IJMMS 25:8 (2001) 517–524
PII. S0161171201003982
http://ijmms.hindawi.com


# TWO-WEIGHT NORM INEQUALITIES FOR THE ROUGH FRACTIONAL INTEGRALS

YONG DING and CHIN-CHENG LIN

(Received 1 October 1999)

Abstract. The authors give the weighted $(L^p,L^q)$-boundedness of the rough fractional integral operator $T_{\Omega,\alpha}$ and the fractional maximal operator $M_{\Omega,\alpha}$ with two different weight functions.

2000 Mathematics Subject Classification. Primary 42B20, 42B25.

**1. Introduction.** Suppose that $0<\alpha<n$, $\Omega(x)$ is homogeneous of degree zero on $\mathbb{R}^n$ and $\Omega(x')\in L^s(S^{n-1})$ $(s>1)$, where $S^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$. Then the fractional integral operator $T_{\Omega,\alpha}$ and the maximal operator $M_{\Omega,\alpha}$ are defined by

$$T_{\Omega,\alpha}f(x)=\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)\,dy,$$
$$M_{\Omega,\alpha}f(x)=\sup_{r>0}\frac{1}{r^{n-\alpha}}\int_{|x-y|<r}|\Omega(x-y)||f(y)|\,dy, \tag{1.1}$$

respectively. It is easy to see that, when $\Omega\equiv 1$, $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ are the usual fractional integral operator $I_\alpha$ and the maximal operator $M_\alpha$. In 1971, Muckenhoupt and Wheeden [6] gave $(L^p,L^q)$-boundedness with power weight of $T_{\Omega,\alpha}$. In 1993, Chanillo, Watson and Wheeden [1] proved that when $s\ge n/(n-\alpha)$, the operator $T_{\Omega,\alpha}$ is weak type $(1,n/(n-\alpha))$. Recently, we gave the weighted $(L^p,L^q)$-boundedness of $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ for general $A(p,q)$ weight [3], and the weak boundedness of $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ with power weight [2].

The purpose of this paper is to study the weighted $(L^p,L^q)$-boundedness of $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ for the two different weights.

Before showing our results, we give the definitions of some weight classes. In the following definitions, the function $\omega$ and the function pair $(u,v)$ are all locally integrable nonnegative functions. Moreover, $C>0$ and $Q$ denotes a cube in $\mathbb{R}^n$ with its sides parallel to the coordinate axes and $\chi_Q(x)$ denotes the characterization function of $Q$.

**The definition of $A_p$ $(1<p<\infty)$.** A function $\omega$ is said to belong to $A_p$ $(1<p<\infty)$ if

$$\sup_{Q\subset\mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q\omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q\omega(x)^{-1/(p-1)}\,dx\right)^{p-1}\le C. \tag{1.2}$$

**The definition of $A(p,q)$ $(1 < p,\ q < \infty)$.** A function $\omega$ is said to belong to $A(p,q)$ $(1 < p,\ q < \infty)$ if

$$\sup_{Q\subset\mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q \omega(x)^q\,dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q \omega(x)^{-p'}\,dx\right)^{1/p'} \le C. \tag{1.3}$$

**The definition of $A_p^*$ $(1 < p < \infty)$.** A function pair $(u,v)$ is said to belong to $A_p^*$ $(1 < p < \infty)$ if

$$\sup_{Q\subset\mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q u(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C. \tag{1.4}$$

**The definition of $A^*(p,q)$ $(1 < p,\ q < \infty)$.** A function pair $(u,v)$ is said to belong to $A^*(p,q)$ $(1 < p,\ q < \infty)$ if

$$\sup_{Q\subset\mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q u(x)^q\,dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q v(x)^{-p'}\,dx\right)^{1/p'} \le C. \tag{1.5}$$

**The definition of $S_p^*$ $(1 < p < \infty)$.** A function pair $(u,v)$ is said to belong to $S_p^*$ $(1 < p < \infty)$ if

$$\int_Q \left[M\left(v^{-1/(p-1)}\chi_Q\right)\right]^p u(x)\,dx \le C\int_Q v(x)^{-1/(p-1)}\,dx. \tag{1.6}$$

In this paper, we prove the following results.

