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ON SOME HYPERBOLIC PLANES FROM FINITE PROJECTIVE PLANES

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ABSTRACT. Let $\Pi = (P, L, I)$ be a finite projective plane of order n, and let $\Pi' = (P', L', I')$ be a subplane of Π with order m which is not a Baer subplane (i.e., $n \geq m^2 + m$). Consider the substructure $\Pi_0 = (P_0, L_0, I_0)$ with $P_0 = P \backslash \{X \in P \mid XIl,\ l \in L'\}$, $L_0 = L \backslash L'$, where I_0 stands for the restriction of I to $P_0 \times L_0$. It is shown that every Π_0 is a hyperbolic plane, in the sense of Graves, if $n \geq m^2 + m + 1 + \sqrt{m^2 + m + 2}$. Also we give some combinatorial properties of the line classes in Π_0 hyperbolic planes, and some relations between its points and lines.

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1. Introduction. In this paper, points are denoted by capital letters (usually P, Q), lines are denoted by lower-case letters (usually l), sets $\mathcal{P}$ and $\mathcal{L}$ denote the sets of points and lines, respectively, $\mathcal{I}$ denotes the incidence relation on points and lines (therefore $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$). The triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is called a geometric structure, if $\mathcal{P} \cap \mathcal{L} = \Phi$. If $(P, l) \in \mathcal{I}$ then P is on l or l passes through P and it is denoted by $P \in l$ or $P\mathcal{I}l$. Similarly if $(P, l) \notin \mathcal{I}$ then P is not on l and it is denoted by $P \notin l$. If $\mathcal{P}$ and $\mathcal{L}$ are finite sets, the geometric structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is called finite.

It is well known that there are alternative systems of axioms for hyperbolic spaces. For instance, Graves [3] introduced the following definition (see [1, 2, 5, 6]).

A *finite hyperbolic plane* is a finite geometric structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ such that

(G1) Two distinct points lie on one and only one line.

(G2) There are at least two points on each line.

(G3) Through each point X not on a line l there pass at least two lines not meeting (parallel to) l.

(G4) There exist at least four points, no three of which are collinear.

(G5) If a subset of $\mathcal{P}$ contains three non-collinear points and all the lines through any pair of its points, then this subset contains all points of $\mathcal{P}$.

In this paper, we construct a class of hyperbolic planes using the non-Baer subplanes of the projective planes of finite order. Thus, in a sense, we find a connection between the non-Baer subplanes of finite projective plane and some hyperbolic planes from that plane by certain deletion.

2. Construction of finite hyperbolic spaces Π_0. Let $\Pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a finite projective plane of order n with a non-Baer subplane $\Pi' = (\mathcal{P}', \mathcal{L}', \mathcal{I}')$ of order m. Then it is well known that

$$n \geq (m^2 + m). \tag{2.1}$$

Let $\mathcal{Q} = \{X \in \mathcal{P} \mid X \in l,\ l \in \mathcal{L}'\}$ and consider the incidence structure $\Pi_0 = (\mathcal{P}_0, \mathcal{L}_0, \mathcal{I}_0)$ obtained by removing all lines of Π_0 with incidence points. Thus, $\mathcal{P}_0 = \mathcal{P}\backslash\mathcal{Q}$, $\mathcal{L}_0 = \mathcal{L}\backslash\mathcal{L}'$, $\mathcal{I}_0 = \mathcal{I} \cap \mathcal{P}_0 \times \mathcal{L}_0$.

The following theorem is an immediate consequence of the construction of Π_0.

Theorem 2.1. *The following properties are valid:*

(i) *Two distinct points of* Π_0 *lie on one and only one line of* Π_0.
(ii) *There are exactly* $n^2 + n - m^2 - m$ *lines in* Π_0.
(iii) *There are exactly* $(n-m)(n-m^2)$ *points in* Π_0.
(iv) *At least* $n - m^2 - m$ *points lie on any line of* Π_0.

A line which contains exactly one point of Π_0 is said to be a *tangent line* and a line which contains no points of Π_0 is called an *exterior line*.

Theorem 2.2. *Any line of* Π_0 *contains exactly either* $n - m^2 - m$ *points or* $n - m^2$ *points.*

Proof. Let $l_0 \in \mathcal{L}_0$ and l denotes the extended line of l_0 in Π. Then l is either a tangent or an exterior line. If l is an exterior line, then l has $m^2 + m + 1$ deleted points. Thus l_0 has $n - m^2 - m$ points. Otherwise l must be a tangent line and therefore it has $m^2 + 1$ deleted points. Thus, if l is a tangent line, then it has $n - m^2$ points. □

It is trivial from Theorem 2.1(i) that, in Π_0, (G1) is satisfied. Any line of Π_0 contains at least $n - m^2 - m$ points, by Theorem 2.1(iv). By (G2), it must be greater than 2, that is,

$$n - m^2 - m \geq 2. \tag{2.2}$$

Notice that (2.2) is stronger than (2.1).

