

EVOLUTIONARY EQUATION OF INERTIAL WAVES IN 3-D MULTIPLY CONNECTED DOMAIN WITH DIRICHLET BOUNDARY CONDITION

PAVEL A. KRUTITSKII

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ABSTRACT. We study initial-boundary value problem for an equation of composite type in 3-D multiply connected domain. This equation governs nonsteady inertial waves in rotating fluids. The solution of the problem is obtained in the form of dynamic potentials, which density obeys the uniquely solvable integral equation. Thereby the existence theorem is proved. Besides, the uniqueness of the solution is studied. All results hold for interior domains and for exterior domains with appropriate conditions at infinity.

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Modern advances in the theory of waves are mostly concerned with nonlinear phenomena [4, 3, 5, 6]. However, there exist some types of linear waves which are not well studied yet, for example, nonsteady inertial waves, that is, internal waves in rotating fluids. These waves are governed by PDEs of composite type and of high order. Such PDEs possess both elliptic and hyperbolic characteristics, and so they share properties of both elliptic and hyperbolic equations [13, 23, 24, 25, 26]. Equations of composite type are not well studied even in linear case and they are not covered by existing classifications of PDEs. In particular, they do not belong to equations of principal part, though elliptic, parabolic, and hyperbolic equations belong [7].

Inertial wave equation (see (1.2)) is an evolutionary equation of fourth order and composite type. This equation comes from ocean dynamics. Different initial-boundary value problems for analogous equations were treated in [8, 9, 10, 14, 15, 16, 17, 18, 19, 20, 30]. In particular, 2-D problems in multiply connected domains were studied in [15, 16, 17, 18, 19, 20]. Initial boundary value problems in simply connected domains were studied for 3-D equation of inertial waves and for similar equations in [8, 9, 10, 14, 30], but 3-D problems were not treated in multiply connected domains before. The aim of the present paper is to consider initial-boundary value problem in 3-D multiply connected domain (interior and exterior) with Dirichlet boundary condition. To prove existence theorem for our problem we use dynamic potentials [14] and the method of boundary integral equations. Uniqueness of a solution is also studied. It should be stressed that the integral equation obtained in the present paper is a uniquely solvable Fredholm equation of the second kind and index zero. This equation can be easily computed by discretization and inversion of a matrix. Initial-boundary value problems for some equations of composite type in a 3-D simply connected domain were reduced to integral equations in [8, 10, 14]. However, the method of reduction to integral equations suggested in these papers for simply connected do-

main is not suitable for multiply connected domain, since obtained integral equations may not have a solution, while the boundary value problem is solvable. Besides, the method of reduction to the integral equation proposed in the present paper is different from those known before, even in case of exterior simply connected domain.

It is a feature of (1.2) that conditions at infinity are required in the exterior domain for the well-posedness of the initial-boundary value problems. However, these conditions were omitted in [14] together with analysis of uniqueness of a solution, and so exterior problems in [14] are not well-posed.

Another approach to studies of initial-boundary value problems for (1.2) in interior domains has been proposed by Sobolev in [27, 28]. This approach enables us to prove formal existence theorem in Sobolev spaces, but it does not provide us with the integral representation for a solution, and so we do not obtain the constructive way for finding a numerical solution.

1. Formulation of the problem and uniqueness theorem. In Cartesian coordinates $x = (x_1, x_2, x_3) \in R^3$ we consider the connected domain \mathcal{D} with the boundary Γ of class $C^{2,\lambda}$, $\lambda \in (0, 1]$. Suppose that Γ consists of simple closed surfaces $\Gamma_0, \Gamma_1, \dots, \Gamma_N$, without common points, so that Γ_0 envelopes all others. The surface Γ_0 may be absent, then the domain \mathcal{D} is unbounded. By \mathbf{n}_x we denote interior normal vector with respect to the domain \mathcal{D} at the point $x \in \Gamma$. Consider surfaces making up Γ as double sided. The side of surfaces Γ , which can be seen if we look towards the normal's tips \mathbf{n}_x ($x \in \Gamma$), is denoted by Γ^- , and the opposite side is denoted by Γ^+ .

DEFINITION 1.1. The function $u(t, x)$ defined in $[0, \infty) \times \mathcal{D}$ belongs to the smoothness class G , if

- (1) $u, u_t \in C^0([0, \infty) \times \mathcal{D})$; $(\partial^k / \partial t^k)(\partial^p / \partial x_j^p)u \in C^0((0, \infty) \times \mathcal{D})$, $k, p = 0, 1, 2$; $j = 1, 2, 3$;
- (2) for any $t \geq 0$ the limiting value of the function u exists on the boundary Γ in the sense of a limit in a normal direction to a boundary.

DEFINITION 1.2. Function $u(t, x)$ defined in an unbounded domain is called regular at infinity if there exist such functions $q_1(t), q_2(t) \in C^0[0, \infty)$, that as $|x| \rightarrow \infty$ the inequalities hold

$$\left| \frac{\partial^j}{\partial t^j} u(t, x) \right| \leq q_1(t) |x|^{-1}, \quad \left| \frac{\partial^j}{\partial t^j} u_{x_k}(t, x) \right| \leq q_2(t) |x|^{-2}, \quad \text{where } j = 0, 1, 2; k = 1, 2, 3. \tag{1.1}$$

Equation of inertial waves in a rotating fluid in the 3-D case has the form [1, 11, 14, 21, 22]:

$$\frac{\partial^2}{\partial t^2} \Delta u + \omega^2 u_{x_3 x_3} = 0, \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \tag{1.2}$$

where $\omega \geq 0$ is twice the angular velocity of a rotation around Ox_3 axis. We assume that ω is a constant.

Let us formulate the initial-boundary value problem for (1.2) in the domain \mathcal{D} for $t \geq 0$.

PROBLEM \mathcal{H} . To find a function $\mathcal{U}(t, x)$ of class G , which obeys (1.2) in $(0, \infty) \times \mathcal{D}$ in a classical sense, that satisfies the initial conditions $\mathcal{U}(0, x) = \mathcal{U}_t(0, x) = 0$, $x \in \mathcal{D}$ and the boundary condition on Γ :

$$\mathcal{U}(t, x)|_{x \in \Gamma} = f_0(t, x)|_{x \in \Gamma}, \quad t \geq 0. \tag{1.3}$$

Besides, if \mathcal{D} is an exterior domain, then the function $\mathcal{U}(t, x)$ must be regular at infinity.

The boundary condition should be satisfied in a sense of a limit in a normal direction to the boundary Γ .

Together with the class G we consider class of smoothness G_1 .

