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# ON CONDITIONS FOR THE STRONG LAW OF LARGE NUMBERS IN GENERAL BANACH SPACES

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Abstract. We give Chung-Teicher type conditions for the SLLN in general Banach spaces under the assumption that the weak law of large numbers holds. An example is provided showing that these conditions can hold when some earlier known conditions fail.

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Let $(\mathcal{B}, \|\,\|)$ be a real, separable Banach space. A strongly measurable mapping $X$ from a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ into $\mathcal{B}$ is said to be a random element.

If $E\|X\| < \infty$, then the expectation $EX$ is defined by the Bochner integral.

Strong laws of large numbers (SLLN) for random elements, i.e., $(X_1 + \cdots + X_n)/n \to 0$ a.s., $n \to \infty$, were investigated by Mourier [14], Fortet and Mourier [8], Beck [3], Beck and Giesy [4], Hoffman-Jørgensen and Pisier [10], Heinkel [9], Taylor [17], Woyczyński [19], and Adler, Rosalsky, and Taylor [1]. Their efforts have concentrated on a complete characterization of all those Banach spaces in which the SLLN holds under conditions of classical probability theory or on finding conditions on the random elements which ensure the SLLN. It is known (Woyczyński [19]) that in certain Banach spaces the Chung's condition (Chung [7]) implies the SLLN for a sequence of independent random elements.

Some handy conditions for the SLLN in Banach spaces were given by Kuelbs and Zinn [13] and by Alt [2].

Extensions of the Chung-Teicher type conditions (cf. Chung [7], Teicher [18], Chow and Teicher [6]) for the SLLN for sequences of independent random elements in Hilbert space were found by Szynal and Kuczmaszewska [16], Choi and Sung [5], and Sung [15].

The aim of our paper is to give conditions for the SLLN in Banach spaces which can be applied in more general cases than those of Choi and Sung [5], Sung [15], Adler, Rosalsky, and Taylor [1] and Kuczmaszewska and Szynal [12]. We present also an example showing that these conditions can be applied when some earlier known conditions fail. We make use of the following lemmas.

**Lemma 1** (cf. Yurinskii [20]). *Let $X_1, X_2, \ldots, X_n$ be independent $\mathcal{B}$-valued random elements with $E\|X_i\| < \infty$, $i = 1, 2, \ldots, n$. Let $\mathcal{F}_k$ be $\sigma$-field generated by $(X_1, X_2, \ldots, X_k)$, $k = 1, 2, \ldots, n$ and let $\mathcal{F}_0 = \{\Omega, \emptyset\}$. Then for $1 \le k \le n$,*

$$\left| E(\|S_n\| \mid \mathcal{F}_k) - E(\|S_n\| \mid \mathcal{F}_{k-1}) \right| \le \|X_k\| + E\|X_k\|, \quad \text{where } S_n = \sum_{i=1}^{n} X_i. \tag{1}$$

**Lemma 2** (cf. Choi and Sung [5]). *Let $\{X_n,\ n \ge 1\}$ be a sequence of independent, $\mathcal{B}$-valued random elements. Then*

$$\frac{S_n}{n} \longrightarrow 0 \quad a.s.,\ n \longrightarrow \infty \tag{2}$$

*if and only if*

$$\frac{S_{2^k}}{2^k} \longrightarrow 0 \quad a.s.,\ k \longrightarrow \infty \qquad and \qquad \frac{S_n}{n} \xrightarrow{P} 0, \quad n \longrightarrow \infty. \tag{3}$$

Let a function $\phi : \mathbb{R} \to \mathbb{R}^+$ be nonnegative, even, continuous and nondecreasing on $(0,\infty)$ with $\lim_{x\to\infty}\phi(x) = \infty$ and such that

(a) $\phi(x)/x \searrow$ or

(b) $\phi(x)/x \nearrow$,

and $\phi(x)/x^p \searrow, x \to \infty$, for some $p$, $1 < p \le 2$.

**Theorem 3.** *Let $\{X_n,\ n \ge 1\}$ be a sequence of independent $\mathcal{B}$-valued random elements. Suppose that in the case (a) for some $p$, $1 < p \le 2$,*

(A) $\sum_{j=2}^{\infty} j^{-p} E\frac{\phi^p(||X_j||)}{\phi^p(j)+\phi^p(||X_j||)} \sum_{i=1}^{j-1} i^p E\frac{\phi^p(||X_i||)}{\phi^p(i)+\phi^p(||X_i||)} < \infty$,

(B) $n^{-p}\sum_{i=1}^{n} i^p E\frac{\phi^p(||X_i||)}{\phi^p(i)+\phi^p(||X_i||)} = o(1)$,

(C) $\sum_{n=1}^{\infty} P(\|X_n\| \ge a_n) < \infty$,

*for some sequence $\{a_n,\ n \ge 1\}$ of positive numbers such that*

(D) $\sum_{n=1}^{\infty}\phi^p(a_n)E\frac{\phi^p(||X_n||)}{\phi^{2p}(n)+\phi^{2p}(||X_n||)} < \infty$

*or in the case (b) for some $p$, $1 < p \le 2$,*

(A$_1$) $\sum_{j=2}^{\infty} j^{-p} E\frac{\phi(||X_j||)}{\phi(j)+\phi(||X_j||)} \sum_{i=1}^{j-1} i^p E\frac{\phi(||X_i||)}{\phi(i)+\phi(||X_i||)} < \infty$,

(B$_1$) $n^{-p}\sum_{i=1}^{n} i^p E\frac{\phi(||X_i||)}{\phi(i)+\phi(||X_i||)} = o(1)$,

*and (C) is satisfied for some sequence $\{a_n,\ n \ge 1\}$ of positive numbers such that*

(D$_1$) $\sum_{n=1}^{\infty}\phi(a_n)E\frac{\phi(||X_n||)}{\phi^2(n)+\phi^2(||X_n||)} < \infty$.

