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SMALL BOUND ISOMORPHISMS OF THE DOMAIN OF A CLOSED $*$-DERIVATION

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Abstract. The domain $\mathfrak{D}(\delta)$ of a closed $*$-derivation δ in $C(K)$ (K : a compact Hausdorff space) is a generalization of the space $C^{(1)}[0,1]$ of differentiable functions on $[0,1]$. In this paper, a problem proposed by Jarosz (1985) is studied in the context of derivations instead of $C^{(1)}[0,1]$.

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Let K_1 and K_2 be two compact Hausdorff spaces. $C(K_i)$ denotes a space of all complex valued continuous functions on K_i $(i = 1,2)$. Let T be a surjective linear isometry from $C(K_1)$ to $C(K_2)$. Then the Banach-Stone theorem states that there exist a homeomorphism τ from K_2 to K_1 and a function w in $C(K_2)$ with $|w(y)| = 1$ $(y \in K_2)$ such that

$$Tf(y) = w(y)f(\tau(y)) \quad \text{for } f \in C(K_1),\ y \in K_2. \tag{1}$$

That is, the existence of a surjective linear isometry between $C(K_1)$ and $C(K_2)$ implies that K_1 and K_2 are homeomorphic. Amir [1] and Cambern [2] extended this theorem from this viewpoint as follows.

Theorem 1 (see [1, 2]). *If there is a surjective linear isomorphism $T : C(K_1) \to C(K_2)$ such that $\|T\|\,\|T^{-1}\| < 2$, then K_1 and K_2 are homeomorphic.*

Let X be a compact subset of the real line $\mathbb{R}$ and $C^{(1)}(X)$ be the space of continuously differentiable functions on X with the Σ-norm defined by $\|f\|_\Sigma = \sup_{x\in X}|f(x)| + \sup_{x\in X}|f'(x)|$.

In [4], Jarosz proposed the following question: "Is there a positive ε such that for any compact subsets X, Y of the real line $\mathbb{R}$ and any linear isomorphism $T : C^{(1)}(X) \to C^{(1)}(Y)$, $\|T\|\|T^{-1}\| < \varepsilon$ implies that X and Y are homeomorphic?"

In [5], Jun and Lee obtained some partial answers for this question.

Theorem 2 (see [5]). *Let X and Y be compact subset of $\mathbb{R}$ and $X \subset [a,b]$ and $Y \subset [c,d]$. If T is a linear isomorphism between $C^1(X)$ and $C^1(Y)$ which satisfies*

(i) *if $f'(t) \equiv 0$, then $(Tf)' \equiv 0$,*
(ii) *$\|fg\| \le \|TfTg\| \le (1+\varepsilon)^2\|fg\|$,*
(iii) *$\|f\| \le \|Tf\| \le (1+\varepsilon)\|f\|$,*
(iv) *$\varepsilon < \min\{1/49, 1/2(b-a+1), 1/2(c-d+1)\}$,*

then X and Y are homeomorphic.

Theorem 3 [5]. *Let X and Y be compact subsets of $\mathbb{R}$ and $X \subset \bigcup_{i=1}^{n}[a_i,b_i]$ ($a_i < b_i < a_{i+1}$) and $\max_i\{|b_i - a_i|\} < k$ and $Y \subset \bigcup_{j=1}^{m}[c_j,d_j]$ ($c_j < d_j < c_{j+1}$) and $\max_i\{|d_j - c_j|\} < k$. If T is a linear map from $C^1(X)$ onto $C^1(Y)$ which satisfies*

(i) $f'(t) \equiv 0$ *if and only if* $(Tf)' \equiv 0$,

(ii) $\|f\| \le \|Tf\| \le (1+\varepsilon)\|f\|$,

(iii) $k < (4-\sqrt{10})/6$ *and* $\varepsilon < 6k^2 - 8k + 1$,

then X and Y are homeomorphic.

In this paper, we consider this problem from another viewpoint. To the end, we recall a closed $*$-derivation.

Let K be a compact Hausdorff space and $C(K)$ denotes the space of all complex valued continuous functions on K with the supremum norm $\|\cdot\|_\infty$. A closed $*$-derivation δ in $C(K)$ is a linear mapping in $C(K)$ satisfying the following conditions:

(1) The domain $\mathfrak{D}(\delta)$ of δ is a norm dense subalgebra of $C(K)$.

(2) $\delta(fg) = \delta(f)g + f\delta(g) (f,g \in \mathfrak{D}(\delta))$.

(3) If $f_n \in \mathfrak{D}(\delta)$, $f_n \to f$, and $\delta(f_n) \to g$ implies $f \in \mathfrak{D}(\delta)$ and $\delta(f) = g$ (i.e., δ is closed as a linear operator).

(4) $f \in \mathfrak{D}(\delta)$ implies $f^* \in \mathfrak{D}(\delta)$ and $\delta(f^*) = \delta(f)^*$, where f^* means the complex conjugate of f.

The differentiation d/dt on the space $C^{(1)}([0,1])$ of continuously differentiable functions on $[0,1]$ is a typical example of closed $*$-derivations. For any closed $*$-derivation δ in $C(K)$, we may regard the domain $\mathfrak{D}(\delta)$ of δ as a generalization of the Banach space $C^{(1)}([0,1])$. Moreover, if $\mathfrak{D}(\delta) = C(K)$, δ is bounded and hence $\delta \equiv 0$.

Properties of the domains of closed $*$-derivations have been studied by many authors.

We summarize useful properties of closed $*$-derivations which is used later frequently without references.

Property 4 [7]. For $f(= f^*) \in \mathfrak{D}(\delta)$ and $h \in C^{(1)}([-\|f\|_\infty, \|f\|_\infty])$, $h(f)(= h \circ f) \in \mathfrak{D}(\delta)$ and $\delta(h(f)) = h'(f)\delta(f)$, where h' means the derivative of h.

Property 5 [7]. If $f \in \mathfrak{D}(\delta)$ is a constant in a neighborhood of $x \in K$, then $\delta(f)(x) = 0$.