**Theorem 1.1.** *Suppose that $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $\Omega$ is homogeneous of degree zero defined on $\mathbb{R}^n$ and $\Omega \in L^s(S^{n-1})$. If $p,q,s$, and $(u,v)$ satisfy one of the following conditions:*

(a) $1 \le s' < p$, $(u^{s'}, v^{s'}) \in A^*(p/s', q/s')$, *in addition* $u(x)^{s'}, v(x)^{s'} \in A(p/s', q/s')$;

(b) $s > q$, $(v^{-s'}, u^{-s'}) \in A^*(q'/s', p'/s')$, *in addition* $v(x)^{-s'}, u(x)^{-s'} \in A(q'/s', p'/s')$;

*then there is a constant $C$, independent of $f$, such that $T_{\Omega,\alpha}$ satisfies*

$$\left(\int_{\mathbb{R}^n} \left|T_{\Omega,\alpha}f(x)u(x)\right|^q dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} \left|f(x)v(x)\right|^p dx\right)^{1/p}. \tag{1.7}$$

**Theorem 1.2.** *Suppose that $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $\Omega$ is homogeneous of degree zero defined on $\mathbb{R}^n$ and $\Omega \in L^s(S^{n-1})$. If $p,q,s$, and $(u,v)$ satisfy one of the following conditions:*

(c) $1 \le s' < p$, $(u^{s'}, v^{s'}) \in A^*(p/s', q/s')$, *in addition* $\sigma = v^{-s'(p/s')'}$ *satisfies the doubling condition;*

(d) $s > q$, $(v^{-s'}, u^{-s'}) \in A^*(q'/s', p'/s')$, *in addition* $v(x)^{-s'}, u(x)^{-s'} \in A(q'/s', p'/s')$;

*then there is a constant $C$, independent of $f$, such that $M_{\Omega,\alpha}$ satisfies*

$$\left(\int_{\mathbb{R}^n} \left[M_{\Omega,\alpha}f(x)u(x)\right]^q dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} \left|f(x)v(x)\right|^p dx\right)^{1/p}. \tag{1.8}$$

**2. Some elementary properties of the weight class.** We begin by giving some properties of the weight classes $A_p, A(p,q), A_p^*, A^*(p,q)$, and $S_p^*$.

**The elementary properties of** $A_p$ $(1 < p < \infty)$

(a) $A_{p_1} \subset A_{p_2}$, if $1 < p_1 < p_2 < \infty$.

(b) $\omega(x) \in A_p$, if and only if $\omega(x)^{1-p'} \in A_{p'}$.

(c) If $\omega(x) \in A_p$, then there is an $\varepsilon > 0$ such that $p - \varepsilon > 1$ and $\omega(x) \in A_{p-\varepsilon}$.

(d) If $\omega(x) \in A_p$, then there is an $\varepsilon > 0$ such that $\omega(x)^{1+\varepsilon} \in A_p$.

(e) If $\omega(x) \in A_p$, then for any $0 < \varepsilon < 1$, $\omega(x)^{\varepsilon} \in A_p$.

(f) If $\omega(x) \in A_p$, then there are $C > 0$ and $\varepsilon > 0$ such that, for any $Q \in \mathbb{R}^n$,

$$\frac{1}{|Q|} \int_Q \omega(x)^{1+\varepsilon}\, dx \le C\left(\int_Q \omega(x)\, dx\right)^{1+\varepsilon}. \tag{2.1}$$

See [4] for the proof.

**The elementary properties of** $A_p^*$ $(1 < p < \infty)$

(i) $A_{p_1}^* \subset A_{p_2}^*$, if $1 < p_1 < p_2 < \infty$.