*Then*

$$n^{-1}\sum_{k=1}^{n}(X_k - EX'_k) \xrightarrow{P} 0, \quad n \longrightarrow \infty, \tag{4}$$

*if and only if*

$$n^{-1}\sum_{k=1}^{n}(X_k - EX'_k) \longrightarrow 0 \quad a.s.,\ n \longrightarrow \infty, \tag{5}$$

*where $X'_k = X_k I[\|X_k\| \le k]$.*

**Proof.** Suppose that (4) holds.

Let $r \ge 1$. We note that

$$\begin{aligned}\sum_{n=1}^{\infty} P(X'_n \ne X_n) &= \sum_{n=1}^{\infty} E[I(||X_n|| \ge n)\cdot I(||X_n|| \ge a_n) + I(||X_n|| \ge n)\cdot I(||X_n|| < a_n)]\\ &\le \sum_{n=1}^{\infty} P(||X_n|| \ge a_n) + 2\sum_{n=1}^{\infty} E\frac{\phi^{2r}(||X_n||)}{\phi^{2r}(n)+\phi^{2r}(||X_n||)} I(||X_n|| < a_n) \end{aligned} \tag{6}$$

$$\leq \sum_{n=1}^{\infty} P(\|X_n\| \geq a_n) + 2\sum_{n=1}^{\infty} \phi^r(a_n) E\frac{\phi^r(\|X_n\|)}{\phi^{2r}(n) + \phi^{2r}(\|X_n\|)} < \infty,$$

where we put $r = p$ in the case (a) and $r = 1$ in the case (b).

Hence $\{X_n',\ n \geq 1\}$ and $\{X_n,\ n \geq 1\}$ are equivalent. Therefore, by (4), we have

$$n^{-1}\left\|\sum_{i=1}^{n}(X_i' - EX_i')\right\| \xrightarrow{P} 0, \quad n \longrightarrow \infty. \tag{7}$$

Write

$$X_k^* = X_k' - EX_k', \quad k \geq 1, \qquad S_n' = \sum_{k=1}^{n} X_k', \qquad S_n^* = \sum_{k=1}^{n} X_k^*. \tag{8}$$

Define

$$Y_{n,i} = E(\|S_n^*\| \mid \mathcal{F}_i) - E(\|S_n^*\| \mid \mathcal{F}_{i-1}), \tag{9}$$

where $\mathcal{F}_i = \sigma(X_1^*, X_2^*, \dots, X_i^*)$ and $\mathcal{F}_0 = \{\Omega, \varnothing\}$.

Note that $\{Y_{n,i},\ 1 \leq i \leq n\}$ is a martingale difference for fixed $n$. Then we have

$$\|S_n^*\| - E\|S_n^*\| = \sum_{i=1}^{n} Y_{n,i}. \tag{10}$$

Now we prove that $E\|S_n^*\|/n \to 0,\ n \to \infty$. Using (9) and Lemma 1, we get

$$\begin{aligned} P(|\|S_n^*\| - E\|S_n^*\|| > n\varepsilon) &\leq \varepsilon^{-2} n^{-2} E(\|S_n^*\| - E\|S_n^*\|)^2 \\ &= \varepsilon^{-2} n^{-2} E\left(\sum_{i=1}^{n} Y_{n,i}\right)^2 = \varepsilon^{-2} n^{-2} \sum_{i=1}^{n} E(Y_{n,i}^2) \\ &\leq \varepsilon^{-2} n^{-2} \sum_{i=1}^{n} E(\|X_i^*\| + E\|X_i^*\|)^2 \\ &\leq 8\varepsilon^{-2} n^{-2} \sum_{i=1}^{n} E\|X_i'\|^2 \leq 8\varepsilon^{-2} n^{-p} \sum_{i=1}^{n} i^p E\frac{\|X_i'\|^p}{i^p}. \end{aligned} \tag{11}$$

Hence in the case (a) by (B), we get

$$P(|\|S_n^*\| - E\|S_n^*\|| > n\varepsilon) \leq 16\varepsilon^{-2} n^{-p} \sum_{i=1}^{n} i^p E\frac{\phi^p(\|X_i\|)}{\phi^p(i) + \phi^p(\|X_i\|)} = o(1), \tag{12}$$

while in the case (b) by ($B_1$)

$$P(|\|S_n^*\| - E\|S_n^*\|| > n\varepsilon) \leq 16\varepsilon^{-2} n^{-p} \sum_{i=1}^{n} i^p E\frac{\phi(\|X_i\|)}{\phi(i) + \phi(\|X_i\|)} = o(1). \tag{13}$$

Therefore, in the case (a) and (b) we have

$$n^{-1}(\|S_n^*\| - E\|S_n^*\|) \xrightarrow{P} 0, \quad n \longrightarrow \infty. \tag{14}$$

Thus we conclude, from $\|S_n^*\|/n \xrightarrow{P} 0$, $n \to \infty$, and (14), that

$$\frac{E\|S_n^*\|}{n} \longrightarrow 0, \quad n \longrightarrow \infty. \tag{15}$$

Now we are going to prove that $\|S_n^*\|/n \to 0$ a.s., $n \to \infty$. By Lemma 2 it is enough to prove that $\|S_{2^k}^*\|/2^k \to 0$ a.s., $k \to \infty$ or equivalently, as $E\|S_{2^k}^*\|/2^k \to 0$, $k \to \infty$, that

$$2^{-k}(\|S_{2^k}^*\| - E\|S_{2^k}^*\|) \longrightarrow 0 \quad \text{a.s., } k \longrightarrow \infty. \tag{16}$$

Taking into account the identity

$$(\|S_n^*\| - E\|S_n^*\|)^2 = \sum_{i=1}^{n} Y_{n,i}^2 + 2\sum_{i=2}^{n} Y_{n,i} \sum_{j=1}^{i-1} Y_{n,j}, \tag{17}$$

we see that

$$2^{-2n}(\|S_{2^n}^*\| - E\|S_{2^n}^*\|)^2 = 2^{-2n}\left(\sum_{i=1}^{2^n} Y_{2^n,i}^2 + 2\sum_{i=2}^{2^n} Y_{2^n,i} \sum_{j=1}^{i-1} Y_{2^n,j}\right). \tag{18}$$