Property 6 [7]. Let J_1 and J_2 be disjoint closed subsets of K. Then there is a function $f \in \mathfrak{D}(\delta)$ such that

$$f = 0 \quad \text{on } J_1, \quad f = 1 \quad \text{on } J_2, \quad (0 \le f \le 1). \tag{2}$$

Now, for any fixed point $x \in K$, we define a linear functional $\eta_x \circ \delta$ on $\mathfrak{D}(\delta)$ by

$$\eta_x \circ \delta(f) := \delta(f)(x) \quad (f \in \mathfrak{D}(\delta)). \tag{3}$$

Let $K(\delta)$ be the set of $x \in K$ such that $\eta_x \circ \delta \ne 0$, i.e.,

$$K(\delta) = \{x \in K : \eta_x \circ \delta \ne 0\} = \{x \in K : \exists f \in \mathfrak{D}(\delta) \text{ such that } \delta(f)(x) \ne 0\}. \tag{4}$$

Then $K(\delta)$ is an open subset of K.

Throughout this paper, the norm $\| \ \|$ in $\mathfrak{D}(\delta)$ is given by

$$\|f\| := \|f\|_\infty + \|\delta(f)\|_\infty \quad (f \in \mathfrak{D}(\delta)). \tag{5}$$

Then we note that for $x_0 \in K(\delta)$, the norm of a linear functional $\eta_{x_0} \circ \delta$ is 1 (see [6]). In [6], we obtained the following result.

Theorem 7. *Let K_i be a compact Hausdorff space and let δ_i be a closed $*$-derivation in $C(K_i)$ $(i = 1,2)$. Let T be a surjective linear isometry between $\mathfrak{D}(\delta_1)$ and $\mathfrak{D}(\delta_2)$. Then, there exist a homeomorphism τ from K_2 to K_1, $w_1 \in \ker(\delta_2)$ and a continuous function w_2 on $K_2(\delta_2)$ such that $\tau(K_2(\delta_2)) = K_1(\delta_1)$, $|w_1(y)| = 1$ for all $y \in K_2$, $|w_2(y)| = 1$ for all $y \in K_2(\delta_2)$,*

$$\begin{aligned} (Tf)(y) &= w_1(y) f(\tau(y)) \quad \text{for } f \in \mathfrak{D}(\delta_1),\ y \in K_2, \\ \delta_2(Tf)(y) &= w_2(y)\delta_1(f)(\tau(y)) \quad \text{for } f \in \mathfrak{D}(\delta_1),\ y \in K_2(\delta_2). \end{aligned} \tag{6}$$

In this paper, we consider Jarosz's problem in the same context as this theorem.

We use the following notation, for a Banach space B, B^* denotes the conjugate space of B. B_1 and B_1^* denote the closed unit balls of B and B^*, respectively. $\mathbf{T}$ denotes the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ in the complex plane.

We shall prove the following theorem.

Theorem 8. *Let K_i be a compact Hausdorff space satisfying the first countable axiom, and let δ_i be a closed $*$-derivation in $C(K_i)$ $(i = 1,2)$. If there exist a linear isomorphism T of $\mathfrak{D}(\delta_1)$ onto $\mathfrak{D}(\delta_2)$ with $\|T\|\,\|T^{-1}\| < 2$ and T, T^{-1} are bounded under the uniform norm, then $K_1(\delta_1)$ and $K_2(\delta_2)$ are homeomorphic. Moreover, if the range $\mathfrak{R}(\delta_i)$ contains 1 $(i = 1,2)$, then K_1 and K_2 are homeomorphic.*

The proof of this theorem is done along the line in [3].

Let K be a compact Hausdorff space satisfying the first countable axiom and let δ be a closed $*$-derivation in $C(K)$.

The following two lemmas will be used in the rest of the paper.

Lemma 9. *For $x_0 \in K(\delta)$, an open neighborhood U of x_0 and ε $(0 < \varepsilon < 1)$, there exists a function $f \in \mathfrak{D}(\delta)$ such that*

$$\begin{gathered} \|f\| \le 1, \qquad \|f\|_\infty \le \varepsilon, \qquad f(x_0) = 0, \\ f = \delta(f) = 0 \quad \text{on } K \backslash U, \quad 1 > |\delta(f)(x_0)| > 1 - \varepsilon. \end{gathered} \tag{7}$$

Proof. We take an open neighborhood V of x_0 such that $\overline{V} \subset U$ and take a function $g \in \mathfrak{D}(\delta)$ such that

$$0 \le g \le 1, \quad g(x_0) = 1, \quad g = 0 \quad \text{on } K \backslash V. \tag{8}$$

Then, $g = \delta(g) = 0$ on $K \backslash U$. Since $x_0 \in K(\delta)$, there is a function $g_\varepsilon (= g_\varepsilon^*) \in \mathfrak{D}(\delta)$ such that

$$||g_\varepsilon|| < 1, \quad 1 - \varepsilon = ||\eta_{x_0} \circ \delta|| - \varepsilon < |\delta(g_\varepsilon)(x_0)|. \tag{9}$$

For $c_\varepsilon := \min\{(1-\|\delta(g_\varepsilon)\|_\infty)/(1+\|\delta(g)\|_\infty),\varepsilon\}$, there is a function $h \in C^1(0[-\|g_\varepsilon\|_\infty, \|g_\varepsilon\|_\infty])$ such that

$$\|h\|_\infty \le c_\varepsilon, \qquad h(g_\varepsilon(x_0)) = 0, \qquad h'(g_\varepsilon(x_0)) = 1, \qquad \|h'\|_\infty = 1. \tag{10}$$

Then $f := h(g_\varepsilon)g \in \mathfrak{D}(\delta)$ has all required properties in Lemma 9. □

Lemma 10. *For $x_0 \in K(\delta)$ and ε $(0 < \varepsilon < 1)$, there exists a sequence $\{f_n\} \subset \mathfrak{D}(\delta)$ such that*

$$\begin{gathered} \|f_n\| \le 1, \qquad \|f_n\|_\infty \le \frac{1}{n}, \qquad f_n(x_0) = 0, \\ \lim_{n\to\infty} \delta(f_n)(x) = 0 \quad (x \ne x_0),\ 1 > |\delta(f_n)(x_0)| > 1-\varepsilon, \end{gathered} \tag{11}$$

and $d_{x_0} := \delta(f_n)(x_0)$ is independent of n.