DEFINITION 1.3. Function $\mathcal{U}(t, x)$ belongs to the class G_1 , if

- (1) $\mathcal{U}, \mathcal{U}_t \in C^0([0, \infty) \times \mathcal{D})$; $\nabla \mathcal{U}, \nabla \mathcal{U}_t \in C^0([0, \infty) \times \mathcal{D})$;
 $(\partial^k / \partial t^k)(\partial^p / \partial x_j^p) \mathcal{U} \in C^0((0, \infty) \times \mathcal{D})$, $k, p = 0, 1, 2$; $j = 1, 2, 3$;
- (2) for any $t \geq 0$, there exists the uniform limit, for all $x \in \Gamma$, of the expression

$$\mathcal{N}_{tx} \mathcal{U}(t, \bar{x}) = \frac{\partial^2}{\partial t^2} (\mathbf{n}_x, \nabla_{\bar{x}}) \mathcal{U}(t, \bar{x}) + \omega^2 \cos(\mathbf{n}_x, x_3) \mathcal{U}_{x_3}(t, \bar{x}) \tag{1.4}$$

as \bar{x} tends to $x \in \Gamma$ along the normal \mathbf{n}_x , that is, the limit is understood in the sense of a limit along a normal to a boundary Γ . The cosine of the angle between the normal vector \mathbf{n}_x in the point $x \in \Gamma$ and the direction of the Ox_3 axis is denoted by $\cos(\mathbf{n}_x, x_3)$.

NOTE. Evidently, any function of class G_1 belongs to the class G . Below $\int_{\Gamma} \dots dS$ denotes the surface integral of the first kind.

Consider the uniqueness theorem for the problem \mathcal{H} .

LEMMA 1.4. Let $\mathcal{U}(t, x)$ be a classical solution of (1.2) in $(0, \infty) \times \mathcal{D}$, and $\mathcal{U}(t, x)$ belongs to class G_1 . Besides, assume that the function $\mathcal{U}(t, x)$ is regular at infinity in case of exterior domain \mathcal{D} . Then for any $t \geq 0$ the identity holds

$$\frac{1}{2} \frac{d}{dt} (\|\nabla \mathcal{U}_t\|^2 + \omega^2 \|\mathcal{U}_{x_3}\|^2) = \int_{\Gamma} \mathcal{U}_t \mathcal{N}_{tx} \mathcal{U} ds, \tag{1.5}$$

where the norm $\|\cdot\|$ is taken in the space $L_2(\mathcal{D})$.

Lemma 1.4 can be proved by direct verification. To do so we construct equidistant surfaces [32] in the domain \mathcal{D} to the boundary Γ . If \mathcal{D} is an exterior domain, then additionally we consider a sphere of enough large radius R with the center in the origin. In the domain bounded by equidistant surfaces (and the sphere if \mathcal{D} is an exterior domain), we write energy equality, which is obtained as a result of multiplying equation (1.2) by \mathcal{U}_t and integration by parts. Tending equidistant surfaces to the boundary Γ and using properties of functions of class G_1 , we prove the identity stated in Lemma 1.4, in case of interior domain \mathcal{D} . If \mathcal{D} is an exterior domain, then additionally we tend R to infinity and use regularity of $\mathcal{U}(t, x)$ at infinity. Finally, we arrive at the identity stated in Lemma 1.4 again.

Next theorem follows from Lemma 1.4.

THEOREM 1.5. *If the problem \mathcal{K} has a solution in the smoothness class G_1 , then this solution is unique.*

Solvability of the problem \mathcal{K} will be studied under weaker smoothness conditions on its solutions, than in [Theorem 1.5](#). More precisely, we drop the condition that solution belongs to the class G_1 . We will prove classical solvability of the problem \mathcal{K} and derive integral representation for a solution in the form of dynamic potentials (see [Theorem 4.2](#)). Density in potentials obeys integral equation, which is solvable.

We now turn to the construction of the classical solution of the problem \mathcal{K} .

2. Reduction of the problem to the integral equation. Let us introduce some classes of functions. Set

$$C_0^2[0, \infty) = \{\varphi(t) : \varphi(t) \in C^2[0, \infty), \varphi(0) = \varphi'(0) = 0\}. \tag{2.1}$$

By $C^k([0, \infty); \mathcal{H})$ we denote class of k times continuously differentiable at $t \geq 0$ abstract functions $\varphi(t)$ with values in the Banach space \mathcal{H} . Besides, we introduce the following class of abstract functions:

$$C_0^2([0, \infty); \mathcal{H}) = \{\varphi(t) : \varphi(t) \in C^2([0, \infty); \mathcal{H}), \varphi(0) = \varphi'(0) = 0\}. \tag{2.2}$$

By the designation $C^0([0, T]; \mathcal{H})$, where $T > 0$, we mean Banach space of abstract functions with the norm

$$\|\cdot\|_{C^0([0, T]; \mathcal{H})} = \|\|\cdot\|_{\mathcal{H}}\|_{C^0[0, T]}. \tag{2.3}$$

Further on, we use the following result from the theory of abstract functions (see proof in [\[29, pages 28–29\]](#)).

LEMMA 2.1. *Classes of functions $C^0([0, \infty); C^0(\Gamma))$ and $C^0([0, \infty) \times \Gamma)$ are equivalent, that is, any function of the first class belongs to the second and conversely.*

Let us construct the solution of the problem \mathcal{K} . Assume that the function $f_0(t, x)$ in the boundary condition [\(1.3\)](#) belongs to the class $C_0^2([0, \infty); C^0(\Gamma))$, then it can be expressed in the form

$$f_0(t, x) = \int_0^t \int_0^{t_1} f(t_2, x) dt_2 dt_1 = \int_0^t (t - \tau) f(\tau, x) d\tau, \tag{2.4}$$

where

$$f(t, x) = \frac{\partial^2}{\partial t^2} f_0(t, x) \in C^0([0, \infty); C^0(\Gamma)). \tag{2.5}$$

By $J_0(\sigma)$ and $J_1(\sigma)$ we denote Bessel functions of zero and first order.