Now, put

$$Z_{2^n,i} = Y_{2^n,i}^2 I(\|X_i\| < a_i) - E[Y_{2^n,i}^2 I(\|X_i\| < a_i) \mid \mathcal{F}_{i-1}], \quad 1 \le i \le 2^n. \tag{19}$$

Then by Chebyshev's inequality, equation (9) and Lemma 1, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{2^n} Z_{2^n,i}\right| > \varepsilon 2^{2n}\right) &\le \varepsilon^{-2}\sum_{n=1}^{\infty} 2^{-4n} E\left(\sum_{i=1}^{2^n} Z_{2^n,i}\right)^2 \\
&= \varepsilon^{-2}\sum_{n=1}^{\infty} 2^{-4n}\sum_{i=1}^{2^n} E[Y_{2^n,i}^4 I(\|X_i\| < a_i)] \\
&\le \varepsilon^{-2}\sum_{n=1}^{\infty} 2^{-4n}\sum_{i=1}^{2^n} E[(\|X_i^*\| + E\|X_i^*\|)^4 I(\|X_i\| < a_i)] \\
&\le \varepsilon^{-2}2^8\sum_{n=1}^{\infty} 2^{-4n}\sum_{i=1}^{2^n} E[\|X_i'\|^4 I(\|X_i\| < a_i)] \\
&\le \varepsilon^{-2}\left(\frac{2^{12}}{15}\right)\sum_{i=1}^{\infty} i^{-4} E[\|X_i'\|^4 I(\|X_i\| < a_i)].
\end{aligned} \tag{20}$$

Hence we see that in the case (a) under the assumptions (C) and (D) we have

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{2^n} Z_{2^n i}\right| > \varepsilon 2^{2n}\right) &\le \varepsilon^{-2}\left(\frac{2^{12}}{15}\right)\sum_{i=1}^{\infty} i^{-2p} E\left[\|X_i'\|^{2p} I(\|X_i\| < a_i)\right] \\
&\le \varepsilon\left(\frac{2^{12}}{15}\right)\sum_{i=1}^{\infty} E\frac{\phi^{2p}(\|X_i'\|)}{\phi^{2p}(i)} I(\|X_i\| < a_i) \\
&\le \varepsilon^{-2}\left(\frac{2^{13}}{15}\right)\sum_{i=1}^{\infty} \phi^p(a_i) E\frac{\phi^p(\|X_i\|)}{\phi^{2p}(i) + \phi^{2p}(\|X_i\|)} < \infty.
\end{aligned} \tag{21}$$

Similarly, in the case (b) under the assumptions (C) and ($D_1$) we lead to

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{2^n} Z_{2^n,i}\right| > \varepsilon 2^{2n}\right) \le \varepsilon^{-2}\left(\frac{2^{12}}{15}\right)\sum_{i=1}^{\infty} i^{-2p} E\left[\|X_i'\|^{2p} I(\|X_i\| < a_i)\right]$$
$$\le \varepsilon^{-2}\left(\frac{2^{13}}{15}\right)\sum_{i=1}^{\infty}\phi(a_i) E\frac{\phi(\|X_i\|)}{\phi^2(i)+\phi^2(\|X_i\|)} < \infty. \tag{22}$$

Therefore, by the Borel-Cantelli lemma, we state that

$$(2^n)^{-2}\left\{\sum_{i=1}^{2^n} Y_{2^n,i}^2 I(\|X_i\| < a_i) - \sum_{i=1}^{2^n} E[Y_{2^n,i}^2 I(\|X_i\| < a_i) \mid \mathcal{F}_{i-1}]\right\} \longrightarrow 0 \quad \text{a.s., } n \longrightarrow \infty. \tag{23}$$

Now, note that in the case (a) the assumption (B) implies

$$(2^n)^{-2}\sum_{i=1}^{2^n} E[Y_{2^n,i}^2 I(\|X_i\| < a_i) \mid \mathcal{F}_{i-1}]$$
$$\le (2^n)^{-2}\sum_{i=1}^{2^n} E(\|X_i^*\| + E\|X_i^*\|)^2 \le 8(2^n)^{-2}\sum_{i=1}^{2^n} E(\|X_i'\|)^2 \tag{24}$$
$$\le 16(2^n)^{-p}\sum_{i=1}^{2^n} i^p E\frac{\phi^p(\|X_i\|)}{\phi^p(i)+\phi^p(\|X_i\|)} = o(1).$$

Similarly, in the case (b), by $B_1$, we get

$$(2^n)^{-2}\sum_{i=1}^{2^n} E[Y_{2^n,i}^2 I(\|X_i\| < a_i) \mid \mathcal{F}_{i-1}] \le 16(2^n)^{-p}\sum_{i=1}^{2^n} i^p E\frac{\phi(\|X_i\|)}{\phi(i)+\phi(\|X_i\|)} = o(1). \tag{25}$$

Therefore, in the case (a) and (b), we obtain

$$(2^n)^{-2}\sum_{i=1}^{2^n} Y_{2^n,i}^2 I(\|X_i\| < a_i) \longrightarrow 0 \quad \text{a.s., } n \longrightarrow \infty. \tag{26}$$

Using the assumption (C), we get

$$(2^n)^{-2}\sum_{i=1}^{2^n} Y_{2^n,i}^2 I(\|X_i\| \ge a_i) \longrightarrow 0 \quad \text{a.s., } n \longrightarrow \infty \tag{27}$$

which proves in the end that

$$(2^n)^{-2}\sum_{i=1}^{2^n} Y_{2^n,i}^2 \longrightarrow 0 \quad \text{a.s., } n \longrightarrow \infty. \tag{28}$$