Proof. Since K satisfies the first countable axiom, there is a family $\{U_n\}$ of open neighborhood of x_0 such that $U_{i+1} \subset U_i$ and $\bigcap_1^\infty U_n = \{x_0\}$. Then there exists a family $\{V_n\}$ of open neighborhood of x_0 such that $\overline{V}_n \subset U_n$, and there is $g_n \in \mathfrak{D}(\delta)$ such that

$$g_n(x_0) = 1, \quad 0 \le g_n \le 1, \quad g_n = 0 \quad \text{on } K\backslash V_n. \tag{12}$$

Then $g_n = \delta(g_n) = 0$ on $K\backslash U_n$. Since x_0 is in $K(\delta)$, there is a function $g_\varepsilon (= g_\varepsilon^*) \in \mathfrak{D}(\delta)$ such that

$$\|g_\varepsilon\| < 1, \quad 1-\varepsilon = \|\eta_{x_0} \circ \delta\| - \varepsilon < |\delta(g_\varepsilon)(x_0)|. \tag{13}$$

For each $c_n := \min\{(1-\|\delta(g_\varepsilon)\|_\infty)/(1+\|\delta(g_n)\|_\infty), 1/n\}$, there is a function $h_n \in C^1([-\|g_\varepsilon\|_\infty, \|g_\varepsilon\|_\infty])$ such that

$$\|h_n\|_\infty \le c_n, \qquad h_n(g_\varepsilon(x_0)) = 0, \qquad h_n'(g_\varepsilon(x_0)) = 1, \qquad \|h_n'\|_\infty = 1. \tag{14}$$

Then every $f_n := h_n(g_\varepsilon)g_n \in \mathfrak{D}(\delta)$ has the properties required in Lemma 10. □

Let W be the compact Hausdorff space $W = K \times K \times \mathbf{T}$ with the product topology. For $f \in \mathfrak{D}(\delta)$, we define $\tilde{f} \in C(W)$ by

$$\tilde{f}(x,x',z) := zf(x) + \delta(f)(x'), \tag{15}$$

for $(x,x',z) \in W$. Then we have $\|\tilde{f}\|_\infty = \|f\|$.

Proof of Theorem 7. Let $W_i := K_i \times K_i \times \mathbf{T}$ and $S_i = \{\tilde{f} \in C(W_i); f \in \mathfrak{D}(\delta_i)\}$ $(i = 1,2)$.

Define a linear isomorphism $\tilde{T}$ of S_1 onto S_2 by

$$\tilde{T}(\tilde{f}) := \widetilde{T(f)} \quad (\tilde{f} \in S_1). \tag{16}$$

Then $\tilde{T}$ is well defined since $f \to \tilde{f}$ is a linear isomorphism.

We may assume that $\|T^{-1}\| = 1$ and $1 < \|T\| < 2$. Then we have $\|\tilde{T}^{-1}\| = \|T^{-1}\| = 1$ and $\|\tilde{T}\| = \|T\| < 2$. For $(y_0, y_0', z_0) \in W_2$, let Φ be a norm-preserving extension of $\tilde{T}^*L_{(y_0,\, y_0',\, z_0)}$ to $C(W_1)$, where $L_{(y_0,\, y_0',\, z_0)}$ denotes the linear functional defined by

$L_{(y_0,y_0',z_0)}(\tilde{f}) = \tilde{f}(y_0,y_0',z_0)$ $(\tilde{f} \in S_2)$. Then, from the Riesz representation theorem, there exists a regular Borel measure $\mu^{y_0,\, y_0',\, z_0}$ on W_1 such that $\|\mu^{y_0,\, y_0',\, z_0}\| = \|\Phi\| = \|\tilde{T}^* L_{(y_0,y_0',z_0)}\| \le \|T\| < 2$ and

$$\Phi(h) = \int_{W_1} h\, d\mu^{y_0,\, y_0',\, z_0} \quad (h \in C(W_1)). \tag{17}$$

Hence we have

$$\begin{aligned} z_0(Tf)(y_0) + \delta_2(Tf)(y_0') &= \int_{W_1} \tilde{f}(x,x',z)\, d\mu^{y_0,\, y_0',\, z_0} \\ &= \int_{W_1} (zf(x) + \delta_1(f)(x'))\, d\mu^{y_0,\, y_0',\, z_0} \end{aligned} \tag{18}$$

for $f \in \mathfrak{D}(\delta_1)$. □

In the following, we identify Φ and $\mu^{y_0,\, y_0',\, z_0}$.

μ^{x_0,x_0',z_0}, where $(x_0,x_0',z_0) \in W_1$, is also defined in a similar way. Then we have $\|\mu^{x_0,x_0',z_0}\| \le 1$.

The following lemma shows that for $x_0 \in K_1(\delta_1)$, $\mu^{y,y',z}(K_1 \times \{x_0\} \times \mathbf{T})$, where $(y,y',z) \in W_2$ depends on y' only, that is, $\mu^{y,y',z}(K_1 \times \{x_0\} \times \mathbf{T})$ is independent of y, z, and any choice of norm-preserving extension of $\tilde{T}^* L_{(y,y',z)}$.

Lemma 11. (1) *For $x_0 \in K_1(\delta_1)$ and ε $(0 < \varepsilon < 1)$, let $\{f_n\} \subset \mathfrak{D}(\delta_1)$ be a sequence in Lemma 10. Then for $(y,y',z) \in W_2$,*

$$\begin{aligned} \mu^{y,y',z}(K_1 \times \{x_0\} \times \mathbf{T}) &= \left(\frac{1}{d_{x_0}}\right) \lim_{n\to\infty} \tilde{T}(\tilde{f}_n)(y,y',z) \\ &= \left(\frac{1}{d_{x_0}}\right) \lim_{n\to\infty} \delta_2(T(f_n))(y'). \end{aligned} \tag{19}$$

(2) *For $y_0 \in K_2(\delta_2)$ and ε $(0 < \varepsilon < 1)$, let $\{g_n\} \subset \mathfrak{D}(\delta_2)$ be a sequence in Lemma 10. Then for $(x,x',z) \in W_1$,*

$$\begin{aligned} \mu^{x,x',z}(K_2 \times \{y_0\} \times \mathbf{T}) &= \left(\frac{1}{d_{y_0}}\right) \lim_{n\to\infty} \tilde{T}^{-1}(\tilde{g}_n)(x,x',z) \\ &= \left(\frac{1}{d_{y_0}}\right) \lim_{n\to\infty} \delta_1(T^{-1}(g_n))(x'). \end{aligned} \tag{20}$$