Consider a function introduced in [\[14\]](#) and called dynamic double layer potential for [\(1.2\)](#),

$$U[\bar{\mu}](t, x) = \int_0^t \int_{\Gamma} \bar{\mu}(\tau, y) \mathcal{N}_{\tau y} W(t - \tau, x - y) dS_y d\tau - \frac{1}{4\pi} \int_{\Gamma} \bar{\mu}(t, y) \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|x - y|} dS_y, \tag{2.6}$$

where $x = (x_1, x_2, x_3) \in \mathcal{D}$, $y = (y_1, y_2, y_3) \in \Gamma$, operator \mathcal{N}_{tx} was introduced in (1.4),

$$W(t, x) = -\frac{1}{4\pi|x|_0} \int_0^{t|x|_0/|x|} J_0(\sigma) d\sigma = -\frac{t}{4\pi|x|} + \frac{|x|_0}{4\pi|x|^2} \left(t * J_1 \left(t \frac{|x|_0}{|x|} \right) \right) \quad (2.7)$$

is a singular solution for (1.2), $|x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, $|x|_0 = \omega(x_1^2 + x_2^2)^{1/2}$. By the symbol $*$ the convolution operation in time is denoted so that for any two functions of time $a(t)$, $b(t)$, we have

$$a(t) * b(t) = \int_0^t a(t-\tau)b(\tau) d\tau = \int_0^t a(\tau)b(t-\tau) d\tau. \quad (2.8)$$

With respect to the density $\bar{\mu}(t, x)$ we assume that it has the form

$$\bar{\mu}(t, x) = \int_0^t \int_0^{t_1} \mu(t_2, x) dt_2 dt_1 = \int_0^t (t-\tau)\mu(\tau, x) d\tau, \quad (2.9)$$

where $\mu(t, x) \in C^0([0, \infty); C^0(\Gamma))$, so that $\bar{\mu}(t, x) \in C_0^2([0, \infty); C^0(\Gamma))$.

As shown in [14], the function $W(t, x)$ for $x \neq 0$ obeys (1.2) in a classical sense, and $W(0, x) = 0$, $W_t(0, x) = -(4\pi|x|)^{-1}$.

For the function $\mathcal{N}_{\tau y}W(t-\tau, x-y)$ the following representation holds [14]:

$$\begin{aligned} &\mathcal{N}_{\tau y}W(t-\tau, x-y) \\ &= \frac{\cos(x-y, \mathbf{n}_y)}{4\pi|x-y|^2} \left[\frac{|x-y|_0}{|x-y|} J'_0(\zeta) + \frac{(\tau-t)(x_3-y_3)^2}{|x-y|^2} \omega^2 J_0(\zeta) \right], \quad \zeta = \frac{(\tau-t)|x-y|_0}{|x-y|}. \end{aligned} \quad (2.10)$$

Here $\cos(x-y, \mathbf{n}_y)$ is a cosine of the angle between the vectors \overrightarrow{yx} and \mathbf{n}_y . The function in the square brackets is continuous for $x \neq y$ and uniformly bounded in all variables on any finite interval of time. Since $|\cos(x-y, \mathbf{n}_y)| \leq \text{const}|x-y|^{\lambda_1}$ for $x, y \in \Gamma$ and $\Gamma \in C^{1, \lambda_1}$ (recall that in our case $\Gamma \in C^{2, \lambda}$), then as shown in [14], the potential $U[\bar{\mu}](t, x)$ is continuous for $x \in \Gamma$.

Next lemma follows from the properties of double layer dynamic potential, presented in [14].

LEMMA 2.2. *If $\mu(t, s) \in C^0([0, \infty); C^0(\Gamma))$ and $\bar{\mu}(t, s)$ is given by (2.9), then the potential $U[\bar{\mu}](t, x)$ belongs to the class G , obeys (1.2) in $(0, \infty) \times \mathcal{D}$ and initial conditions of the problem \mathcal{K} . Besides, if \mathcal{D} is exterior domain, then the potential $U[\bar{\mu}](t, x)$ is regular at infinity.*

By Y_1, \dots, Y_N we denote an arbitrary fixed points placed inside surfaces $\Gamma_1, \dots, \Gamma_N$ and consider a function

$$\begin{aligned} Q[\bar{\mu}](t, x) &= \sum_{k=1}^N \int_0^t \int_{\Gamma_k} \mu(\tau, y) dS_y W(t-\tau, x-Y_k) d\tau \\ &= \frac{1}{4\pi} \sum_{k=1}^N \left\{ -\frac{1}{|x-Y_k|} \int_{\Gamma_k} \bar{\mu}(t, y) dS_y + \frac{|x-Y_k|_0}{|x-Y_k|^2} \right. \\ &\quad \left. \times \int_0^t \int_{\Gamma_k} \bar{\mu}(\tau, y) dS_y J_1 \left(\frac{|x-Y_k|_0}{|x-Y_k|} (t-\tau) \right) d\tau \right\}, \end{aligned} \quad (2.11)$$

where $W(t, x)$ was defined above.

Function $Q[\bar{\mu}](t, \mathbf{x})$ is a linear combination of point sources for (1.2) placed in the points Y_k , $k = 1, \dots, N$. If conditions of Lemma 2.2 hold, then $Q[\bar{\mu}](t, \mathbf{x})$ belongs to the class G , obeys (1.2) in $(0, \infty) \times \mathcal{D}$ and satisfies initial conditions of the problem \mathcal{H} . Moreover, $Q[\bar{\mu}](t, \mathbf{x})$ is infinitely differentiable in \mathbf{x} in $R^3 \setminus (\cup_{k=1}^N Y_k)$ and is regular at infinity.

The solution of the problem \mathcal{H} we seek in the form

$$\mathcal{U}[\bar{\mu}](t, \mathbf{x}) = U[\bar{\mu}](t, \mathbf{x}) + Q[\bar{\mu}](t, \mathbf{x}), \quad (2.12)$$

where $\bar{\mu}(t, \mathbf{x})$ is defined by (2.9), for $U[\bar{\mu}](t, \mathbf{x})$ the representation (2.6) holds, and $Q[\bar{\mu}](t, \mathbf{x})$ is given by expression (2.11). If we drop the function $Q[\bar{\mu}](t, \mathbf{x})$ in (2.12), then the integral equation obtained below may be not solvable.

The above properties of functions occurring in (2.12) yield the following assertion.