(ii) $(u,v) \in A_p^*$, if and only if $(v^{1-p'}, u^{1-p'}) \in A_{p'}^*$.

(iii) $S_p^* \subset A_p^*$, for $1 < p < \infty$.

(iv) If $(u,v) \in A_p^*$, then for any $0 < \varepsilon < 1$, $(u^{\varepsilon}, v^{\varepsilon}) \in S_p^*$.

(v) If $(u,v) \in A_p^*$ and $u(x), v(x) \in A_p$, then $(u,v) \in S_p^*$ and $(v^{1-p'}, u^{1-p'}) \in S_{p'}^*$.

(vi) If $(u,v) \in A_p^*$ and $u(x), v(x) \in A_p$, then there is an $\varepsilon > 0$ such that $(u^{1+\varepsilon}, v^{1+\varepsilon}) \in A_p^*$ and $(v^{(1-p')(1+\varepsilon)}, u^{(1-p')(1+\varepsilon)}) \in A_{p'}^*$.

(vii) If $(u,v) \in A_p^*$ and $u(x), v(x) \in A_p$, then there is an $\varepsilon > 0$ such that $p - \varepsilon > 1$ and $(u,v) \in A_{p-\varepsilon}^*$.

**Proof.** The proof of (i) and (ii) can be deduced from the definition of $A_p^*$.

For (iii), by [8], we know that the Hardy-Littlewood maximal operator $M$ is bounded from $L^p(v)$ to $L^p(u)$ $(1 < p < \infty)$ if and only if $(u,v) \in S_p^*$. On the other hand, by [5], the operator $M$ is bounded from $L^p(v)$ to weak $L^p(u)$ if and only if $(u,v) \in A_p^*$. Hence, (iii) holds.

The proof of (iv) is a conclusion in [7].

Now, we prove (v). Since $v(x) \in A_p$, we have $v(x)^{1-p'} \in A_{p'}$, by (b). Thus, from (f) there are $C > 0$ and $\eta > 0$ such that, for any $Q \in \mathbb{R}^n$,

$$\frac{1}{|Q|} \int_Q u(x)^{1+\eta}\, dx \le C\left(\int_Q u(x)\, dx\right)^{1+\eta}, \tag{2.2}$$

$$\frac{1}{|Q|} \int_Q v(x)^{(1-p')(1+\eta)}\, dx \le C\left(\int_Q v(x)^{1-p'}\, dx\right)^{1+\eta}, \tag{2.3}$$

that is,

$$\frac{1}{|Q|} \int_Q v(x)^{-(1+\eta)/(p-1)}\, dx \le C\left(\int_Q v(x)^{-1/(p-1)}\, dx\right)^{1+\eta}. \tag{2.4}$$

Hence, by $(u,v) \in A_p^*$, (2.2), and (2.4) we have

$$\begin{aligned} &\sup_Q \left(\frac{1}{|Q|} \int_Q u(x)^{1+\eta}\, dx\right)\left(\frac{1}{|Q|} \int_Q v(x)^{-(1+\eta)/(p-1)}\, dx\right)^{p-1} \\ &\quad \le C\left(\sup_Q \left(\frac{1}{|Q|} \int_Q u(x)\, dx\right)\left(\frac{1}{|Q|} \int_Q v(x)^{-1/(p-1)}\, dx\right)^{p-1}\right)^{1+\eta} < \infty, \end{aligned} \tag{2.5}$$

that is,

$$(u^{1+\eta},v^{1+\eta}) \in A_p^*. \tag{2.6}$$

Taking $\delta = 1/(1+\eta)$, then $0<\delta<1$ and $(u,v) = (u^{(1+\eta)\delta}, v^{(1+\eta)\delta}) \in S_p^*$ by (2.6) and (iv). On the other hand, by (2.6) and (ii) we get $(v^{(1+\eta)(1-p')}, u^{(1+\eta)(1-p')}) \in A_{p'}^*$. As above, we take $\delta = 1/(1+\eta)$, then $(v^{1-p'}, u^{1-p'}) \in S_{p'}^*$. This is (v).