Proof. (1) Let $\mu^{y,y',z}$ be a norm-preserving extension of $\tilde{T}^* L_{(y,y',z)}$.

$$\begin{aligned} \lim_{n\to\infty} \tilde{T}(\tilde{f}_n)(y,y',z) &= \lim_{n\to\infty} \int_{W_1} \tilde{f}_n\, d\mu^{y,y',z} = \int_{W_1} \lim_{n\to\infty} \tilde{f}_n\, d\mu^{y,y',z} \\ &= \int_{K_1 \times \{x_0\} \times \mathbf{T}} d_{x_0}\, d\mu^{y,y',z} = d_{x_0} \mu^{y,y',z}(K_1 \times \{x_0\} \times \mathbf{T}). \end{aligned} \tag{21}$$

From the uniform boundedness of T,

$$\lim_{n\to\infty} \tilde{T}(\tilde{f}_n)(y,y',z) = \lim_{n\to\infty} (z(Tf_n)(y) + \delta_2(Tf_n)(y')) = \lim_{n\to\infty} \delta_2(Tf_n)(y'). \tag{22}$$

Thus, we have

$$d_{x_0}\mu^{y,y',z}(K_1\times\{x_0\}\times\mathbf{T}) = \lim_{n\to\infty}\delta_2(Tf_n)(y') \tag{23}$$

which implies that for $x_0\in K_1(\delta_1)$, $\mu^{y,y',z}(K_1\times\{x_0\}\times\mathbf{T})$ depends on $y'\in K_2$ only.

The statement (2) is also shown by the same argument as above.

Now, let M_1 be any real number with $(1<)\|T\|<2M_1<2$. Let $\tilde{K}_2 := \{y\in K_2 : \exists x\in K_1$ such that $|\mu^{y,y,z}(K_1\times\{x\}\times\mathbf{T})|>M_1$ for every $z\in\mathbf{T}$ and every norm-preserving extension $\mu^{y,y,z}$ of $\tilde{T}^*L_{(y,y,z)}\}$. Since $\|\mu^{y,y,z}\| = \|\tilde{T}^*L_{(y,y,z)}\|\le\|T\|<2M_1$, for $y\in\tilde{K}_2$, there can be at most one $x\in K_1$ with the property in the definition of $\tilde{K}_2$. Thus the map ρ_1 of $\tilde{K}_2$ to K_1 is well defined by $\rho_1(y) := x$ if x is related to y as above.

Next, we set $M_2 := 1/(2M_1)$. Let $\tilde{K}_1 := \{x\in K_1 : \exists y\in K_2$ such that $|\mu^{x,x,z}(K_2\times\{y\}\times\mathbf{T})|>M_2$ for every $z\in\mathbf{T}$ and for every norm-preserving extension $\mu^{x,x',z}$ of $(\tilde{T}^{-1})^*L_{(x,x,z)}\}$. Since $\|\mu^{x,x,z}\| = \|(\tilde{T}^{-1})^*L_{(x,x,z)}\|\le\|T^{-1}\|\le 1$, for $x\in\tilde{K}_1$, there can be at most one $y\in K_2$ with the property in the definition of $\tilde{K}_1$. Thus, the map ρ_2 of $\tilde{K}_1$ to K_2 is well defined by $\rho_2(x) := y$ if y is related to x as above. □

The following lemma shows that $\tilde{K}_i$ contains sufficiently many elements (hence, is nonempty).

Lemma 12. (1) *For $x_0\in K_1(\delta_1)$, there exists $y_0\in\tilde{K}_2\cap K_2(\delta_2)$ such that $\rho_1(y_0) = x_0$.*
(2) *For $y_0\in K_2(\delta_2)$, there exists $x_0\in\tilde{K}_1\cap K_1(\delta_1)$ such that $\rho_2(x_0) = y_0$.*

Proof. (1) For $x_0\in K_1(\delta_1)$ and $0<\varepsilon<1-M_1$, there exists a family $\{f_n\}\subset\mathfrak{D}(\delta_1)$ in Lemma 10 such that

$$\begin{gathered}\|f_n\|\le 1,\qquad \|f_n\|_\infty\le\frac{1}{n},\qquad f_n(x_0)=0,\\ \lim_{n\to\infty}\delta_1(f_n)(x)=0\quad(\forall x\ne x_0),\quad 1-\varepsilon<|d_{x_0}|<1,\end{gathered}\tag{24}$$

where $d_{x_0} = \delta_1(f_n)(x_0)$. If $\lim_{n\to\infty}|\tilde{T}(\tilde{f}_n)(y,y',z)|\le M_1$ for every $(y,y',z)\in W_2$, then

$$\begin{aligned}1-\varepsilon<|d_{x_0}| &= \lim_{n\to\infty}|f_n(x_0)+\delta_1(f_n)(x_0)| = \lim_{n\to\infty}|\tilde{f}_n(x_0,x_0,1)|\\ &= \lim_{n\to\infty}|(\tilde{T}^{-1})^*L_{(x_0,x_0,1)}(\tilde{T}(\tilde{f}_n))|\\ &= \lim_{n\to\infty}\left|\int_{W_2}\tilde{T}(\tilde{f}_n)(y,y',z)\,d\mu^{x_0,x_0,1}\right|\\ &\le\int_{W_2}\lim_{n\to\infty}|\tilde{T}(\tilde{f}_n)(y,y',z)|\,d|\mu^{x_0,x_0,1}|\\ &\le M_1\|\mu^{x_0,x_0,1}\|\le M_1.\end{aligned}\tag{25}$$

This contradicts with $1-\varepsilon>M_1$.