LEMMA 2.3. *If conditions of Lemma 2.2 hold, then the function (2.12) belongs to the class G , satisfies (1.2) in $(0, \infty) \times \mathcal{D}$ and initial conditions of the problem \mathcal{H} . In addition, if \mathcal{D} is an exterior domain, then the function (2.12) is regular at infinity.*

Thus, $\mathcal{U}[\bar{\mu}](t, \mathbf{x})$ satisfies all conditions of the problem \mathcal{H} , except for the boundary condition (1.3). Using formulas for limiting values of double layer dynamic potential at the boundary of a domain [14] and satisfying the boundary condition (1.3), we arrive at integral equation with respect to the function $\mu(t, \mathbf{x})$,

$$-\frac{1}{2}\mu(t, \mathbf{x}) - (A\mu)(t, \mathbf{x}) + (B\mu)(t, \mathbf{x}) = f(t, \mathbf{x}), \quad \mathbf{x} \in \Gamma, \quad (2.13)$$

where $f(t, \mathbf{x}) = (f_0(t, \mathbf{x}))_{tt} \in C^0([0, \infty); C^0(\Gamma))$, $f_0(t, \mathbf{x}) \in C_0^2([0, \infty); C^0(\Gamma))$. By A and B the following operators are denoted:

$$\begin{aligned} (A\mu)(t, \mathbf{x}) &= \int_{\Gamma} \mu(t, \mathbf{y}) \mathcal{A}(\mathbf{x}, \mathbf{y}) dS_{\mathbf{y}}, \\ (B\mu)(t, \mathbf{x}) &= \int_0^t \int_{\Gamma} \mu(\tau, \mathbf{y}) \mathcal{B}(t - \tau, \mathbf{x}, \mathbf{y}) dS_{\mathbf{y}} d\tau, \\ \mathcal{B}(t - \tau, \mathbf{x}, \mathbf{y}) &= \mathcal{N}_{\tau, \mathbf{y}} W(t - \tau, \mathbf{x} - \mathbf{y}) + \frac{1}{4\pi} \sum_{k=1}^N \delta_k(\mathbf{y}) \frac{|x - Y_k|_0}{|x - Y_k|^2} J_1 \left((t - \tau) \frac{|x - Y_k|_0}{|x - Y_k|} \right), \\ \mathcal{A}(\mathbf{x}, \mathbf{y}) &= \frac{1}{4\pi} \frac{\partial}{\partial \mathbf{n}_{\mathbf{y}}} \frac{1}{|\mathbf{x} - \mathbf{y}|} + \frac{1}{4\pi} \sum_{k=1}^N \delta_k(\mathbf{y}) \frac{1}{|\mathbf{x} - Y_k|}. \end{aligned} \quad (2.14)$$

Here $\mathcal{N}_{\tau, \mathbf{y}} W(t - \tau, \mathbf{x} - \mathbf{y})$ is a function from (2.10) and

$$\delta_k(\mathbf{y}) = \begin{cases} 1, & \mathbf{y} \in \Gamma_k, \\ 0, & \mathbf{y} \notin \Gamma_k. \end{cases} \quad (2.15)$$

Let us study some properties of operators A and B .

LEMMA 2.4. (1) *Kernels $\mathcal{A}(x, y), \mathcal{B}(t, x, y)$ are continuous for $x \neq y$ and have a weak singularity if $x = y$.*

(2) *The operator A acts from $C^0(\Gamma)$ into $C^0(\Gamma)$ and bounded.*

(3) *The operator B acts from $C^0([0, \infty); C^0(\Gamma))$ into $C^0([0, \infty); C^0(\Gamma))$.*

(4) *If $h(t, x) \in C^0([0, \infty); C^0(\Gamma))$, then for any $t \geq 0$ the following inequality holds:*

$$\|(Bh)(t, \cdot)\|_{C^0(\Gamma)} \leq c_B \left[\int_0^t \|h(\tau, \cdot)\|_{C^0(\Gamma)} d\tau + \int_0^t \int_0^{t_1} \|h(\tau, \cdot)\|_{C^0(\Gamma)} d\tau dt_1 \right], \tag{2.16}$$

where c_B is a constant.

PROOF. Item (1) can be verified directly. Kernel $\mathcal{A}(x, y)$ has been studied in [32]. Properties of kernel $\mathcal{B}(t, x, y)$ follow from the representation (2.10).

Item (2) follows from [32, Section 17.4] and item (1), since the kernel of the operator A is polar.

For item (3), let $h(t, x) \in C^0([0, \infty); C^0(\Gamma))$. According to Lemma 2.1, $h(t, x) \in C^0([0, \infty) \times \Gamma)$. Set

$$\begin{aligned} (Bh)(t, x) &= (\hat{B}h)(t, x) + (\check{B}h)(t, x), \\ (\hat{B}h)(t, x) &= \int_0^t \int_{\Gamma} h(\tau, y) \mathcal{N}_{\tau y} W(t - \tau, x - y) dS_y d\tau, \\ (\check{B}h)(t, x) &= \int_0^t \int_{\Gamma} h(\tau, y) (\mathcal{B}(t - \tau, x, y) - \mathcal{N}_{\tau y} W(t - \tau, x - y)) dS_y d\tau. \end{aligned} \tag{2.17}$$

It follows from the results of [14] that the operator \hat{B} acts from $C^0([0, \infty); C^0(\Gamma))$ into $C^0([0, \infty); C^0(\Gamma))$. From the explicit expression for the kernel $(\mathcal{B}(t - \tau, x, y) - \mathcal{N}_{\tau y} W(t - \tau, x - y))$ it follows that this kernel is continuous. Hence the operator \check{B} acts from $C^0([0, \infty) \times \Gamma)$ into $C^0([0, \infty) \times \Gamma)$. Taking into account Lemma 2.1, we are convinced that the operator \check{B} acts from $C^0([0, \infty); C^0(\Gamma))$ into $C^0([0, \infty); C^0(\Gamma))$. Consequently, item (3) of lemma holds for operator $B = \hat{B} + \check{B}$.

Let us prove (4). Let $\Gamma \in C^{1, \lambda_1} \subset C^{2, \lambda}$, $\lambda_1 \in (0, 1]$. If $x, y \in \Gamma$, then the following estimates hold:

$$\begin{aligned} |\mathcal{N}_{\tau y} W(t - \tau, x - y)| &\leq c_1 \frac{|\cos(x - y, \mathbf{n}_y)|}{|x - y|^2} [1 + (t - \tau)], \\ \mathcal{B}(t - \tau, x, y) &\leq c_1 \frac{|\cos(x - y, \mathbf{n}_y)|}{|x - y|^2} [1 + (t - \tau)] + c_2, \end{aligned} \tag{2.18}$$

where c_1, c_2 are some constants and first estimate follows from (2.10).