By (2.6), we have $(u^{1+\eta},v^{1+\eta}) \in A_p^*$. Now, we take $0<\delta<1$ such that $\delta(1+\eta)>1$. Let $1+\varepsilon = \delta(1+\eta)$, then $\varepsilon>0$ and

$$(u^{1+\varepsilon},v^{1+\varepsilon}) = (u^{\delta(1+\eta)},v^{\delta(1+\eta)}) \in S_p^* \subset A_p^*, \tag{2.7}$$

by (iii) and (iv). From (2.7) and (ii), we can get $(v^{(1-p')(1+\varepsilon)}, u^{(1-p')(1+\varepsilon)}) \in A_{p'}^*$. Thus, we prove (vi). Finally, we prove (vii). By (vi), there is an $\eta>0$ such that

$$(v^{(1-p')(1+\eta)}, u^{(1-p')(1+\eta)}) \in A_{p'}^*. \tag{2.8}$$

Taking $\varepsilon = \eta(p-1)/(1+\eta)$, then we can see easily that $\varepsilon>0$ and $1<p-\varepsilon<p$. Hence, we have $p' < (p-\varepsilon)'$ and $(v^{(1-p')(1+\eta)}, u^{(1-p')(1+\eta)}) \in A_{(p-\varepsilon)'}^*$, by (i). From (ii), we get $(u^{(1-p')(1+\eta)[1-(p-\varepsilon)]}, v^{(1-p')(1+\eta)[1-(p-\varepsilon)]}) \in A_{p-\varepsilon}^*$. However, $(1-p')(1+\eta)[1-(p-\varepsilon)] = 1$. Thus, we have $(u,v) \in A_{p-\varepsilon}^*$. □

**The relations between $A_p$ and $A(p,q)$, $A_p^*$, and $A^*(p,q)$.** Suppose that $0<\alpha<n$, $1<p<n/\alpha$, $1/q = 1/p-\alpha/n$, then we have the following conclusions:

$$\omega(x) \in A(p,q) \Longleftrightarrow \omega(x)^q \in A_{q(n-\alpha)/n} \Longleftrightarrow \omega(x)^{-p'} \in A_{1+p'/q}, \tag{2.9}$$

$$(u,v) \in A^*(p,q) \Longleftrightarrow (u^q,v^q) \in A_{q(n-\alpha)/n}^* \Longleftrightarrow (v^{-p'},u^{-p'}) \in A_{1+p'/q}^*. \tag{2.10}$$

Equations (2.9) and (2.10) can be deduced from the definitions of $A_p$ and $A(p,q)$, $A_p^*$ and $A^*(p,q)$, respectively. Here we omit the details.

**3. Proofs of the theorems.** The proofs of the theorems are based on Wheeden's a result in [9] and some lemmas.

**Theorem 3.1** (see [9]). *For $0<\alpha<n$, $1<p<n/\alpha$, and $1/q = 1/p-\alpha/n$, $M_\alpha$ is bounded from $L^p(v^p)$ to $L^q(u^q)$ if and only if the weights pair $(u,v) \in A^*(p,q)$ and $\sigma = v^{-p'}$ satisfies the doubling condition. That is, there is a constant $C>0$ such that $\sigma(2B) \le C\sigma(B)$ for all ball $B$ in $\mathbb{R}^n$.*

The following lemma gives a pointwise relation between $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$.