Now we see that $\{Y_{n,i}\sum_{j=1}^{i-1} Y_{n,j},\ 2 \le i \le n\}$ is a martingale difference for fixed $n$. Therefore, in the case (a), after using Chebyshev's inequality, (9), and Lemma 1, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\left|2^{-2n}\sum_{i=2}^{2^n} Y_{2^n,i}\sum_{j=1}^{i-1} Y_{2^n,j}\right| > \varepsilon\right) \\
&\quad\le \varepsilon^{-2}\sum_{n=1}^{\infty} 2^{-4n}\sum_{i=2}^{2^n} E\left(Y_{2^n,i}\sum_{j=1}^{i-1} Y_{2^n,j}\right)^2 \\
&\quad\le \varepsilon^{-2}\sum_{n=1}^{\infty} 2^{-4n}\sum_{i=2}^{2^n} E\left[\left(||X_i^*||+E||X_i^*||\right)^2\left(\sum_{j=1}^{i-1} Y_{2^n,j}\right)^2\right] \\
&\quad\le 2^8\varepsilon^{-2}\sum_{n=1}^{\infty} 2^{-4n}\sum_{i=2}^{2^n} E||X_i'||^2\sum_{j=1}^{i-1} E||X_j'||^2 \qquad (29)\\
&\quad\le \left(\frac{2^{12}}{15}\right)\varepsilon^{-2}\sum_{i=1}^{\infty} i^{-4}E||X_i'||^2\sum_{j=1}^{i-1} E||X_j'||^2 \\
&\quad\le \left(\frac{2^{12}}{15}\right)\varepsilon^{-2}\sum_{i=1}^{\infty} i^{-2p}E||X_i'||^p\sum_{j=1}^{i-1} E||X_j'||^p \\
&\quad\le \left(\frac{2^{14}}{15}\right)\varepsilon^{-2}\sum_{i=1}^{\infty} i^{-p}E\frac{\phi^p(||X_i||)}{\phi^p(i)+\phi^p(||X_i||)}\sum_{j=1}^{i-1} j^pE\frac{\phi^p(||X_j||)}{\phi^p(j)+\phi^p(||X_j||)} < \infty.
\end{aligned}$$

Similarly, in the case (b), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\left|2^{-2n}\sum_{i=2}^{2^n} Y_{2^n,i}\sum_{j=1}^{i-1} Y_{2^n,j}\right| \ge \varepsilon\right) \\
&\quad\le \left(\frac{2^{12}}{15}\right)\varepsilon^{-2}\sum_{i=2}^{\infty} i^{-2p}E||X_i'||^p\sum_{j=1}^{i-1} E||X_j'||^p \qquad (30)\\
&\quad\le \left(\frac{2^{14}}{15}\right)\varepsilon^{-2}\sum_{i=2}^{\infty} i^{-p}E\frac{\phi(||X_i||)}{\phi(i)+\phi(||X_i||)}\sum_{j=1}^{i-1} j^pE\frac{\phi(||X_j||)}{\phi(j)+\phi(||X_j||)} < \infty.
\end{aligned}$$

Now using the Borel-Cantelli lemma we obtain

$$2^{-2n}\sum_{i=2}^{2^n} Y_{2^n,i}\sum_{j=1}^{i-1} Y_{2^n,j} \longrightarrow 0 \quad \text{a.s., } n\longrightarrow\infty. \qquad (31)$$

Thus by (15) and (18) we see that

$$\frac{S_{2^n}^*}{2^n} \longrightarrow 0 \quad \text{a.s., } n\longrightarrow\infty, \qquad (32)$$

so we have

$$n^{-1}\left\|\sum_{i=1}^{n}(X_i'-EX_i')\right\| \longrightarrow 0 \quad \text{a.s., } n\longrightarrow\infty. \qquad (33)$$

But $\{X_n',\ n\ge 1\}$ and $\{X_n,\ n\ge 1\}$ are equivalent, so that

$$n^{-1}\left\|\sum_{n=1}^{\infty}(X_i-EX_i')\right\| \longrightarrow 0 \quad \text{a.s., } n\longrightarrow\infty \qquad (34)$$

which completes to proof of Theorem 3. □

Theorem 3 generalizes results of [2, 5, 10].

Before giving an example showing that the presented conditions under which WLLN is equivalent to the SLLN can be applied when some earlier known ones fail we quote the following results in this subject.

**Theorem 4** [13]. *Let $\{X_n,\ n \geq 1\}$ be a sequence of independent $\mathcal{B}$-valued random variables such that*

(a) $X_j/j \to 0$ *a.s.*, $j \to \infty$,

(b) *for some* $p \in [1,2]$ *and some* $r \in (0,\infty)$

$$\sum_{n=1}^{\infty}\left(\sum_{j=2^n+1}^{2^{n+1}} E\|X_j\|^p/2^{(n+1)p}\right)^r < \infty. \tag{35}$$

*Then*

$$\frac{S_n}{n} \xrightarrow{P} 0 \quad \text{if and only if } \frac{S_n}{n} \longrightarrow 0 \text{ a.s., } n \longrightarrow \infty. \tag{36}$$

**Theorem 5** [13]. *Let $\{X_n,\ n \geq 1\}$ be a sequence of independent $\mathcal{B}$-valued random variables such that*

(a) $|X_j| \leq M_j/LLj$ *for some constant* $M < \infty$, *where* $LLj = \log(\log(j \vee e^e))$, *and*

(b) $\sum_{n=1}^{\infty}\{-\varepsilon/\Lambda(n)\} < \infty$ *for all* $\varepsilon > 0$ *where* $\Lambda(n) = \sum_{j=2^n+1}^{2^{n+1}} E\|X_j\|^2/2^{2(n+1)}$. *Then*

$$\frac{S_n}{n} \xrightarrow{P} 0 \quad \text{if and only if } \frac{S_n}{n} \longrightarrow 0 \text{ a.s., } n \longrightarrow \infty. \tag{37}$$

**Theorem 6** [15]. *Let $\{X_n,\ n \geq 1\}$ be a sequence of independent $B$-valued random variables, and let $\{a_n\}$ and $\{b_n\}$ be constants that $0 < b_n \nearrow$. Suppose that*

(i) $\sum_{i=2}^{\infty} \frac{a_i^2 E\phi(\|X_i\|)}{b_i^4 \phi(a_i)} \sum_{j=1}^{i-1} \frac{a_j^2 E\phi(\|X_j\|)}{\phi(a_j)} < \infty$

(ii) $\frac{1}{b_n^2}\sum_{i=1}^{n} \frac{a_i^2 E\phi(\|X_i\|)}{\phi(a_i)} \to 0$,

(iii) $\sum_{i=1}^{\infty} P(\|X_i\| > a_i) < \infty$,

(iv) $\sum_{i=1}^{\infty} \frac{a_i^4 E\phi(\|X_i\|)}{b_i^4 \phi(a_i)} < \infty$.