Let $h(t, x) \in C^0([0, \infty); C^0(\Gamma))$, then for operator B the following estimate holds:

$$\begin{aligned} |(Bh)(t, x)| &\leq \int_0^t \int_{\Gamma} |h(\tau, y)| \left\{ c_1 \frac{|\cos(x - y, \mathbf{n}_y)|}{|x - y|^2} [1 + (t - \tau)] + c_2 \right\} dS_y d\tau \\ &\leq \int_0^t \|h(\tau, \cdot)\|_{C^0(\Gamma)} \left\{ c_1 \int_{\Gamma} \frac{|\cos(x - y, \mathbf{n}_y)|}{|x - y|^2} dS_y [1 + (t - \tau)] + c_2 \int_{\Gamma} 1 dS_y \right\} d\tau \\ &\leq \int_0^t \|h(\tau, \cdot)\|_{C^0(\Gamma)} \{ \tilde{c}_1 [1 + (t - \tau)] + \tilde{c}_2 \} d\tau \\ &= \tilde{c}_1 \int_0^t \int_0^{t_1} \|h(\tau, \cdot)\|_{C^0(\Gamma)} d\tau dt_1 + (\tilde{c}_1 + \tilde{c}_2) \int_0^t \|h(\tau, \cdot)\|_{C^0(\Gamma)} d\tau, \end{aligned} \tag{2.19}$$

where \tilde{c}_1, \tilde{c}_2 are some constants and Lemma 2.1 from [32, Section 27.4] has been used. This lemma states that the integral

$$\int_{\Gamma} \frac{|\cos(x - y, \mathbf{n}_y)|}{|x - y|^2} dS_y \tag{2.20}$$

is uniformly bounded by a constant for all $x \in \Gamma$. Taking $\sup_{x \in \Gamma}$ on both sides of the inequality (2.19), we obtain

$$\|(Bh)(t, \cdot)\|_{C^0(\Gamma)} \leq (\tilde{c}_1 + \tilde{c}_2) \left\{ \int_0^t \int_0^{t_1} \|h(\tau, \cdot)\|_{C^0(\Gamma)} d\tau dt_1 + \int_0^t \|h(\tau, \cdot)\|_{C^0(\Gamma)} d\tau \right\}, \tag{2.21}$$

and so we complete the proof of estimate (2.16). □

In the next section, we continue analysis of operators included in integral equation (2.13).

3. Properties of operators included in the integral equation. On the contour Γ we consider integral equation with respect to the function $h(x) \in C^0(\Gamma)$,

$$-\frac{1}{2}h(x) - (Ah)(x) = g(x), \quad x \in \Gamma, \tag{3.1}$$

where A is an operator from (2.13), and $g(x)$ belongs to $C^0(\Gamma)$. According to Lemma 2.4, the kernel of the operator A is polar, therefore (3.1) is a Fredholm equation of the second kind and index zero.

Let us prove that (3.1) is uniquely solvable in $C^0(\Gamma)$. To this end, it is sufficient to show that homogeneous equation (3.1) has only a trivial solution. We give a proof by contradiction. Let the homogeneous equation have a nontrivial solution $h_0(x) \in C^0(\Gamma)$ for which the identity holds

$$-\frac{1}{2}h_0(x) - (Ah_0)(x) = 0, \quad x \in \Gamma. \tag{3.2}$$

We write this identity in the form

$$-\frac{1}{2}h_0(x) = \int_{\Gamma} h_0(y) \mathcal{A}_1(x, y) dS_y + \int_{\Gamma} h_0(y) \mathcal{A}_2(x, y) dS_y, \tag{3.3}$$

where

$$\mathcal{A}_1(x, y) = \frac{1}{4\pi} \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|x - y|}, \quad \mathcal{A}_2(x, y) = \frac{1}{4\pi} \sum_{k=1}^N \delta_k(y) \frac{1}{|x - Y_k|}. \tag{3.4}$$

We show that $h_0(x) \in C^{1,\omega}(\Gamma)$ with $\omega \in (0, 1)$. Indeed, the second integral term in the right part of (3.3) is infinitely differentiable in x since it does not have any singularity at $x = y$. According to [2, Theorem 2.15], the first integral term in the right-hand side of (3.3) is a Hölder function in x on Γ . It follows from the identity (3.3) for the function $h_0(x)$ that $h_0(x)$ is also Hölder function in x on Γ . Using [2, Theorem 2.22], we obtain that the first integral term in (3.3) belongs to $C^{1,\omega}(\Gamma)$ in x for some $\omega \in (0, 1)$. Now from the identity (3.3) for the function $h_0(x)$ we get: $h_0(x) \in C^{1,\omega}(\Gamma)$, $\omega \in (0, 1)$. So, we complete the proof.

LEMMA 3.1. *Any solution of the homogeneous equation (3.1) (with $g \equiv 0$) in $C^0(\Gamma)$ automatically belongs to $C^{1,\omega}(\Gamma)$, where $\omega \in (0, 1)$.*

Consider a harmonic function in \mathcal{D} constructed with the help of $h_0(x)$ in the form of a sum of double layer potential and point sources placed inside surfaces Γ_k in the points Y_k ,

$$v[h_0](x) = -\frac{1}{4\pi} \int_{\Gamma} h_0(y) \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|x-y|} dS_y - \frac{1}{4\pi} \sum_{k=1}^N \int_{\Gamma_k} h_0(y) dS_y \frac{1}{|x-Y_k|}. \quad (3.5)$$

This function belongs to $C^0(\bar{\mathcal{D}}) \cap C^2(\mathcal{D})$ and meets all conditions of the homogeneous Dirichlet problem

$$\Delta v(x) = 0, \quad x \in \mathcal{D}; \quad v|_{\Gamma} = 0. \quad (3.6)$$

Besides, if \mathcal{D} is an exterior domain, then

$$|v(x)| = O(|x|^{-1}), \quad \text{as } |x| \rightarrow \infty. \quad (3.7)$$

Note that when satisfying the boundary condition $v|_{\Gamma} = 0$, we obtain identity (3.3). According to the maximum principle for harmonic functions, the solution of this Dirichlet problem is unique. Consequently,

$$v[h_0](x) \equiv 0, \quad x \in \bar{\mathcal{D}}. \quad (3.8)$$

By \mathcal{D}_j we denote the interior simply connected domain, bounded by the surface Γ_j ($j = 1, \dots, N$). Let \mathcal{F} be simple smooth closed surface, lying in the domain \mathcal{D} and enveloping only the surface Γ_j among all surfaces making up Γ . Assuming that \mathbf{n}_x is continuous normal on the surface \mathcal{F} , we compute

$$\begin{aligned} \int_{\mathcal{F}} \frac{\partial v[h_0](x)}{\partial \mathbf{n}_x} dS_x = 0 = & - \int_{\Gamma_j} h_0(y) \frac{\partial}{\partial \mathbf{n}_y} \left(\frac{1}{4\pi} \int_{\mathcal{F}} \frac{\partial}{\partial \mathbf{n}_x} \frac{1}{|x-y|} dS_x \right) dS_y, \\ & - \int_{\Gamma_j} h_0(y) dS_y \frac{1}{4\pi} \int_{\mathcal{F}} \frac{\partial}{\partial \mathbf{n}_x} \frac{1}{|x-Y_j|} dS_x. \end{aligned} \quad (3.9)$$