Hence, any line l of Π_0 has at least $m^2 + 1$ deleted points, in Π_0 there are at least $m^2 + 1$ parallel lines through any point X, $X \in l$. Since $m \geq 2$, through each point X not on a line l there pass at least five lines parallel to l. Hence, Π_0 satisfies properties (G1), (G2), and (G3), if (2.2) holds. That existence of four points no three of which are collinear is obvious from the definition of Π_0.

Finally, we investigate when the last axiom is satisfied in Π_0. Let $\mathcal{S} \subset \mathcal{P}_0$ contain three non-collinear points A, B, C. We consider the lines AB, AC, and BC. Then $\mathcal{S}$ contains all of the points on the lines AB, AC, and BC, and all points on the lines through pairs of distinct points of $\mathcal{S}$. Each of the lines has at least $n - m^2 - m$ points in $\mathcal{S}$. Thus, there are at least $n - m^2 - m$ lines in Π_0 through A and meeting the line BC. $\mathcal{S}$ contains at least, $(n - m^2 - m)(n - m^2 - m - 1) + 1$ points, since each of the above lines contains at least $n - m^2 - m - 1$ points other than the point A. Now, let X be a point of Π_0 not on a line that joins the point A to a point of BC. X is in $\mathcal{S}$ if there exists a line which contains X and at least two of the above $(n - m^2 - m)(n - m^2 - m - 1) + 1$ points. This is possible if $(n - m^2 - m)(n - m^2 - m - 1) + 1 \geq n + 2$, since X is on exactly $n + 1$ lines and these lines contains all points of Π_0. This inequality is valid when

$$n \leq m^2 + m + 1 - \sqrt{m^2 + m + 2}, \tag{2.3}$$

or

$$n \geq m^2 + m + 1 + \sqrt{m^2 + m + 2}. \tag{2.4}$$

But, equations (2.2) and (2.3) cannot be true at the same time. Therefore (2.3) is eliminated. Thus the following theorem is obtained.

Theorem 2.3. *Let* $\Pi = (\mathcal{P},\mathcal{L},\mathcal{I})$ *be a finite projective plane of order* n *with a non-Baer subplane* $\Pi' = (\mathcal{P}',\mathcal{L}',\mathcal{I}')$ *of order* m*. Then the substructure* $\Pi_0 = (\mathcal{P}_0,\mathcal{L}_0,\mathcal{I}_0)$*,*

$$\mathcal{P}_0 = \mathcal{P}\backslash\{X \in \mathcal{P} \mid X \in l,\ l \in \mathcal{L}'\}, \qquad \mathcal{L}_0 = \mathcal{L}\backslash\mathcal{L}', \qquad \mathcal{I}_0 = \mathcal{I}\cap(\mathcal{P}_0\times\mathcal{L}_0) \tag{2.5}$$

is a hyperbolic plane, in the sense of Graves, if

$$n \geq m^2+m+1+\sqrt{m^2+m+2}. \tag{2.6}$$

3. Some properties of Π_0**.** The following theorem is an immediate consequence of the construction of Π_0.

Theorem 3.1. (i) *Through any point of* Π' *there pass* $n-m$ *lines in* Π_0.

(ii) *There are exactly* $(m^2+m+1)(n-m)$ *tangent lines in* Π_0*.*

(iii) *There are exactly* $n^2+n+1-(m^2+m+1)(n+1-m)$ *exterior lines of* Π_0.

(iv) Π_0 *is not regular.*

(v) *Through any points of* Π_0 *there pass* m^2+m+1 *tangent lines.*

(vi) *Through any points of* Π_0 *there pass* $n-m^2-m$ *exterior lines.*

We define the following line classes;

$$\mathbf{C}_t = \{l \in \mathcal{L} \mid l \notin \mathcal{L}',\ P \in l,\ P \in \mathcal{P}'\}, \qquad \mathbf{C}_e = \{l \in \mathcal{L} \mid l \notin \mathcal{L}',\ P \notin l,\ \forall P \in \mathcal{P}'\}, \tag{3.1}$$

which consist of tangent and exterior lines of Π_0, respectively. We call $\mathbf{C}_t$ as tangent lines class, $\mathbf{C}_e$ as exterior lines class.