*Then* $S_n/b_n \xrightarrow{P} 0$, *if and only if* $S_n/b_n \to 0$ *a.s.*, $n \to \infty$.

**Example 7.** Let

$$l^2 = \left\{x = \{x_n,\ n \geq 1\} \in \mathbb{R}^\infty,\ \|x\|^2 = \sum_{n=1}^{\infty} |x_n|^2 < \infty\right\} \tag{38}$$

and let $e^n$ denotes the element having 1 for its $n$th coordinate and 0 in the other coordinates.

Assume that $\{\xi_n,\ n \geq 1\}$ is a sequence of independent random variables such that

$$\begin{aligned} P(\xi_1 = 0) = 1, \qquad & P\left(\xi_j = \pm\frac{j}{(LLj)^{2/(1+\delta)}}\right) = \frac{(LLj)^2+1}{j\log j}, \\ P(\xi_j = \pm j^3) = \frac{1}{j^{1+\delta}}, \qquad & P(\xi_j = 0) = 1 - \frac{2}{j^{1+\delta}} - \frac{2(LLj)^2+2}{j\log j}, \end{aligned} \tag{39}$$

$j \geq 1$, $0 < \delta < 1$, and define $X_n = \xi_n e_n$.

We see that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\sum_{j=2^n+1}^{2^{n+1}}\frac{E\|X_j\|^p}{2^{(n+1)p}}\right)^r \\
&\quad= 2\sum_{n=1}^{\infty}\left(\sum_{j=2^n+1}^{2^{n+1}}\frac{\left[j^p/(LLj)^{2p/(1+\delta)}\cdot((LLj)^2+1)/j\log j+j^{3p}/j^{(1+\delta)}\right]}{2^{(n+1)p}}\right)^r \\
&\quad\geq 2\sum_{n=1}^{\infty}\left(\sum_{j=2^n+1}^{2^{n+1}}\frac{j^{3p-(1+\delta)}}{2^{(n+1)p}}\right)^r \geq 2\sum_{n=1}^{\infty}\left(2^n\cdot\frac{2^{n(3p-(1+\delta))}}{2^{np}2^p}\right)^r \\
&\quad= \frac{2}{2^{pr}}\sum_{n=1}^{\infty}2^{n(2p-\delta)r} = \infty.
\end{aligned} \tag{40}$$

Moreover, we note that the condition (a) of Theorem 5 (see [13]) is not satisfied. Therefore neither Theorem 4 nor Theorem 5 can be applied in this case.

Now we state that the series (i) with $\phi(x) = |x|^{1+\delta}$, $0 < \delta < 1$, $b_n = n$ and $a_n = n/LLn$ in Theorem 6 (see [15]) can be written in the form

$$\begin{aligned}
&\sum_{j=2}^{\infty}\frac{a_j^2E\phi(\|X_j\|)}{j^4\phi(a_j)}\sum_{i=1}^{j-1}a_i^2\frac{E\phi(\|X_i\|)}{\phi(a_i)} \\
&\quad= \sum_{j=3}^{\infty}\frac{j^2}{(LLj)^2}\frac{(j^{1+\delta}/(LLj)^2)\cdot(((LLj)^2+1)/j\log j)+(j^{3(1+\delta)})/(j^{1+\delta})}{j^4(j^{1+\delta}/(LLj)^{1+\delta})} \\
&\qquad\times\sum_{i=2}^{j-1}\frac{i^2}{(LLi)^2}\frac{((i^{1+\delta}/(LLi)^2)\cdot(((LLi)^2+1)/i\log i)+(i^{3(1+\delta)}/i^{1+\delta}))}{(i^{1+\delta}/(LLi)^{1+\delta})} \\
&\quad\geq \sum_{j=3}^{\infty}\frac{(j^2/(LLj)^2)\cdot j^{2(1+\delta)}}{j^4(j^{1+\delta}/(LLj)^{1+\delta})}\sum_{i=1}^{j-1}\frac{i^2}{(LLi)^2}\frac{i^{2(1+\delta)}}{(i^{1+\delta}/(LLi)^{1+\delta})} \\
&\quad\geq \sum_{j=3}^{\infty}\frac{j^{\delta-1}}{(LLj)^{1+\delta}}\sum_{i=1}^{j-1}\frac{i^{3+\delta}}{(LLi)^{1+\delta}} \geq \sum_{j=3}^{\infty}\frac{j^{\delta-1}}{(LLj)^{1-\delta}}\frac{j^{3+\delta}}{(LLj)^{1+\delta}} = \sum_{j=3}^{\infty}\frac{j^{2+2\delta}}{(LLj)^{2-2\delta}} = \infty.
\end{aligned} \tag{41}$$

Thus we cannot also use Theorem 6.

But we can show that the assumptions of Theorem 3 are fulfiled as

(A) $\sum_{j=2}^{\infty}j^{-2}E\frac{\phi(\|X_j\|)}{\phi(j)+\phi(\|X_j\|)}\sum_{i=1}^{j-1}i^2E\frac{\phi(\|X_i\|)}{\phi(i)+\phi(\|X_i\|)} \leq \sum_{j=2}^{\infty}j^{-2}\left(\frac{1}{j\log j}+\frac{1}{j^{(1+\delta)}}\right)\sum_{i=1}^{j-1}i^2 \times\left(\frac{1}{i\log i}+\frac{1}{i^{(1+\delta)}}\right) < \infty$,

(B) $n^{-2}\sum_{i=1}^{n}i^2E\frac{\phi(\|X_i\|)}{\phi(i)+\phi(\|X_i\|)} \leq n^{-2}\sum_{i=1}^{n}i^2\left(\frac{1}{i\log i}+\frac{1}{i^{(1+\delta)}}\right) = o(1)$,

(C) $\sum_{n=1}^{\infty}P\left[\|X_n\| \geq \frac{n}{LLn}\right] = \sum_{n=1}^{\infty}\frac{1}{n^{1+\delta}} < \infty$,

(D) $\sum_{n=1}^{\infty}\phi(a_n)E\frac{\phi(\|X_n\|)}{\phi(n)+\phi(\|X_n\|)} \leq C\sum_{n=2}^{\infty}\left(\frac{2}{n\log n(LLn)^{1+\delta}}+\frac{1}{n^{3(1+\delta)}(LLn)^{1+\delta}}\right) < \infty$.