Hence there exists $(y_0,y_0',z_0)\in W_2$ such that

$$\lim_{n\to\infty}|\tilde{T}(\tilde{f}_n)(y_0,y_0',z_0)|>M_1. \tag{26}$$

Then, from Lemma 11 we have for arbitrary $z \in \mathbf{T}$ and any norm-preserving extension $\mu^{y_0,y_0',z}$ of $\tilde{T}^* L_{(y_0,y_0',z)}$,

$$\begin{aligned} M_1 &< \lim_{n\to\infty} |\tilde{T}(\tilde{f}_n)(y_0,y_0',z_0)| = \lim_{n\to\infty} |\delta_2(Tf_n)(y_0')| \\ &= \lim_{n\to\infty} |\tilde{T}(\tilde{f}_n)(y_0',y_0',z_0)| = |d_{x_0}\mu^{y_0',y_0',z}(K_1\times\{x_0\}\times\mathbf{T})| \\ &< |\mu^{y_0',y_0',z}(K_1\times\{x_0\}\times\mathbf{T})|. \end{aligned} \tag{27}$$

Thus, $y_0' \in \tilde{K}_2 \cap K_2(\delta_2)$ and $\rho_1(y_0') = x_0$.

(2) For $y_0 \in K_2(\delta_2)$ and $0 < \varepsilon < 1 - M_2\|T\|$, we take a family $\{g_n\} \subset \mathfrak{D}(\delta_2)$ in Lemma 10. The remainder of the proof is completed by the same way as above. □

Now, we state another important lemma which holds without the first countability axiom.

Lemma 13. *If $x_0 \in \tilde{K}_1$ and $\rho_2(x_0) \in K_2(\delta_2)$, then $x_0 \in K_1(\delta_1)$.*

Proof. Let $\mu^{x_0,x_0,1}$ be a norm-preserving extension of $(\tilde{T}^{-1})^* L_{(x_0,x_0,1)}$. Since $\mu^{x_0,x_0,1}$ is regular, Since for all ε such that $0 < \varepsilon < M_2/(M_2 + 3 + \|T^{-1}\|_\infty)$ there is an open neighborhood U_ε of $\rho_2(x_0)$ such that

$$|\mu^{x_0,x_0,1}|(K_2\times(U_\varepsilon\backslash\{\rho_2(x_0)\})\times\mathbf{T}) < \varepsilon. \tag{28}$$

For $\varepsilon, U_\varepsilon$ and $\rho_2(x_0)$, we take a function $f \in \mathfrak{D}(\delta_2)$ in Lemma 9, then

$$\begin{gathered} \|f\| \le 1, \qquad \|f\|_\infty \le \varepsilon, \qquad f(\rho_2(x_0)) = 0, \\ f = \delta_2(f) = 0 \quad \text{on } K_2\backslash U_\varepsilon, \quad 1 > |\delta_2(f)(\rho_2(x_0))| > 1-\varepsilon. \end{gathered} \tag{29}$$

Since

$$\begin{aligned} &\left|\int_{K_2\times\{\rho_2(x_0)\}\times\mathbf{T}} zf(y)\,d\mu^{x_0,x_0,1}\right| \le \|f\|_\infty \|\mu^{x_0,x_0,1}\| \le \varepsilon, \\ &\qquad\left|\int_{K_2\times\{\rho_2(x_0)\}\times\mathbf{T}} \delta_2(f)(\rho_2(x_0))\,d\mu^{x_0,x_0,1}\right| \\ &\qquad\qquad = |\delta_2(f)(\rho_2(x_0))||\mu^{x_0,x_0,1}||(K_2\times\{\rho_2(x_0)\}\times\mathbf{T})| \\ &\qquad\qquad > (1-\varepsilon)M_2, \end{aligned} \tag{30}$$

we have

$$\begin{aligned} \left|\int_{K_2\times\{\rho_2(x_0)\}\times\mathbf{T}} \tilde{f}\,d\mu^{x_0,x_0,1}\right| &\ge \left|\int_{K_2\times\{\rho_2(x_0)\}\times\mathbf{T}} \delta_2(f)(\rho_2(x_0))\,d\mu^{x_0,x_0,1}\right| \\ &\quad - \left|\int_{K_2\times\{\rho_2(x_0)\}\times\mathbf{T}} zf(y)\,d\mu^{x_0,x_0,1}\right| \\ &> (1-\varepsilon)M_2 - \varepsilon > 0. \end{aligned} \tag{31}$$

From this and

$$\left|\int_{K_2\times(U_\varepsilon\backslash\{\rho_2(x_0)\})\times\mathbf{T}} \tilde{f}\,d\mu^{x_0,x_0,1}\right| \le \|\tilde{f}\|_\infty |\mu^{x_0,x_0,1}|(K_2\times(U_\varepsilon\backslash\{\rho_2(x_0)\})\times\mathbf{T}) \le \varepsilon, \tag{32}$$

we have

$$
\begin{aligned}
\left| \int_{K_2\times U_\varepsilon\times\mathbf{T}} \tilde{f}\, d\mu^{x_0,x_0,1} \right| &\geq \left| \int_{K_2\times\{\rho_2(x_0)\}\times\mathbf{T}} \tilde{f}\, d\mu^{x_0,x_0,1} \right| \\
&\quad - \left| \int_{K_2\times(U_\varepsilon\setminus\{\rho_2(x_0)\})\times\mathbf{T}} \tilde{f}\, d\mu^{x_0,x_0,1} \right| \\
&> (1-\varepsilon)M_2 - 2\varepsilon > 0.
\end{aligned} \tag{33}
$$

Since

$$
\begin{aligned}
\left| \int_{K_2\times(K_2\setminus U_\varepsilon)\times\mathbf{T}} \tilde{f} d\mu^{x_0,x_0,1} \right| &= \left| \int_{K_2\times(K_2\setminus U_\varepsilon)\times\mathbf{T}} z f(y) d\mu^{x_0,x_0,1} \right| \\
&\leq \|f\|_\infty \|\mu^{x_0,x_0,1}\| \leq \varepsilon,
\end{aligned} \tag{34}
$$

we get

$$
\begin{aligned}
|(\tilde{T}^{-1}\tilde{f})(x_0,x_0,1)| &= |(\tilde{T}^{-1})^* L_{(x_0,x_0,1)}(\tilde{f})| = \left| \int_{W_2} \tilde{f}\, d\mu^{x_0,x_0,1} \right| \\
&\geq \left| \int_{K_2\times U_\varepsilon\times\mathbf{T}} \tilde{f}\, d\mu^{x_0,x_0,1} \right| - \left| \int_{K_2\times(K_2\setminus U_\varepsilon)\times\mathbf{T}} \tilde{f}\, d\mu^{x_0,x_0,1} \right| \\
&\geq (1-\varepsilon)M_2 - 3\varepsilon > 0.
\end{aligned} \tag{35}
$$