Here we take into account the following property of harmonic functions: if the function $V(x)$ is harmonic inside the surface \mathcal{F} and $\nabla V(x)$ can be continuously extended on \mathcal{F} from inside, then

$$\int_{\mathcal{F}} \frac{\partial V}{\partial \mathbf{n}_x} dS_x = 0. \quad (3.10)$$

We keep in mind that thanks to the Ostrogradsky-Gauss formula

$$\frac{1}{4\pi} \int_{\mathcal{F}} \frac{\partial}{\partial \mathbf{n}_x} \frac{1}{|x-y|} dS_x = \frac{1}{4\pi} \int_{\mathcal{F}} \frac{\partial}{\partial \mathbf{n}_x} \frac{1}{|x-Y_j|} dS_x = \pm 1, \quad (3.11)$$

where the sign depends on the selection of the direction of the normal \mathbf{n}_x on the surface \mathcal{F} . Hence,

$$\int_{\mathcal{F}} \frac{\partial v[h_0](x)}{\partial \mathbf{n}_x} dS_x = \mp \int_{\Gamma_j} h_0(y) dS_y = 0. \quad (3.12)$$

So, we obtain the identity

$$\int_{\Gamma_j} h_0(\mathcal{Y}) dS_{\mathcal{Y}} = 0, \tag{3.13}$$

which guarantees that the function $v[h_0](x)$ does not have a point source inside the surface Γ_j . Therefore $v[h_0](x)$ is a harmonic function in \mathcal{D}_j . Proceeding from [Lemma 3.1](#), $h_0(x) \in C^{1,\omega}(\Gamma)$. As shown in [2, Theorem 2.23] under such density $h_0(x)$ the derivatives of the double layer potential in $v[h_0](x)$ are continuously extended on Γ_j from inside and outside. Moreover,

$$\lim_{\substack{x^1 \rightarrow x \in \Gamma_j \\ x^1 \in \mathcal{D}}} \frac{\partial v[h_0](x^1)}{\partial \mathbf{n}_x} = \lim_{\substack{x^1 \rightarrow x \in \Gamma_j \\ x^1 \in \mathcal{D}_j}} \frac{\partial v[h_0](x^1)}{\partial \mathbf{n}_x} = 0, \tag{3.14}$$

where (3.8) has been used. Clearly, the function $v[h_0](x)$ satisfies the following Neumann problem in \mathcal{D}_j :

$$\Delta v(x) = 0, \quad x \in \mathcal{D}_j; \quad \left. \frac{\partial v}{\partial \mathbf{n}_x} \right|_{\Gamma_j} = 0. \tag{3.15}$$

It is well known [32] that the solution of such problem is a constant, that is,

$$v[h_0](x) \equiv \text{const} = c, \quad x \in \mathcal{D}_j. \tag{3.16}$$

Employing the theorem on the jump of the double layer potential and using (3.8), we obtain

$$\lim_{\substack{x^1 \rightarrow x \in \Gamma_j \\ x^1 \in \mathcal{D}}} v[h_0](x^1) - \lim_{\substack{x^1 \rightarrow x \in \Gamma_j \\ x \in \mathcal{D}_j}} v[h_0](x^1) = -h_0(x) = -c, \tag{3.17}$$

so that $h_0(x) = c$ on Γ_j . It follows from (3.13), that $c = 0$. Therefore $h_0(x) \equiv 0$ on Γ_j . Recall that Γ_j is an arbitrary closed surface ($j = 1, \dots, N$), belonging to Γ . Hence, $h_0(x) \equiv 0$ on $\Gamma_1, \dots, \Gamma_N$. Thus, $h_0(x) \equiv 0$ on Γ if \mathcal{D} is an exterior domain and $h_0(x) \equiv 0$ on $\Gamma \setminus \Gamma_0$ if \mathcal{D} is an interior domain. Finally, we should clarify the value of $h_0(x)$ on Γ_0 in case of interior domain \mathcal{D} .

Let \mathcal{D} be an interior domain, then $h_0(x) \equiv 0$ on $\Gamma \setminus \Gamma_0$, and the function $v[h_0](x)$ take the form of double layer potential

$$v[h_0](x) = -\frac{1}{4\pi} \int_{\Gamma_0} h_0(\mathcal{Y}) \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|x - \mathcal{Y}|} dS_{\mathcal{Y}}. \tag{3.18}$$

Let \mathcal{D}_0 be an exterior domain, bounded by the surface Γ_0 . The potential $v[h_0](x)$ is harmonic in \mathcal{D}_0 and $v(x) = O(|x|^{-1})$, $\nabla v(x) = O(|x|^{-2})$ as $|x| \rightarrow \infty$. Taking into account smoothness of $h_0(x)$ on Γ , ensured by the [Lemma 3.1](#) and employing [22, Theorem 2.23], we obtain

$$\lim_{\substack{x^1 \rightarrow x \in \Gamma_0 \\ x^1 \in \mathcal{D}}} \frac{\partial v[h_0](x^1)}{\partial \mathbf{n}_x} = \lim_{\substack{x^1 \rightarrow x \in \Gamma_0 \\ x^1 \in \mathcal{D}_0}} \frac{\partial v[h_0](x^1)}{\partial \mathbf{n}_x} = 0, \tag{3.19}$$

where (3.8) has been used. Consequently, the function $v[h_0](x)$ satisfies the following Neumann problem in \mathcal{D}_0 :

$$\begin{aligned} \Delta v(x) &= 0, \quad x \in \mathcal{D}_0; & \frac{\partial v}{\partial \mathbf{n}_x} \Big|_{\Gamma_0} &= 0; \\ v(x) &= O(|x|^{-1}), \quad \nabla v(x) = O(|x|^{-2}), \quad |x| \rightarrow \infty. \end{aligned} \tag{3.20}$$

It is known that (see [32]) this problem has only the trivial solution: $v[h_0](x) \equiv 0$ in \mathcal{D}_0 . Using the theorem on the jump of the double layer potential [32] and using (3.8), we obtain

$$\lim_{\substack{x^1 \rightarrow x \in \Gamma_0 \\ x^1 \in \mathcal{D}}} v[h_0](x^1) - \lim_{\substack{x^1 \rightarrow x \in \Gamma_0 \\ x^1 \in \mathcal{D}_0}} v[h_0](x^1) = -h_0(x) = 0, \tag{3.21}$$

so that $h_0(x) \equiv 0$ on Γ_0 , if $\Gamma_0 \subset \Gamma$. On the basis of previous results $h_0(x) \equiv 0$ on Γ in case of both interior and exterior domain \mathcal{D} .

Thereby we arrive at the contradiction to the assumption that $h_0(x)$ is a nontrivial solution of the homogeneous equation (3.1) (with $g \equiv 0$). Thus, the homogeneous equation (3.1) has only the trivial solution. Since Fredholm alternative holds for (3.1), we arrive at the following statement.