Theorem 3.2. *The line of* Π_0 *which is contained in* $\mathbf{C}_t$ *or* $\mathbf{C}_e$ *contains at most* $n-4$ *or* $n-6$ *points, respectively.*

Proof. It is clear, if the reality of $m \geq 2$ is used with the definition of $\mathbf{C}_t$ and $\mathbf{C}_e$. □

In the next section, we give some combinatorial properties of the line classes in Π_0 by using the technique of [4].

4. Parallel line classes of Π_0 **hyperbolic planes.** A class of the lines every two of which are parallel is called *parallel line class*. All lines of Π_0 passing through any deleted point P of Π form a parallel line class. This parallel line class is called *parallel class determined by* P or *parallel class of type* (P). A line together with all lines passing through a deleted point Q which is not on l and cutting l in the deleted points in Π form a parallel line class. But many parallel classes can be found containing this parallel line class. The intersection of all parallel classes containing the mentioned class of lines is called *parallel class determined by* l *and* Q, or *parallel class of type* (l,Q).

As, there might be other parallel classes apart from the above ones, it is convenient to call the parallel classes of type (P) and (l,Q) as obvious parallel classes.

Theorem 4.1. *There are* n *or* $n-m$ *lines in a parallel line class of type* (P) *of* Π_0.

Proof. The necessary and sufficient condition for a point P to be a deleted point is that either $P \in \mathcal{P}'$ or $P \notin \mathcal{P}'$, $P\mathcal{I}l \in \mathcal{L}'$. Therefore,

(i) if $P \in \mathcal{P}'$, then the number of lines of $\mathcal{L}_0$ passing through P is $n-m$. As all of these $n-m$ lines pass through the deleted point P, $|(P)| = n-m$.

(ii) If $P \notin \mathcal{P}'$, $P\mathcal{I}l \in \mathcal{L}'$, then the number of lines of Π_0 passing through a deleted external point is the required number. $n+1$ lines pass through P except one, none of these lines do not belong to Π'. Therefore, the number of lines of Π_0 passing through P in Π is n. □

Theorem 4.2. *We denote the minimum number of lines belonging to the parallel class of* (l,P) *type by* $\min|(l,P)|$. *Then,*

$$\min|(l,p)| = \begin{cases} m^2+1 & \text{if } P \notin \mathcal{P}',\ l \in \mathbf{C}_t \text{ or } P \in \mathcal{P}',\ l \in \mathbf{C}_d, \\ m^2+m+1 & \text{if } P \notin \mathcal{P}',\ l \in \mathbf{C}_d, \\ m^2-m+1 & \text{if } P \in \mathcal{P}',\ l \in \mathbf{C}_t. \end{cases} \tag{4.1}$$

Proof. Let l be any line of Π_0. Then either $l \in \mathbf{C}_t$ or $l \in \mathbf{C}_d$, since $\mathcal{L}_0 = \mathbf{C}_t \cup \mathbf{C}_d$, $\mathbf{C}_t \cap \mathbf{C}_d = \Phi$.

(i) If $l \in \mathbf{C}_t$, then

(a) if $P \in \mathcal{P}'$, the number of deleted points on l is m^2+1. Furthermore, $m+1$ lines pass through P in Π' and these lines are the deleted lines. Therefore, together with the line l at least $m^2+1-(m+1)+1 = m^2-m+1$ lines belong to (l,P) type.

(b) If $P \notin \mathcal{P}'$, the number of deleted points on l is m^2+1. If we join this m^2+1 points with P not incident on l, then the obtained m^2+1 lines meet l at deleted points in Π. Since one of these lines is a deleted line, there are at least m^2+1 (l,P)-type lines.

(ii) If $l \in \mathbf{C}_d$, then

(a) if $P \notin \mathcal{P}'$, then m^2+m+1 points are deleted from l. Join these points to $P \notin l$. The m^2+m+1 lines which are obtained by joining $P \notin l$ to deleted points from l meets l on deleted points in Π. Since one of these lines is a deleted line, together with l at least m^2+m+1 lines are in type (l,P).

(b) If $P \in \mathcal{P}'$, then there are $m+1$ lines passing through P in Π'. For this, there are $m+1$ deleted lines among the m^2+m+1 lines obtained by joining P to the deleted m^2+m+1 points from l. Hence, together with l there are at least m^2+1 lines in type (l,P). □

Theorem 4.3. *The line* $l \in \mathbf{C}_t$ *of* Π_0 *belongs to* m^2+1 (P)*-type and* $m(m+1)(n-m)+n$ (l,P)*-type parallel classes.*

Proof. Since $l \in \mathbf{C}_t$, the intersection of l and Π' has one and only one point. Since $m+1$ lines pass through this point, the remaining m^2 lines meet l on different points. Since every deleted point corresponds to a parallel line class of type (P) and l has m^2+1 deleted points, line l_0 belongs to $m+1(P)$-type parallel class.