**Lemma 3.2** (see [3]). *For any $\varepsilon>0$ with $0<\alpha-\varepsilon<\alpha+\varepsilon<n$, we have*

$$|T_{\Omega,\alpha}f(x)| \le C[M_{\Omega,\alpha+\varepsilon}f(x)]^{1/2} \cdot [M_{\Omega,\alpha-\varepsilon}f(x)]^{1/2}, \quad x \in \mathbb{R}^n, \tag{3.1}$$

*where $C$ depends only on $\alpha,\varepsilon,n$.*

The following two lemmas characterize an important property of $A^*(p,q)$ weights.

**Lemma 3.3.** *Suppose that $0<\alpha<n$, $1<p<n/\alpha$, $1/q = 1/p-\alpha/n$, $(u,v) \in A^*(p,q)$, and $u(x),v(x) \in A(p,q)$. Then there is an $\varepsilon>0$ such that $\varepsilon<\alpha<\alpha+\varepsilon<n$; $1/p > (\alpha+\varepsilon)/n$, $1/q < (n-\varepsilon)/n$, and $(u,v) \in A^*(p,q_\varepsilon)$, $(u,v) \in A^*(p,\bar{q}_\varepsilon)$ hold at the same time, where $1/q_\varepsilon = 1/p-(\alpha+\varepsilon)/n$, $1/\bar{q}_\varepsilon = 1/p-(\alpha-\varepsilon)/n$.*

**Proof.** For $\alpha > 0$, $1/q < 1$, we may take $\delta_1 > 0$ such that $\delta_1 < \alpha$ and $1/q + \delta_1/n < 1$. Let

$$\frac{1}{q_{\delta_1}} = \frac{1}{p} - \frac{\alpha - \delta_1}{n} = \frac{1}{q} + \frac{\delta_1}{n}, \tag{3.2}$$

then $q > q_{\delta_1} > 1$ and $1 + p'/q < 1 + p'/q_{\delta_1}$. By $(u,v) \in A^*(p,q)$, (2.10), and (i), we have

$$(v^{-p'}, u^{-p'}) \in A^*_{1+p'/q} \subset A^*_{1+p'/q_{\delta_1}}. \tag{3.3}$$

Since $0 < \alpha - \delta_1 < n$, $1 < p < n/(\alpha - \delta_1)$, then by (3.2) and (2.10), we know that (3.3) is equivalent to

$$(u,v) \in A^*(p, q_{\delta_1}). \tag{3.4}$$

On the other hand, by $(u,v) \in A^*(p,q)$ and $u(x), v(x) \in A(p,q)$, we get

$$(v^{-p'}, u^{-p'}) \in A^*_{1+p'/q}, \qquad v(x)^{-p'}, u(x)^{-p'} \in A_{1+p'/q}, \tag{3.5}$$

by (2.10) and (2.9), respectively. From (3.5) and (vii), we know that there is an $\eta$ satisfying $0 < \eta < 1/q$ such that

$$(v^{-p'}, u^{-p'}) \in A^*_{1+p'(1/q-\eta)}. \tag{3.6}$$

Obviously, we can also choose $\delta_2 > 0$ small enough such that $\delta_2 < \min\{\alpha, n-\alpha\}$, $1/p > (\alpha+\delta_2)/n$, and $\delta_2/n < \eta$ hold at the same time. Now, let $1/q_{\delta_2} = 1/p - (\alpha+\delta_2)/n$, then by $1/p > (\alpha+\delta_2)/n$ and $\delta_2/n < \eta$, we get $0 < 1/q_{\delta_2} < 1$ and $1/q_{\delta_2} = 1/q - \delta_2/n > 1/q - \eta$. From this and (3.6), we have

$$(v^{-p'}, u^{-p'}) \in A^*_{1+p'(1/q-\eta)} \subset A^*_{1+p'/q_{\delta_2}}. \tag{3.7}$$