LEMMA 3.2. *Equation (3.1) is uniquely solvable in $C^0(\Gamma)$ for any right-hand side $g(x) \in C^0(\Gamma)$.*

The inverse operator to $(-1/2 - A)$ is denoted by $(-1/2 - A)^{-1}$, then the solution of (3.1) can be expressed in the form

$$h(x) = \left(\left(-\frac{1}{2} - A \right)^{-1} g \right) (x). \tag{3.22}$$

Let $g(t, x) \in C^0([0, \infty); C^0(\Gamma))$. Consider an analogue of (3.1) with respect to the function $h(t, x) \in C^0([0, \infty); C^0(\Gamma))$,

$$-\frac{1}{2}h(t, x) - (Ah)(t, x) = g(t, x). \tag{3.23}$$

For any fixed t , the operator $(-1/2 - A)$ is invertible,

$$h(t, x) = \left(\left(-\frac{1}{2} - A \right)^{-1} g \right) (t, x), \tag{3.24}$$

and for any t , $h(t, x) \in C^0(\Gamma)$.

Since the operator $(-1/2 - A)$ is linear and bounded in $C^0(\Gamma)$ (see item (2) of Lemma 2.4), then thanks to Banach theorem [31, page 59], the linear operator $(-1/2 - A)^{-1}$ is also bounded in $C^0(\Gamma)$. Therefore, for any $t \geq 0$,

$$\|h(t, \cdot)\|_{C^0(\Gamma)} \leq c_A \|g(t, \cdot)\|_{C^0(\Gamma)}, \tag{3.25}$$

where c_A is a constant. From the boundedness of the operator $(-1/2 - A)^{-1}$ it follows its continuity. This means that if

$$\|g(t, \cdot) - g(t_0, \cdot)\|_{C^0(\Gamma)} \rightarrow 0, \quad t \rightarrow t_0, \tag{3.26}$$

then for any functions $h(t, x)$, $h(t_0, x)$, constructed on the basis of formula (3.24), the following relationship holds

$$\|h(t, \cdot) - h(t_0, \cdot)\|_{C^0(\Gamma)} \rightarrow 0, \quad t \rightarrow t_0. \tag{3.27}$$

Hence $h(t, x) \in C^0([0, \infty); C^0(\Gamma))$. Thereby the lemma holds.

LEMMA 3.3. *If $g(t, x) \in C^0([0, \infty); C^0(\Gamma))$, then (3.23) has the unique solution $h(t, x)$ of class $C^0([0, \infty); C^0(\Gamma))$, and this solution obeys estimate (3.25).*

4. Solution of the integral equation and the existence theorem. We seek the solution of the integral equation (2.13) in the form of the series [10],

$$\mu(t, x) = \sum_{m=0}^{\infty} \mu_m(t, x), \tag{4.1}$$

where the functions $\mu_m(t, x) \in C^0([0, \infty); C^0(\Gamma))$ are subsequently defined from the equalities, which are obtained when substituting (4.1) into (2.13) formally

$$-\frac{1}{2}\mu_0(t, x) - (A\mu_0)(t, x) = f(t, x), \tag{4.2}$$

$$-\frac{1}{2}\mu_m(t, x) - (A\mu_m)(t, x) = -(B\mu_{m-1})(t, x), \quad m = 1, 2, \dots \tag{4.3}$$

Since operator A does not depend on t , this variable in (4.2) and (4.3) is a parameter.

Since the function $f(t, x) \in C^0([0, \infty); C^0(\Gamma))$, then according to Lemma 3.3, equation (4.2) has the unique solution $\mu_0(t, x) \in C^0([0, \infty); C^0(\Gamma))$, which satisfies the following inequality

$$\|\mu_0(t, \cdot)\|_{C^0(\Gamma)} \leq c_A \|f(t, \cdot)\|_{C^0(\Gamma)} \leq c_A \|f\|_{C^0([0, t]; C^0(\Gamma))}. \tag{4.4}$$

The norm in the Banach space $C^0([0, T]; C^0(\Gamma))$ is defined by

$$\|\cdot\|_{C^0([0, T]; C^0(\Gamma))} = \|\|\cdot\|_{C^0(\Gamma)}\|_{C^0[0, T]}. \tag{4.5}$$

Substituting $\mu_0(t, x)$ found above into the right side of (4.3), we obtain the equation with respect to $\mu_1(t, x)$:

$$-\frac{1}{2}\mu_1(t, x) - (A\mu_1)(t, x) = -(B\mu_0)(t, x). \tag{4.6}$$

Using item (3) of Lemma 2.4, $(B\mu_0)(t, x) \in C^0([0, \infty); C^0(\Gamma))$.

According to Lemma 3.3, equation (4.6) has the unique solution

$$\mu_1(t, x) \in C^0([0, \infty); C^0(\Gamma)). \tag{4.7}$$

Besides, on the basis of Lemmas 2.4 and 3.3, the function $\mu_1(t, x)$ satisfies the estimate

$$\begin{aligned} \|\mu_1(t, \cdot)\|_{C^0(\Gamma)} &\leq c_A \|(B\mu_0)(t, \cdot)\|_{C^0(\Gamma)} \\ &\leq c_A c_B \left[\int_0^t \|\mu_0(\tau, \cdot)\|_{C^0(\Gamma)} d\tau + \int_0^t \int_0^{t_1} \|\mu_0(\tau, \cdot)\|_{C^0(\Gamma)} d\tau dt_1 \right]. \end{aligned} \tag{4.8}$$

We continue the proof by induction. Let $\mu_{m-1}(t, x) \in C^0([0, \infty); C^0(\Gamma))$, then $\mu_m(t, x)$ must satisfy (4.3). On the basis of Lemma 2.4,

$$(B\mu_{m-1})(t, x) \in C^0([0, \infty); C^0(\Gamma)). \tag{4.9}$$

Using Lemma 3.3, equation (4.3) has the unique solution $\mu_m(t, x) \in C^0([0, \infty); C^0(\Gamma))$. Using Lemmas 2.4 and 3.3 this solution satisfies the estimate