Now it is enough to see that (4) holds. Taking into account that $\sum_{n=1}^{\infty} P[X'_n \neq X_n] < \infty$, we need only to verify that

$$P\left[\left\|\sum_{i=1}^{n}(X'_i - EX'_i)\right\| \geq n\varepsilon\right] \longrightarrow 0, \quad n \longrightarrow \infty. \tag{42}$$

Using Chebyshev's inequality and the fact that $l^2$ is a space of the type 2, we have

$$P\left[\left\|\sum_{i=1}^{n}(X'_i - EX'_i)\right\| \geq n\varepsilon\right] \leq C(\varepsilon)\sum_{i=1}^{n}\frac{E\|X'_i\|^2}{n^2} = o(1), \tag{43}$$

which completes the proof that

$$\frac{S_n}{n} \longrightarrow 0 \quad \text{a.s., } n \longrightarrow \infty \tag{44}$$

**Corollary 8.** *If $(B)$ and $(B_1)$ are replaced by the condition*

$$n^{-1}\sum_{i=1}^{n} iE\left[\frac{\phi(\|X_i\|)}{\phi(i)}\right] = o(1), \tag{45}$$

*and in the case (b) additionally $EX_k = 0$, $k \geq 1$, then*

$$\frac{S_n}{n} \xrightarrow{P} 0 \quad \text{a.s., } n \longrightarrow \infty \text{ if and only if } \frac{S_n}{n} \longrightarrow 0 \text{ a.s., } n \longrightarrow \infty. \tag{46}$$

**Proof.** It is enough to show that the condition

$$n^{-1}\sum_{i=1}^{n} iE\left[\frac{\phi(\|X_i\|)}{\phi(i)}\right] = o(1) \tag{47}$$

implies

$$n^{-1}\left\|\sum_{i=1}^{n} EX_iI(\|X_i\| < i)\right\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{48}$$

Indeed in the case (a) we have

$$\begin{aligned} n^{-1}\left\|\sum_{i=1}^{n} EX_iI(\|X_i\| < i)\right\| &= n^{-1}\left\|\sum_{i=1}^{n}\frac{iEX_iI(\|X_i\| < i)}{i}\right\| \\ &\leq n^{-1}\sum_{i=1}^{n} iE\left[\frac{\phi(\|X_i\|)}{\phi(i)}\right] = o(1), \end{aligned} \tag{49}$$

while in the case (b),

$$\begin{aligned} n^{-1}\left\|\sum_{i=1}^{n} EX_iI(\|X_i\| < i)\right\| &= n^{-1}\left\|\sum_{i=1}^{n}\frac{iEX_iI(\|X_i\| \geq i)}{i}\right\| \\ &\leq n^{-1}\sum_{i=1}^{n} iE\left[\frac{\phi(\|X_i\|)}{\phi(i)}\right] = o(1). \end{aligned} \tag{50}$$

□

**Corollary 9.** *Let* $\{X_n,\ n \ge 1\}$ *be a sequence of independent* $\mathcal{B}$*-valued random elements. If*

$$\sum_{j=2}^{\infty} j^{-2} E \frac{\|X_j\|^2}{j^2 + \|X_j\|^2} \sum_{i=1}^{j-1} i^2 E \frac{\|X_i\|^2}{i^2 + \|X_i\|^2} < \infty,$$

$$n^{-2} \sum_{i=1}^{n} i^2 E \frac{\|X_i\|^2}{i^2 + \|X_i\|^2} = o(1), \tag{51}$$

$$\sum_{n=1}^{\infty} P(\|X_n\| \ge a_n) < \infty$$

*for some sequence* $\{a_n,\ n \ge 1\}$ *of positive numbers with*

$$\sum_{n=1}^{\infty} a_n^2 E \frac{\|X_n\|^2}{n^4 + \|X_n\|^4} < \infty, \tag{52}$$

*then*

$$n^{-1} \sum_{k=1}^{n} (X_k - EX_k') \xrightarrow{P} 0, \quad n \longrightarrow \infty \tag{53}$$

*if and only if*

$$n^{-1} \sum_{k=1}^{n} (X_k - EX_k') \longrightarrow 0 \quad \text{a.s.}, n \longrightarrow \infty. \tag{54}$$

To prove the above-given assertion it is enough to use in the case (b) of the Theorem 3 the function $\phi(x) = x^2$.

**Corollary 10.** *Let* $\{X_n,\ n \ge 1\}$ *be a sequence of independent random elements in a Banach space of* $\mathcal{G}_\alpha$, $0 < \alpha \le 1$ [12]. *Suppose that in the case (a) the conditions (A), (B), (C) and (D) are satisfied with* $p = 1 + \alpha$, *or in the case (b) the conditions* $(A_1)$, $(B_1)$, $(C)$, *and* $(D_1)$ *are satisfied with* $p = 1 + \alpha$. *Then*

$$n^{-1} \sum_{k=1}^{n} (X_k - EX_k') \longrightarrow 0 \quad \text{a.s.}, n \longrightarrow \infty. \tag{55}$$