On the other hand, the number of deleted points which is not on l is,

$$(m^2+m+1)(n+1-m)-m^2-1 = m(m+1)(n-m)+n. \tag{4.2}$$

Therefore line l belongs to $m(m+1)(n-m)+n(l,P)$-type parallel class. □

Now we give a theorem which can be proved like the previous theorem.

Theorem 4.4. *The line $l \in \mathbf{C}_d$ of Π_0 belongs to $m^2+m+1(P)$-type and $(m^2+m+1)(n-m)(l,P)$-type parallel classes.*

Theorem 4.5. *Let $\mathcal{L}_0$ denotes the set of lines of Π_0, i be any parallel line class type and $f_i(l)$, $l \in \mathcal{L}_0$ be the number of parallel line classes type i. Then the following equations are valid:*

$$\begin{gathered} f_{(P)}(l)+f_{(l,P)}(l)=(m^2+m+1)(n-m+1), \\ \sum_{l\in\mathcal{L}} f_{(P)}(l)=(m^2+m+1)(n+1)(n-m), \\ \sum_{l\in\mathcal{L}} f_{(l,P)}(l)=(m^2+m+1)(n-m)(n^2-m^2+n). \end{gathered} \tag{4.3}$$

Proof. Any line $l \in \mathcal{L}_0$ belongs to a parallel line class of type (P), as much as the number of deleted points from l and type (l,P) as much as the number of deleted points from Π which is not on l. Therefore, there are parallel line classes of type (P) or type (l,P) as much as the number of deleted points from Π. Since the number of deleted points from Π is $(m^2+m+1)(n+1-m)$, we obtain

$$f_{(P)}(l)+f_{(l,P)}(l)=(m^2+m+1)(n-m+1); \quad \forall l \in \mathcal{L}_0. \tag{4.4}$$

The sum $\sum_{l\in\mathcal{L}} f_{(P)}(l)$ is the number of total flags which are obtained from deleted points and the lines of Π_0 passing through these deleted points. This sum can be written as follows:

$$\sum_{l\in\mathcal{L}_0} f_{(P)}(l)=\sum_{\substack{l\in\mathcal{L}_0\\ P\in\mathcal{P}'}} f_{(P)}(l)+\sum_{\substack{l\in\mathcal{L}_0\\ P\notin\mathcal{P}' \text{ deleted points}}} f_{(P)}(l). \tag{4.5}$$

Therefore,

$$\sum_{l\in\mathcal{L}_0} f_{(P)}(l)=|\mathcal{P}'|(n-m)+(|\mathcal{Q}|-|\mathcal{P}'|)n=(m^2+m+1)(n-m)(n+1). \tag{4.6}$$

Total anti-flag numbers of deleted points and lines of Π_0 not passing through these points is $\sum_{l\in\mathcal{L}_0} f_{(l,P)}(l)$. Hence,

$$\begin{aligned} \sum_{l\in\mathcal{L}_0} f_{(l,P)}(l) &= \sum_{l\in\mathbf{C}_t} f_{(l,P)}(l)+\sum_{l\in\mathbf{C}_d} f_{(l,P)}(l) \\ &= |\mathbf{C}_t|\left[(m^2+m+1)(n-m)+m\right]+|\mathbf{C}_d|(m^2+m+1)(n-m) \\ &= (m^2+m+1)(n-m)(n^2-m^2+n). \end{aligned} \tag{4.7}$$

□

5. Isomorphism. Let Π be a projective plane of order n and Π', Π'' be subplanes of Π with order m, and $n \geq m^2+m+1+\sqrt{m^2+m+2}$. Then according to Theorem 2.1, we can construct the hyperbolic planes Π_0', Π_0'' by deleting, respectively, the lines of Π', Π'' together with incident points. Then we can give the following obvious consequence.

Consequence 5.1. If there exists a collination of Π which transforms Π' to Π'', then the hyperbolic planes Π'_0 and Π''_0 are isomorphic.

6. Some open questions. In this paper, it is shown that a structure obtained by deletion of a subplane from a projective plane of finite order is a hyperbolic plane, when the order of the subplane is suitably small relative to the order of superplane (see Theorem 2.3). But now we give some outstanding problems.

(1) When is a hyperbolic plane with appropriate order restriction a subplane-deleted projective plane?

(2) Is there a way to distinguish the subplane deleted Desarguesian hyperbolic plane from all other such hyperbolic planes?

(3) Is there a way to distinguish subplane-deleted translation hyperbolic planes from other planes?

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