$$\begin{aligned} \|\mu_m(t, \cdot)\|_{C^0(\Gamma)} &\leq c_A c_B \left[\int_0^t \|\mu_{m-1}(\tau, \cdot)\|_{C^0(\Gamma)} d\tau + \int_0^t \int_0^{\tau_1} \|\mu_{m-1}(\tau, \cdot)\|_{C^0(\Gamma)} d\tau dt_1 \right] \\ &\leq c_A^2 c_B^2 \left[\int_0^t \int_0^{\tau_1} \|\mu_{m-2}(\tau, \cdot)\|_{C^0(\Gamma)} d\tau dt_1 \right. \\ &\quad \left. + \int_0^t \int_0^{\tau_1} \int_0^{\tau_2} \int_0^{\tau_3} \|\mu_{m-2}(\tau, \cdot)\|_{C^0(\Gamma)} d\tau dt_3 dt_2 dt_1 \right] \\ &\leq \dots \\ &\leq c_A^m c_B^m \left[\int_0^t \int_0^{\tau_1} \dots \int_0^{\tau_{m-1}} \|\mu_0(\tau, \cdot)\|_{C^0(\Gamma)} d\tau dt_{m-1} \dots dt_1 \right. \\ &\quad \left. + \int_0^t \int_0^{\tau_1} \dots \int_0^{\tau_{2m-1}} \|\mu_0(\tau, \cdot)\|_{C^0(\Gamma)} d\tau dt_{2m-1} \dots dt_1 \right] \\ &\leq c_A^{m+1} c_B^m \|f\|_{C^0([0,t]; C^0(\Gamma))} \left[\int_0^t \int_0^{\tau_1} \dots \int_0^{\tau_{m-1}} d\tau dt_{m-1} \dots dt_1 \right. \\ &\quad \left. + \int_0^t \int_0^{\tau_1} \dots \int_0^{\tau_{2m-1}} d\tau dt_{2m-1} \dots dt_1 \right] \\ &= c_A \|f\|_{C^0([0,t]; C^0(\Gamma))} \left(\frac{(c_B c_A t)^m}{m!} + \frac{(\sqrt{c_B c_A t})^{2m}}{(2m)!} \right), \end{aligned} \tag{4.10}$$

where the inequality (4.4) has been used.

Thus, all functions $\mu_m(t, x)$, making up series (4.1), are found subsequently. On the basis of Lemma 2.1 each function $\mu_m(t, x)$ belongs to $C^0([0, \infty) \times \Gamma)$. Besides, each function meets inequality (4.10).

Let us show that series (4.1) converges absolutely and uniformly in $[0, T] \times \Gamma$ for any $T > 0$. To prove this we use the Weierstrass theorem [12] and majorize the terms of series (4.1) by terms of the convergent number series. Employing inequality (4.10), we obtain that for any $(t, x) \in [0, T] \times \Gamma$ the following inequality holds

$$\begin{aligned} |\mu_m(t, x)| &\leq \|\mu_m(t, \cdot)\|_{C^0(\Gamma)} \leq \sup_{t \in [0, T]} \|\mu_m(t, \cdot)\|_{C^0(\Gamma)} = \|\mu_m\|_{C^0([0, T]; C^0(\Gamma))} \\ &\leq c_A \left[\frac{(c_A c_B T)^m}{m!} + \frac{(\sqrt{c_A c_B T})^{2m}}{(2m)!} \right] \times \|f\|_{C^0([0, T]; C^0(\Gamma))}. \end{aligned} \tag{4.11}$$

Thereby any term of the series (4.1) is majorized on the set $[0, T] \times \Gamma$ by the term of the number series

$$c_A \|f\|_{C^0([0, T]; C^0(\Gamma))} \left[\frac{(c_A c_B T)^m}{m!} + \frac{(\sqrt{c_A c_B T})^{2m}}{(2m)!} \right], \tag{4.12}$$

which converges and its sum is equal to

$$c_A \|f\|_{C^0([0,T];C^0(\Gamma))} [\exp(c_{AC_B} T) + \text{ch}(\sqrt{c_{AC_B} T})]. \quad (4.13)$$

Consequently, using the Weierstrass theorem, series (4.1) converges absolutely and uniformly on the set $[0, T] \times \Gamma$ for any $T > 0$. These arguments and continuity of the terms of the series prove that the series can be integrated term-by-term when substituting this series in (2.13). Moreover, the integrated series is also absolutely and uniformly convergent [12] on $[0, T] \times \Gamma$ for any $T > 0$, and so its terms can be summed in an arbitrary order. Using the subsequence of equalities (4.2) and (4.3), equation (2.13) becomes an identity. Thereby the function $\mu(t, x)$, defined as a sum of series (4.1), obeys (2.13), because all formal operations used in the substitution of (4.1) into (2.13) are justified.

According to the theorem on continuity of the sum of a series [12], from uniform convergence of series (4.1) in $[0, T] \times \Gamma$ for any $T > 0$, and from continuity of its terms in $[0, \infty) \times \Gamma$ it follows that $\mu(t, x) \in C^0([0, \infty) \times \Gamma)$. By Lemma 2.1, $\mu(t, x) \in C^0([0, \infty); C^0(\Gamma))$. Thus, the desired solution of (2.13) has been constructed.

THEOREM 4.1. *For any function $f(t, x) \in C^0([0, \infty); C^0(\Gamma))$ equation (2.13) has a solution $\mu(t, x) \in C^0([0, \infty); C^0(\Gamma))$.*

Note that a function $f(t, x) = [f_0(t, x)]_{tt} \in C^0([0, \infty); C^0(\Gamma))$ in a right-hand side of the integral equation (2.13) corresponds to an arbitrary function $f_0(t, x) \in C_0^2([0, \infty); C^0(\Gamma))$ in the boundary condition (1.3) of the problem \mathcal{H} .

Theorem 4.1 ensures the existence of the solution $\mu(t, x)$ of the integral equation (2.13) in the required class of smoothness. Hence, the potential (2.12), constructed for this function $\mu(t, x)$, satisfies the boundary condition of the problem \mathcal{H} . Besides, according to Lemma 2.3, the potential (2.12) constructed in such a way belongs to the smoothness class G and satisfies all other conditions of the problem \mathcal{H} . We arrive at the following theorem.

THEOREM 4.2. *For any function $f_0(t, x) \in C_0^2([0, \infty); C^0(\Gamma))$ the classical solution of the problem \mathcal{H} exists and is given by formula (2.12), where the density $\mu(t, x) \in C^0([0, \infty); C^0(\Gamma))$ is the solution of the integral equation (2.13), which is always solvable.*

One can show that if $f_0(t, x) \in C_0^2([0, \infty); C^{1,\lambda}(\Gamma))$, $\lambda \in (0, 1]$, then the solution of the problem \mathcal{H} constructed in Theorem 4.2 belongs to the smoothness class G_1 . In this case, the solution of problem \mathcal{H} is unique, by Theorem 1.5.

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PAVEL A. KRUTITSKII: DEPARTMENT OF MATHEMATICS, FACULTY OF PHYSICS, MOSCOW STATE UNIVERSITY, MOSCOW 117234, RUSSIA

E-mail address: krutitsk@math.phys.msu.su