**Proof.** It is enough to show that (4) holds. Indeed in the case (a), we get

$$\begin{aligned} P\left[\left\|\sum_{k=1}^{n} (X_k' - EX_k')\right\| > n\varepsilon\right] &\le \varepsilon^{-2} n^{-2} E\left\|\sum_{k=1}^{n} (X_k' - EX_k')\right\|^2 \le 4\varepsilon^{-2} n^{-2} \sum_{k=1}^{n} E\|X_k'\|^2 \\ &\le 8\varepsilon^{-2} n^{-2} \sum_{k=1}^{n} k^p E \frac{\phi^p(\|X_k\|)}{\phi^p(k) + \phi^p(\|X_k\|)} = o(1), \end{aligned} \tag{56}$$

and in the case (b), we have

$$\begin{aligned} P\left[\left\|\sum_{k=1}^{n} (X_k' - EX_k')\right\| > n\varepsilon\right] &\le 4\varepsilon^{-2} n^{-p} \sum_{k=1}^{n} k^p E\left(\frac{\|X_k'\|^p}{k^p}\right) \\ &\le 8\varepsilon^{-2} n^{-p} \sum_{k=1}^{n} k^p E \frac{\phi(\|X_k\|)}{\phi(k) + \phi(\|X_k\|)} = o(1). \end{aligned} \tag{57}$$

But $\{X'_n,\ n \ge 1\}$ and $\{X_n,\ n \ge 1\}$ are equivalent, therefore we have (4) which completes the proof of Corollary 10. □

Now we give a generalization of Theorem 3 replacing the condition (A) or $(A_1)$ by less restrictive ones.

We need the following lemma (see [6, page 329]).

**Lemma 11.** *If $x_j$, $1 \le j \le n$, are real numbers, $S_n = \sum_{j=1}^n x_j$ and*

$$Q_{k,n} = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdot \cdots \cdot x_{i_k}, \quad 1 \le k \le n, \tag{58}$$

*then for $2 \le k \le n$,*

$$Q_{k,n} = \sum_{j=k}^{n} x_j Q_{k-1,j-1} \qquad \text{and} \qquad S_n^k = k! \cdot Q_{k,n} + c_k, \tag{59}$$

*where $c_k$ is a generic designation for a finite linear combination (coefficients independent of $n$) of terms $\prod_{j=1}^m \left( \sum_{i=1}^n x_i^{h_j} \right)$ of order $k$, that is,*

$$\sum_{j=1}^{m} h_j = k, \quad 1 \le h_j \le k,\ 1 \le m \le k. \tag{60}$$

Using this lemma we can prove the following result.

**Theorem 12.** *Let $\{X_n,\ n \ge 1\}$ be a sequence of independent $\mathcal{B}$-valued random elements. Suppose that in the case (a) for some $p$, $1 < p \le 2$,*

(A′) $\sum_{j_k=k}^{\infty} j_k^{-p(k-1)} E \frac{\phi^p(\|X_{j_k}\|)}{\phi^p(j_k)+\phi^p(\|X_{j_k}\|)} \sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^p \cdot E \frac{\phi^p(\|X_{j_{k-1}}\|)}{\phi^p(j_{k-1})+\phi^p(\|X_{j_{k-1}}\|)} \cdots \sum_{j_1=1}^{j_2-1} j_1^p E \frac{\phi^p(\|X_{j_1}\|)}{\phi^p(j_1)+\phi^p(\|X_{j_1}\|)} < \infty$ *for* $k \ge 2$

*(B) and (C)–(D) are satisfied, or the case (b) for some $p$, $1 < p \le 2$,*

($\text{A}_1'$) $\sum_{j_k=k}^{\infty} j_k^{-p(k-1)} E \frac{\phi(\|X_{j_k}\|)}{\phi(j_k)+\phi(\|X_{j_k}\|)} \sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^p \cdot E \frac{\phi(\|X_{j_{k-1}}\|)}{\phi(j_{k-1})+\phi(\|X_{j_{k-1}}\|)} \cdots \sum_{j_1=1}^{j_2-1} j_1^p E \frac{\phi(\|X_{j_1}\|)}{\phi(j_1)+\phi(\|X_{j_1}\|)} < \infty$,

*$(B_1)$, and (C)–$(D_1)$ are satisfied.*

*Then (4) holds if and only if (5) does.*

**Proof.** Using Lemma 11 we get

$$\begin{aligned}(2^n)^{-k} \left( \|S_{2^n}^*\| - E\|S_{2^n}^*\| \right)^k &= \left( 2^{-n} \sum_{i=1}^{2^n} Y_{2^n,i} \right)^k \\ &= \sum_{h=1}^{k-2} c_h \left( 2^{-n} \sum_{i=1}^{2^n} Y_{2^n,i} \right)^h A_{k-h,2^n} + c_k A_{k,2^n} + k!(2^n)^{-k} Q_{k,2^n},\end{aligned} \tag{61}$$

where $c_1, c_2, \dots, c_{k-2}$ are constants, $0 \le h \le k-2$, $A_{k-h,2^n}$ is finite linear combination of terms $\prod_{i=1}^{m}[(2^n)^{-h_i}\sum_{i=1}^{2^n} Y_{2^n,i}^{h_i}]$ satisfying $h_i \ge 2$ for $1 \le i \le i < m < k$, $\sum_{i=1}^{m} h_i = k-h$ and

$$Q_{0,2^n} = 1, \qquad Q_{k,2^n} = \sum_{1 \le i_1 < \cdots < i_k \le 2^n} Y_{2^n,i_1} \cdot \cdots \cdot Y_{2^n,i_k}. \tag{62}$$

Now note that, for $h_i \ge 2$,

$$\left| \prod_{i=1}^{m} \left[ (2^n)^{-h_i} \sum_{j=1}^{2^n} Y_{2^n,j}^{h_i} \right] \right| \le \prod_{i=1}^{m} \left[ (2^n)^{-2} \sum_{j=1}^{2^n} Y_{2^n,j}^{2} \right]^{h_i/2} \longrightarrow 0 \quad \text{a.s., } n \longrightarrow \infty, \tag{63}$$

as

$$(2^n)^{-2} \sum_{i=1}^{2^n} Y_{2^n,i}^2 \longrightarrow 0 \quad \text{a.s., } n \longrightarrow \infty, \tag{64}$$

(cf. the proof of Theorem 3).

