \in \mathbb{Z}. \end{aligned} \tag{2.4}$$

Any fundamental region F of the covering space $(\mathbb{R}^2,p)$ of K is a closed square of diameter 2Δ with the boundary identified as shown in Figure 3(b). Since φ_* is trivial, for all $x \in R^2$, $\psi(G \cdot x) = \psi(x)$.

Now, let x and y be distinct points of $\mathbb{R}^2$ such that $y \ne g \cdot x$ for all $g \in G$, and let $d(x,y) = \alpha_1$. Thus, there exists a point $x^* = g \cdot x$ such that

$$d(y,x^*) = \min(\alpha_i), \qquad \alpha_i = d(y, g_i \cdot x), \quad i = 1,\dots,4. \tag{2.5}$$

Thus, there are always four equivalent points $g_i \cdot x$ which form the vertices of a parallelogram such that the shortest diameter is of length less than 2Δ.

Now, the point y is either inside or on the boundary of a triangle of vertices $g_1 \cdot x = x$, $g_2 \cdot x$, $g_3 \cdot x$. Let y' be a point equidistant from the vertices of this triangle, i.e.,

$$d(y',x) = d(y', g_2 \cdot x) = d(y', g_3 \cdot x). \tag{2.6}$$

From Figure 3(b), it is clear that $d(y',x) < \Delta$ and, hence, $d(x^*,y) < \Delta$. Therefore,

$$d\left(\psi(x),\psi(y)\right) = d\left(\psi(g_i \cdot x),\psi(y)\right) \le d(g \cdot x_i, y) = d(x^*,y) < \Delta \tag{2.7}$$

and the result follows.

Now, let N be the connected sum of three projective planes obtained as the quotient space of a hexagon with the sides identified in pairs as indicated in Figure 4(a). In this case, $(\mathbb{R}^2, p)$ is the universal cover of N and $G \simeq \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_2$. Using the same method as that used above, we can always have equivalent points $g_i \cdot x$, $i = 1,\dots,4$ which form the vertices of a parallelogram whose shortest diameter is of length less than 2Δ. From Figure 4(b), we can see that $\max d(y,x^*) < \Delta$ and the theorem is proved.

In general and by using the same technique, the theorem is also true for the connected sum of n-projective planes. □

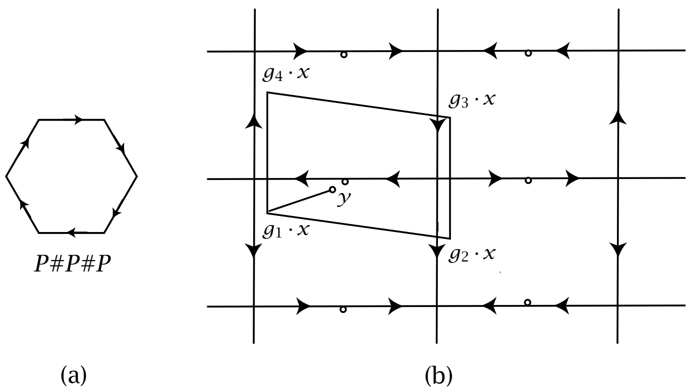


FIGURE 4.

3. Volume and convex folding. The following theorem succeeds in estimating the maximum volume we may have if we convexly folded a compact 2-manifold into itself.

THEOREM 3.1. *The infimum of the ratio*

$$e_N = \frac{\operatorname{Vol} N}{\operatorname{Vol} \varphi(N)}, \tag{3.1}$$

where N is a compact 2-manifold over all convex isometric foldings $\varphi \in C(N)$ of degree zero, is 4.

PROOF. Robertson [4] has shown that if N is a compact 2-manifold, and $\varphi : N \to N$ is a convex isometric folding, any convex isometric folding is an isometric folding, then $\deg\varphi$ is ± 1 or 0. We consider only the case for which $\deg\varphi$ is zero otherwise $\varphi(N)$ cannot be embedded as a convex subset of $\mathbb{R}^2$ unless N is. In this case, the set of singularities of φ decomposes N into an even number of strata, say k, each of which is homeomorphic to $\varphi(N)$ and, hence,

$$\operatorname{Vol} N = k \operatorname{Vol} \varphi(N), \tag{3.2}$$

that is, e_N should be an even number. To calculate the exact value of e_N, consider first an orientable 2-compact manifold N. By using (1.1)

$$e_N = \frac{\operatorname{Vol} F}{\operatorname{Vol} \varphi(F)} \tag{3.3}$$

and this means that e_N can be calculated by calculating the volume of F and of its image $\varphi(F)$, but F is a closed square of diameter 2Δ and $\varphi(F)$ is a closed subset of F such that the distance $d(x, x')$ between any two points x, $x' \in \varphi(F)$ is at most Δ. The supremum of 2-dimensional volume of such set is $\phi(\Delta/2)^2$ and, hence, $2 < e_N$. But e_N is an even number. Hence, $e_N = 4$.

Now, let N be a nonorientable 2-compact manifold, i.e., a connected sum of n-projective planes. Elkholy [2] proved the theorem for $n = 1$.

The fundamental region in this case is a square or a rectangle of diameter 2Δ according to whether n is even or odd. If n is an even number, then

$$\operatorname{Vol} F = 2\Delta^2 \tag{3.4}$$

and the result follows. Now, let n be an odd number. Then F is a rectangle of lengths $((n+1)/2)a$, $((n-1)/2)a$ and hence

$$\operatorname{Vol} F = 4\Delta^2 \sin\theta\cos\theta = 4\Delta^2 \frac{a(n+1)/2}{a\sqrt{(n^2+1)/2}} \frac{a(n-1)/2}{a\sqrt{(n^2+1)/2}} = \frac{n^2-1}{n^2+1} 2\Delta^2. \tag{3.5}$$

Therefore, $e_N > 2$ for all $n > 1$. Since e_N is an even number, $e_N = 4$. □

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