Thus

$$
\begin{aligned}
|\delta_1(T^{-1}(f))(x_0)| &= |\tilde{T}^{-1}(\tilde{f})(x_0,x_0,1) - T^{-1}(f)(x_0)| \\
&\geq |\tilde{T}^{-1}(\tilde{f})(x_0,x_0,1)| - |T^{-1}(f)(x_0)| \\
&\geq (1-\varepsilon)M_2 - 3\varepsilon - \varepsilon\|T^{-1}\|_\infty > 0,
\end{aligned} \tag{36}
$$

that is, $x_0 \in K_1(\delta_1)$. This completes the proof. □

Lemma 14. *If $y_0 \in \tilde{K}_2 \cap K_2(\delta_2)$, then $\rho_1(y_0) \in \tilde{K}_1 \cap K_1(\delta_1)$ and $\rho_2(\rho_1(y_0)) = y_0$.*

Proof. Let $\rho_1(y_0) = x_0$ ($y_0 \in \tilde{K}_2 \cap K_2(\delta_2)$). If $x_0 \in \tilde{K}_1$ and $\rho_2(x_0) = y_0$, then $x_0 \in K_1(\delta_1)$ from Lemma 13. Hence, suppose that either x_0 is not in $\tilde{K}_1$ or $x_0 \in \tilde{K}_1$ and $\rho_2(x_0) \neq y_0$. Then there exists $z_0 \in \mathbf{T}$ such that $|\mu^{x_0,x_0,z_0}(K_2\times\{y_0\}\times\mathbf{T})| \leq M_2$.

Let $P := \sup\{|\mu^{x,x,z}(K_2\times\{y_0\}\times\mathbf{T})|; (x,x,z) \in W_1\} (\leq 1)$. Since $y_0 \in K_2(\delta_2)$, we have $P = \sup\{|\mu^{x,x',z}(K_2\times\{y_0\}\times\mathbf{T})|; (x,x',z) \in W_1\}$ by Lemma 11. Since $P > M_2$ by Lemma 12 and $0 < \|T\| - M_1 < M_1$, there exists $(x_1,x_1,z_1) \in W_1$ such that

$$
|\mu^{x_1,x_1,z_1}(K_2\times\{y_0\}\times\mathbf{T})| > \max\{M_2, (\|T\| - M_1)P/M_1\}. \tag{37}
$$

Then, for arbitrary $z \in \mathbf{T}$ and any norm-preserving extension $\mu^{x_1,x_1,z}$,

$$
|\mu^{x_1,x_1,z}(K_2\times\{y_0\}\times\mathbf{T})| > M_2, \tag{38}
$$

by Lemma 11. Thus, $x_1 \in \tilde{K}_1$, $\rho_2(x_1) = y_0$, and $x_1 \neq x_0$. Therefore, $x_1 \in K_1(\delta_1)$ by Lemma 13. Since $x_1 \neq x_0$, there exist $y_1 (\neq y_0) \in \tilde{K}_2 \cap K_2(\delta_2)$ such that $\rho_1(y_1) = x_1$

by Lemma 12. For $y_0 \in K_2(\delta_2)$ and ε $(0 < \varepsilon < 1)$, there exists a family $\{g_n\} \subset \mathcal{D}(\delta_2)$ in Lemma 10. Then, since $y_1 \neq y_0$,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left(z_1 g_n(y_1) + \delta_2(g_n)(y_1)\right) = \lim_{n\to\infty} \tilde{g}_n(y_1, y_1, z_1) \\ &= \lim_{n\to\infty} \tilde{T}^* L_{(y_1,y_1,z_1)}(\tilde{T}^{-1}(\tilde{g}_n)) = \lim_{n\to\infty} \int_{W_1} \tilde{T}^{-1}(\tilde{g}_n)\, d\mu^{y_1,y_1,z_1} \\ &= \lim_{n\to\infty} \int_{K_1\times\{x_1\}\times\mathbf{T}} \tilde{T}^{-1}(\tilde{g}_n)\, d\mu^{y_1,y_1,z_1} + \lim_{n\to\infty} \int_{K_1\times(K_1\setminus\{x_1\})\times\mathbf{T}} \tilde{T}^{-1}(\tilde{g}_n)\, d\mu^{y_1,y_1,z_1}. \end{aligned} \tag{39}$$

Now, by Lemma 11,

$$\begin{aligned} &\left| \lim_{n\to\infty} \int_{K_1\times\{x_1\}\times\mathbf{T}} \tilde{T}^{-1}(\tilde{g}_n) d\mu^{y_1,y_1,z_1} \right| \\ &\qquad = \left| \int_{K_1\times\{x_1\}\times\mathbf{T}} \lim_{n\to\infty} \tilde{T}^{-1}(\tilde{g}_n)\, d\mu^{y_1,y_1,z_1} \right| \\ &\qquad = \left| \int_{K_1\times\{x_1\}\times\mathbf{T}} d_{y_0}\, \mu^{x,x_1,z}(K_2\times\{y_0\}\times\mathbf{T})\, d\mu^{y_1,y_1,z_1} \right| \\ &\qquad = \left| \int_{K_1\times\{x_1\}\times\mathbf{T}} d_{y_0}\, \mu^{x_1,x_1,z_1}(K_2\times\{y_0\}\times\mathbf{T})\, d\mu^{y_1,y_1,z_1} \right| \\ &\qquad = \left| d_{y_0} \mu^{x_1,x_1,z_1}(K_2\times\{y_0\}\times\mathbf{T}) \mu^{y_1,y_1,z_1}(K_1\times\{x_1\}\times\mathbf{T}) \right| \\ &\qquad > |d_{y_0}| \cdot \frac{(\|T\|-M_1)P}{M_1} \cdot M_1 = |d_{y_0}| P(\|T\|-M_1). \end{aligned} \tag{40}$$