Thus for $0 \le h \le k-2$,

$$A_{k-h,2^n} \longrightarrow 0 \quad \text{a.s., } n \longrightarrow \infty. \tag{65}$$

Now we show that

$$2^{-kn} \sum_{j_k=k}^{2^n} Y_{2^n,j_k} \sum_{j_{k-1}=k-1}^{j_{k-1}} Y_{2^n,j_{k-1}} \dots \sum_{j_1=1}^{j_{2-1}} Y_{2^n,j_1} \longrightarrow 0 \quad \text{a.s., } n \longrightarrow \infty. \tag{66}$$

To prove (66) it is enough to see that in the case (a)

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left( 2^{-kn} \left| \sum_{j_k=k}^{2^n} Y_{2^n,j_k} \sum_{j_{k-1}=k-1}^{j_k-1} Y_{2^n,j_{k-1}} \cdots \sum_{j_1=1}^{j_2-1} Y_{2^n,j_i} \right| > \varepsilon \right) \\
&\quad \le \varepsilon^{-2} \sum_{n=1}^{\infty} 2^{-2kn} E \left| \sum_{j_k=k}^{2^n} Y_{2^n,j_k} \sum_{j_{k-1}=k-1}^{j_k-1} Y_{2^n,j_{k-1}} \cdots \sum_{j_1=1}^{j_2-1} Y_{2^n,j_1} \right|^2 \\
&\quad = \varepsilon^{-2} \sum_{n=1}^{\infty} 2^{-2kn} \sum_{j_k=k}^{2^n} EY_{2^n,j_k}^2 \sum_{j_{k-1}=k-1}^{j_k-1} EY_{2^n,j_{k-1}}^2 \cdots \sum_{j_1=1}^{j_2-1} EY_{2^n,j_1}^2 \\
&\quad \le \varepsilon^{-2} 2^{4k} \sum_{n=1}^{\infty} 2^{-2kn} \sum_{j_k=k}^{2^n} E\|X'_{j_k}\|^2 \sum_{j_{k-1}=k-1}^{j_k-1} E\|X'_{j_{k-1}}\|^2 \cdots \sum_{j_1=1}^{j_2-1} E\|X'_{j_1}\|^2 \\
&\quad \le \frac{\varepsilon^{-2} 2^{6k}}{2^{2k}-1} \sum_{j_k=k}^{\infty} j_k^{-p(k-1)} E \frac{\|X'_{jk}\|^p}{j_k^p} \sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^p E \frac{\|X'_{jk-1}\|^p}{j_{k-1}^p} \cdots \sum_{j_1=1}^{j_2-1} j_1^p E \frac{\|X'_{j_1}\|^p}{j_1^p} \\
&\quad \le \frac{\varepsilon^{-2} 2^{7k}}{2^{2k}-1} \sum_{j_k=k}^{\infty} j_k^{-p(k-1)} E \frac{\phi^p(\|X_{j_k}\|)}{\phi^p(j_k) + \phi^p(\|X_{j_k}\|)} \\
&\quad\quad \times \sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^p E \frac{\phi^p(\|X_{j_{k-1}}\|)}{\phi^p(j_{k-1}) + \phi^p(\|X_{j_{k-1}}\|)} \cdots \sum_{j_1=1}^{j_2-1} j_1^p E \frac{\phi^p(\|X_{j_1}\|)}{\phi^p(j_1) + \phi^p(\|X_{j_1}\|)} < \infty,
\end{aligned} \tag{67}$$

while in the case (b)

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(2^{-kn}\left|\sum_{j_k=k}^{2^n} Y_{2^n,j_k}\sum_{j_{k-1}=k-1}^{j_{k-1}} Y_{2^n,j_{k-1}}\cdots\sum_{j_1=1}^{j_{2-1}} Y_{2^n,j_1}\right|>\varepsilon\right)\\
&\quad\le \frac{\varepsilon^{-2}2^{6k}}{2^{2k}-1}\sum_{j_k=k}^{\infty} j_k^{-p(k-1)}E\frac{\|X'_{j_k}\|^p}{j_k^p}\sum_{j_{k-1}=k-1}^{j_{k-1}} j_{k-1}^p E\frac{\|X'_{j_{k-1}}\|^p}{j_{k-1}^p}\cdots\sum_{j_1=1}^{j_{2-1}} j_1^p E\frac{\|X'_{j_1}\|^p}{j_1^p}\\
&\quad\le \frac{\varepsilon^{-2}2^{7k}}{2^{2k}-1}\sum_{j_k=k}^{\infty} j_k^{-p(k-1)}E\frac{\phi(\|X_{j_k}\|)}{\phi(j_k)+\phi(\|X_{j_k}\|)} \qquad (68)\\
&\qquad\times\sum_{j_{k-1}=k-1}^{j_{k-1}} j_{k-1}^p E\frac{\phi(\|X_{j_{k-1}}\|)}{\phi(j_{k-1})+\phi(\|X_{j_{k-1}}\|)}\cdots\sum_{j_1=1}^{j_{2-1}} j_1^p E\frac{\phi(\|X_{j_1}\|)}{\phi(j_1)+\phi(\|X_{j_1}\|)}<\infty.
\end{aligned}$$

By (61), we see that $2^{-n}\sum_{i=1}^{2^n} Y_{2^n,i}$ is a root of a $k$th degree polynomial in which the leading coefficient is unity and the remaining coefficients (cf. (65) and (66)) converge almost surely to zero. Therefore, the conclusion

$$2^{-n}\left(\|S_{2^n}^*\|-E\|S_{2^n}^*\|\right)=2^{-n}\sum_{i=1}^{2^n} Y_{2^n,i}\longrightarrow 0 \quad \text{a.s., } n\longrightarrow\infty, \qquad (69)$$

follows from the well-known relations between the roots and coefficients of a polynomial. Similarly, as in the proof of Theorem 3 we can complete the proof of Theorem 12.

□

**Remark 13.** The presented results extend also to random elements theorems of [18] and [11].

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