Since $0 < \alpha + \delta_2 < n$, $1 < p < n/(\alpha+\delta_2)$, and $1/q_{\delta_2} = 1/p - (\alpha+\delta_2)/n$, then by (2.10) we know that (3.7) is equivalent to

$$(u,v) \in A^*(p, q_{\delta_2}). \tag{3.8}$$

Finally, let $\varepsilon = \min\{\delta_1, \delta_2\}$ and $1/q_\varepsilon = 1/p - (\alpha+\varepsilon)/n$, $1/\bar{q}_\varepsilon = 1/p - (\alpha-\varepsilon)/n$. Then by (3.4) and (3.8), we get $(u,v) \in A^*(p, q_\varepsilon)$ and $(u,v) \in A^*(p, \bar{q}_\varepsilon)$. Thus, the proof of Lemma 3.3 is complete. □

**Lemma 3.4.** *Suppose that $0 < \alpha < n$, $1 \le s' < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $(u^{s'}, v^{s'}) \in A^*(p/s', q/s')$. Moreover, $u(x)^{s'}, v(x)^{s'} \in A(p/s', q/s')$. Then there is an $\varepsilon > 0$ such that*

$$\varepsilon < \alpha < \alpha + \varepsilon < n; \tag{3.9}$$

$$\frac{1}{p} > \frac{\alpha+\varepsilon}{n}, \qquad \frac{1}{q} < \frac{n-\varepsilon}{n}, \tag{3.10}$$

*and $(u^{s'}, v^{s'}) \in A^*(p/s', q_\varepsilon/s')$, $(u^{s'}, v^{s'}) \in A^*(p/s', \bar{q}_\varepsilon/s')$ hold at the same time. Where $1/q_\varepsilon = 1/p - (\alpha+\varepsilon)/n$ and $1/\bar{q}_\varepsilon = 1/p - (\alpha-\varepsilon)/n$.*

**Proof.** Since $1/(q/s') = 1/(p/s') - \alpha s'/n$, from Lemma 3.3, there is an $\eta > 0$ such that $\eta < \alpha s' < \alpha s' + \eta < n$, $1/(p/s') > (\alpha s' + \eta)/n$, $1/(q/s') < (n-\eta)/n$ and

$$(u^{s'}, v^{s'}) \in A^*\left(\frac{p}{s'}, q_\eta\right), \qquad (u^{s'}, v^{s'}) \in A^*\left(\frac{p}{s'}, \bar{q}_\eta\right), \tag{3.11}$$

hold at the same time, where

$$\frac{1}{q_\eta} = \frac{1}{p/s'} - \frac{\alpha s' + \eta}{n}, \qquad \frac{1}{\bar{q}_\eta} = \frac{1}{p/s'} - \frac{\alpha s' - \eta}{n}. \tag{3.12}$$

Let $\varepsilon = \eta/s'$, $q_\varepsilon = s' q_\eta$, and $\bar{q}_\varepsilon = s' \bar{q}_\eta$, then it is easy to see that $\varepsilon$ satisfies (3.9) and (3.10). Moreover, by (3.11) and (3.12) we know that

$$(u^{s'}, v^{s'}) \in A^*\left(\frac{p}{s'}, \frac{q_\varepsilon}{s'}\right), \qquad (u^{s'}, v^{s'}) \in A^*\left(\frac{p}{s'}, \frac{\bar{q}_\varepsilon}{s'}\right), \tag{3.13}$$

hold at the same time, where $1/q_\varepsilon = 1/p - (\alpha+\varepsilon)/n$ and $1/\bar{q}_\varepsilon = 1/p - (\alpha-\varepsilon)/n$. This completes the proof of Lemma 3.4. □

Now, we turn to the proofs of Theorems 1.1 and 1.2. We alternatively prove them.