On the other hand,

$$\begin{aligned} &\left| \lim_{n\to\infty} \int_{K_1\times(K_1\setminus\{x_1\})\times\mathbf{T}} \tilde{T}^{-1}(\tilde{g}_n)\, d\mu^{y_1,y_1,z_1} \right| \\ &\quad = \left| \int_{K_1\times(K_1\setminus\{x_1\})\times\mathbf{T}} \lim_{n\to\infty} \tilde{T}^{-1}(\tilde{g}_n)\, d\mu^{y_1,y_1,z_1} \right| \\ &\quad = \left| \int_{K_1\times(K_1\setminus\{x_1\})\times\mathbf{T}} d_{y_0}\, \mu^{x,x',z}(K_2\times\{y_0\}\times\mathbf{T})\, d\mu^{y_1,y_1,z_1} \right| \\ &\quad \le |d_{y_0}| P |\mu^{y_1,y_1,z_1}| (K_1\times(K_1\setminus\{x_1\})\times\mathbf{T}) \\ &\quad = |d_{y_0}| P (|\mu^{y_1,y_1,z_1}|(K_1\times K_1\times\mathbf{T}) - |\mu^{y_1,y_1,z_1}|(K_1\times\{x_1\}\times\mathbf{T})) \\ &\quad \le |d_{y_0}| P (\|T\| - |\mu^{y_1,y_1,z_1}|(K_1\times\{x_1\}\times\mathbf{T})) < |d_{y_0}| P(\|T\|-M_1). \end{aligned} \tag{41}$$

This contradicts to

$$0 = \lim_{n\to\infty} \int_{K_1\times\{x_1\}\times\mathbf{T}} \tilde{T}^{-1}(\tilde{g}_n)\, d\mu^{y_1,y_1,z_1} + \lim_{n\to\infty} \int_{K_1\times(K_1\setminus\{x_1\})\times\mathbf{T}} \tilde{T}^{-1}(\tilde{g}_n)\, d\mu^{y_1,y_1,z_1}. \tag{42}$$

Thus $x_0 \in \tilde{K}_1$ and $y_0 = \rho_2(x_0) = \rho_2(\rho_1(y_0))$. □

By Lemmas 12 and 14, we have $K_1(\delta_1) \subseteq \rho_1(\tilde{K}_2 \cap K_2(\delta_2)) \subseteq \tilde{K}_1 \cap K_1(\delta_1) \subseteq K_1(\delta_1)$ and $K_2(\delta_2) \subseteq \rho_2(\tilde{K}_1 \cap K_1(\delta_1)) = \rho_2(K_1(\delta_1)) = \rho_2(\rho_1(\tilde{K}_2 \cap K_2(\delta_2))) \subseteq \tilde{K}_2 \cap K_2(\delta_2) \subseteq K_2(\delta_2)$. Thus, $K_1(\delta_1) \subseteq \tilde{K}_1, K_1(\delta_1) = \rho_1(\tilde{K}_2 \cap K_2(\delta_2))$, and $K_2(\delta_2) = \tilde{K}_2 \cap K_2(\delta_2) \subseteq \tilde{K}_2$. Therefore, $\rho_1(K_2(\delta_2)) = K_1(\delta_1)$ and $\rho_2(K_1(\delta_1)) = K_2(\delta_2)$. Since $\rho_2(\rho_1(y)) = y$ for $y \in K_2(\delta_2)$ from Lemma 14, ρ_1 is injective on $K_2(\delta_2)$. Moreover, we have $\rho_1(\rho_2(x)) = x$ for $x \in K_1(\delta_1)$ and hence ρ_2 is injective on $K_1(\delta_1)$.

Lemma 15. *ρ_i is continuous on $K_i(\delta_i)$ $(i = 1,2)$.*

Proof. We show that ρ_1 is continuous. Suppose that ρ_1 is discontinuous at $y_0 \in K_2(\delta_2)$. Then there exists a sequence $\{y_n\} \subset K_2(\delta_2)$ such that $y_n \to y_0 \in K_2(\delta_2)$, but $x_n := \rho_1(y_n)$ is not converge to $\rho_1(y_0) = x_0$. There exists an open neighborhood $V_1(\subset K_1(\delta_1))$ of x_0 such that for every n_0 there is $n(\geq n_0)$ with x_n outside V_1. Since $\mu^{y_0,y_0,1}$ is regular, for ε $(0 < \varepsilon < (2M_1 - \|T\|)/(\|T\| + 2M_1 + 10))$ there exists an open neighborhood $U_1(\subset V_1)$ of x_0 such that

$$|\mu^{y_0,y_0,1}|(K_1 \times (U_1 \setminus \{x_0\}) \times \mathbf{T}) < \varepsilon, \quad \overline{U}_1 \subset V_1. \tag{43}$$

For x_0, U_1, and ε, by Lemma 9, there exists a function $f \in \mathfrak{D}(\delta_1)$ such that

$$\begin{gathered} \|f\| \leq 1, \qquad \|f\|_\infty \leq \varepsilon, \qquad f(x_0) = 0, \\ 1 > |\delta_1(f)(x_0)| > 1 - \varepsilon, \quad f = \delta_1(f) = 0 \quad \text{on } K_1 \setminus U_1. \end{gathered} \tag{44}$$

Since

$$\begin{aligned} \left| \int_{K_1 \times \{x_0\} \times \mathbf{T}} z f(x)\, d\mu^{y_0,y_0,1} \right| &\leq \|f\|_\infty \|\mu^{y_0,y_0,1}\| \leq 2\varepsilon, \\ \left| \int_{K_1 \times \{x_0\} \times \mathbf{T}} \delta_1(f)(x_0)\, d\mu^{y_0,y_0,1} \right| &= |\delta_1(f)(x_0)| |\mu^{y_0,y_0,1}| (K_1 \times \{x_0\} \times \mathbf{T})| \\ &> (1-\varepsilon) M_1, \end{aligned} \tag{45}$$

we have

$$\left| \int_{K_1 \times \{x_0\} \times \mathbf{T}} \tilde{f}\, d\mu^{y_0,y_0,1} \right| > (1-\varepsilon) M_1 - 2\varepsilon > \varepsilon. \tag{46}$$