**The proof of (1.8) for the condition (c) in Theorem 1.2.** Note that, for $r > 0$, $\left(\int_{|x-y|<r} |\Omega(x-y)|^s \, dy\right)^{1/s} \le C r^{n/s} \|\Omega\|_{L^s(S^{n-1})}$. Hence, we have

$$M_{\Omega,\alpha} f(x) \le C\left[M_{\alpha s'}\left(|f|^{s'}\right)(x)\right]^{1/s'}. \tag{3.14}$$

Since $1 \le s' < p < n/\alpha$ and $1/q = 1/p - \alpha/n$, we get $0 < \alpha s' < n$, $1 < p/s' < n/\alpha s'$, and $1/(q/s') = 1/(p/s') - \alpha s'/n$. Moreover, $(u^{s'}, v^{s'}) \in A^*(p/s', q/s')$ and $\sigma = (v^{s'})^{-(p/s')'}$ satisfies the doubling condition by (c). From Theorem 3.1, we know that the operator $M_{\alpha s'}$ is bounded from $L^{p/s'}((v^{s'})^{p/s'})$ to $L^{q/s'}((u^{s'})^{q/s'})$. Thus, we get

$$\begin{aligned}\left(\int_{\mathbb{R}^n} [M_{\Omega,\alpha} f(x) u(x)]^q \, dx\right)^{1/q} &\le C\left(\int_{\mathbb{R}^n} \left[M_{\alpha s'}\left(|f|^{s'}\right)(x) u(x)^{s'}\right]^{q/s'} dx\right)^{1/q} \\ &\le C\left(\int_{\mathbb{R}^n} |f(x)|^p v(x)^p \, dx\right)^{1/p}.\end{aligned} \tag{3.15}$$

This completes the proof of (1.8) for the condition (c).

**The proof of (1.7) for the condition (a) in Theorem 1.1.** First, we show that, under the condition (a) in Theorem 1.1, $\sigma = v^{-s'(p/s')'}$ still satisfies the doubling condition. In fact, by $v(x)^{s'} \in A(p/s', q/s')$ and (2.9), we have $(v(x)^{s'})^{-(p/s')'} \in A_{1+(p/s')'/(q/s')} \subset A_\infty$. Since, every weight function in $A_\infty$ satisfies the doubling condition, and so does $\sigma = v^{-s'(p/s')'}$.

By Lemma 3.4, there is an $\varepsilon > 0$ satisfying (3.9) and (3.10) such that

$$(u^{s'}, v^{s'}) \in A^*\left(\frac{p}{s'}, \frac{q_\varepsilon}{s'}\right), \qquad (u^{s'}, v^{s'}) \in A^*\left(\frac{p}{s'}, \frac{\bar{q}_\varepsilon}{s'}\right), \tag{3.16}$$

hold at the same time. Where $1/q_\varepsilon = 1/p - (\alpha+\varepsilon)/n$ and $1/\bar{q}_\varepsilon = 1/p - (\alpha-\varepsilon)/n$. Let $l_1 = 2q_\varepsilon/q$, $l_2 = 2\bar{q}_\varepsilon/q$, then $1/l_1 + 1/l_2 = 1$. For given $\varepsilon > 0$ above, using (3.1) and

Hölder's inequality, we have

$$\begin{aligned}\|T_{\Omega,\alpha}f\|_{q,u^q} &\le C\Big(\int_{\mathbb{R}^n}[M_{\Omega,\alpha+\varepsilon}f(x)u(x)]^{ql_1/2}dx\Big)^{1/ql_1}\Big(\int_{\mathbb{R}^n}[M_{\Omega,\alpha-\varepsilon}f(x)u(x)]^{ql_2/2}dx\Big)^{1/ql_2}\\ &= C\Big(\int_{\mathbb{R}^n}[M_{\Omega,\alpha+\varepsilon}f(x)u(x)]^{q_\varepsilon}dx\Big)^{1/2q_\varepsilon}\Big(\int_{\mathbb{R}^n}[M_{\Omega,\alpha-\varepsilon}f(x)u(x)]^{\bar q_\varepsilon}dx\Big)^{1/2\bar q_\varepsilon}.\end{aligned} \tag{3.17}$$

Note that $\sigma = v^{-s'(p/s')'}$ satisfies the doubling condition, applying Lemma 3.4 and the conclusion of Theorem 1.2 under the condition (c), we get

$$\|T_{\Omega,\alpha}f\|_{q,u^q} \le C\|f\|_{p,v^p}. \tag{3.18}$$

This is the conclusion of Theorem 1.1 for the case (a).

**The proof of (1.7) for the condition (b) in Theorem 1.1.** Since the fractional integral operator $T_{\Omega,\alpha}$ is a linear operator, we denote $T^* := (T_{\Omega,\alpha})^*$ as the adjoint operator of $T_{\Omega,\alpha}$. Then $(T_{\Omega,\alpha})^* = T_{\Omega^*,\alpha}$, where $\Omega^*(x) = \overline{\Omega(-x)}$. Clearly, $\Omega^*$ satisfies the same conditions as $\Omega$. We have

$$\|T_{\Omega,\alpha}f\|_{q,u^q} = \sup_g \Big|\int_{\mathbb{R}^n} T_{\Omega,\alpha}f(x)g(x)\,dx\Big|, \tag{3.19}$$

where the supremum is taken over all $g$ with $\|g\|_{q',u^{-q'}} \le 1$. On the other hand,

$$\int_{\mathbb{R}^n} T_{\Omega,\alpha}f(x)g(x)\,dx = \int_{\mathbb{R}^n} f(x)T^*g(x)\,dx. \tag{3.20}$$

Thus, by Hölder's inequality, we get

$$\|T_{\Omega,\alpha}f\|_{q,u^q} = \sup_g \Big|\int_{\mathbb{R}^n} T_{\Omega,\alpha}f(x)g(x)\,dx\Big| \le \|f\|_{p,v^p}\sup_g\|T^*g\|_{p',v^{-p'}}. \tag{3.21}$$

From condition (b) of Theorem 1.1, we see that $1/p' = 1/q' - \alpha/n$ and $s' < q' < n/\alpha$. Since $(v^{-s'}, u^{-s'}) \in A^*(q'/s', p'/s')$ and $v(x)^{-s'}, v(x)^{-s'} \in A(q'/s', p'/s')$, using the conclusion of Theorem 1.1 for the case (a), we get

$$\|T^*g\|_{p',v^{-p'}} \le C\|g\|_{q',u^{-q'}}. \tag{3.22}$$

Therefore,

$$\|T_{\Omega,\alpha}f\|_{q,u^q} \le \|f\|_{p,v^p}\cdot\sup_g\|T^*g\|_{p',v^{-p'}} \le C\|f\|_{p,v^p}. \tag{3.23}$$

This is the inequality (1.7) in Theorem 1.1.

**The proof of (1.8) for the condition (d) in Theorem 1.2.** Finally, we show how to obtain the weighted inequality (1.8) for the case (d) in Theorem 1.2. Note that the conclusions of Theorem 1.1 hold also for $T_{|\Omega|,\alpha}(|f|)$, hence inequality (1.8) for the case (d) is a direct result of the following lemma and the conclusion of Theorem 1.1 for the case (b).

**Lemma 3.5** (see [2]). *Let $0 < \alpha < n$, $\Omega \in L^1(S^{n-1})$. Then we have*

$$M_{\Omega,\alpha}f(x) \le T_{|\Omega|,\alpha}(|f|)(x), \quad x \in \mathbb{R}^n. \tag{3.24}$$

**Acknowledgement.** Yong Ding was partially supported by NSF of China (Grant No. 19971010).

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Yong Ding: Department of Mathematics, Beijing Normal University, Beijing 100875, China

*E-mail address*: dingy@bnu.edu.cn

Chin-Cheng Lin: Department of Mathematics, National Central University, Chung-Li, 320, Taiwan

*E-mail address*: clin@math.ncu.edu.tw