From (46) and

$$\left| \int_{K_1 \times (U_1 \setminus \{x_0\}) \times \mathbf{T}} \tilde{f}\, d\mu^{y_0,y_0,1} \right| \leq \|\tilde{f}\|_\infty |\mu^{y_0,y_0,1}| (K_1 \times (U_1 \setminus \{x_0\}) \times \mathbf{T}) \leq \varepsilon, \tag{47}$$

we have

$$\begin{aligned} \left| \int_{K_1 \times U_1 \times \mathbf{T}} \tilde{f}\, d\mu^{y_0,y_0,1} \right| &\geq \left| \int_{K_1 \times \{x_0\} \times \mathbf{T}} \tilde{f}\, d\mu^{y_0,y_0,1} \right| \\ &\quad - \left| \int_{K_1 \times (U_1 \setminus \{x_0\}) \times \mathbf{T}} \tilde{f}\, d\mu^{y_0,y_0,1} \right| \\ &\geq (1-\varepsilon) M_1 - 3\varepsilon > 2\varepsilon \\ \left| \int_{K_1 \times (K_1 \setminus U_1) \times \mathbf{T}} \tilde{f}\, d\mu^{y_0,y_0,1} \right| &= \left| \int_{K_1 \times (K_1 \setminus U_1) \times \mathbf{T}} z f(x)\, d\mu^{y_0,y_0,1} \right| \\ &\leq \|f\|_\infty \|\mu^{y_0,y_0,1}\| \leq 2\varepsilon. \end{aligned} \tag{48}$$

Thus

$$\begin{aligned} |\tilde{T}(\tilde{f})(y_0,y_0,1)| &= |\tilde{T}^* L_{(y_0,y_0,1)}(\tilde{f})| = \left| \int_{W_1} \tilde{f} d\mu^{y_0,y_0,1} \right| \\ &\geq \left| \int_{K_1\times U_1\times \mathbf{T}} \tilde{f} d\mu^{y_0,y_0,1} \right| - \left| \int_{K_1\times (K_1\setminus U_1)\times \mathbf{T}} \tilde{f} d\mu^{y_0,y_0,1} \right| \\ &> (1-\varepsilon)M_1 - 5\varepsilon > 0. \end{aligned} \tag{49}$$

Now, since $y_n \to y_0$ in K_2, then $(y_n,y_n,1) \to (y_0,y_0,1)$ in W_2. There exists n_0 such that $\forall n (> n_0)$ implies $|\tilde{T}(\tilde{f})(y_n,y_n,1)| > (1-\varepsilon)M_1 - 5\varepsilon$. Fix $n_1 (\geq n_0)$ such that $x_{n_1} = \rho_1(y_{n_1})$ lies outside V_1. Since $\mu^{y_{n_1},y_{n_1},1}$ is regular, there exists an open neighborhood $U_2 (\subset K_1)$ of x_{n_1} such that

$$|\mu^{y_{n_1},y_{n_1},1}|(K_1\times (U_2\setminus\{x_{n_1}\})\times \mathbf{T}) < \varepsilon, \qquad \overline{U}_1 \cap \overline{U}_2 = \phi. \tag{50}$$

For x_{n_1}, U_2, and ε, we take $g(\in \mathfrak{D}(\delta_1))$ in Lemma 9 such that

$$\begin{gathered} \|g\| \leq 1, \qquad \|g\|_\infty \leq \varepsilon, \qquad g(x_{n_1}) = 0, \\ 1 > |\delta_1(g)(x_{n_1})| > 1-\varepsilon, \quad g = \delta_1(g) = 0 \quad \text{on } K_1\setminus U_2. \end{gathered} \tag{51}$$

By the same way as above, we have

$$\begin{aligned} \left| \int_{K_1\times U_2\times \mathbf{T}} \tilde{g} d\mu^{y_{n_1},y_{n_1},1} \right| &> (1-\varepsilon)M_1 - 3\varepsilon > 0, \\ \left| \int_{K_1\times (K_1\setminus U_2)\times \mathbf{T}} \tilde{g} d\mu^{y_{n_1},y_{n_1},1} \right| &= \left| \int_{K_1\times (K_1\setminus U_2)\times \mathbf{T}} z g(x) d\mu^{y_{n_1},y_{n_1},1} \right| \\ &\leq \|g\|_\infty \|\mu^{y_{n_1},y_{n_1},1}\| \leq 2\varepsilon. \end{aligned} \tag{52}$$

Then

$$\begin{aligned} |\tilde{T}(\tilde{g})(y_{n_1},y_{n_1},1)| &= |\tilde{T}^* L_{(y_{n_1},y_{n_1},1)}(\tilde{g})| = \left| \int_{W_1} \tilde{g}\, d\mu^{y_{n_1},y_{n_1},1} \right| \\ &\geq \left| \int_{K_1\times U_2\times \mathbf{T}} \tilde{g}\, d\mu^{y_{n_1},y_{n_1},1} \right| - \left| \int_{K_1\times (K_1\setminus U_2)\times \mathbf{T}} \tilde{g}\, d\mu^{y_{n_1},y_{n_1},1} \right| \\ &> (1-\varepsilon)M_1 - 5\varepsilon > 0. \end{aligned} \tag{53}$$

Thus, if we choose a complex number $\lambda_0 \in \mathbf{T}$ such that $\tilde{T}(\tilde{f})(y_{n_1},y_{n_1},1)$ and λ_0 $(\tilde{T}(\tilde{g}))(y_{n_1},y_{n_1},1)$ have equal arguments, then

$$||f+\lambda_0 g|| = \max\{||f||_\infty, ||g||_\infty\} + \max\{||\delta_1(f)||_\infty, ||\delta_1(g)||_\infty\} \leq 1+\varepsilon, \tag{54}$$

This is a contradiction. Therefore, ρ_1 is continuous on $K_2(\delta_2)$. A similar argument shows that ρ_2 is continuous on $K_1(\delta_2)$. □

From Lemma 15, it follows that $K_1(\delta_1)$ and $K_2(\delta_2)$ are homeomorphic. Thus, all proofs of Theorem are completed.

Remark 16. There is not a nonzero closed $*$-derivation in $C(D)$ (D is the Cantor set). However, we can obtain similar results for $C^{(1)}(X)$ (X : a compact subset of $\mathbb{R}$) by the same way as above.

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