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SOME REMARKS ON THE SEVERI VARIETIES OF SURFACES IN $\mathbb{P}^3$

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Abstract. Continuing the work of Chiantini and Ciliberto (1999) on the Severi varieties of curves on surfaces in $\mathbb{P}^3$, we complete the proof of the existence of regular components for such varieties.

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1. Introduction. Let $\mathscr{L}$ be a line bundle on a smooth surface S. Fix a positive integer δ and the Severi variety $V^0_{\mathscr{L},\delta}$ is defined as the subscheme of $|\mathscr{L}|$ consisting of irreducible curves in $|\mathscr{L}|$ with δ nodes and no other singularities. This scheme, or sometimes, its closure $V_{\mathscr{L},\delta}$ in $|\mathscr{L}|$ is called the Severi variety of curves with δ nodes in $|\mathscr{L}|$. The Severi variety $V^0_{\mathscr{L},\delta}$, if not empty, has the dimension at least $\max\{\dim|\mathscr{L}|-\delta,-1\}$, where $\max\{\dim|\mathscr{L}|-\delta,-1\}$ is called the expected dimension of $V^0_{\mathscr{L},\delta}$. A reduced component of $V^0_{\mathscr{L},\delta}$ that has the expected dimension is called a regular component of $V^0_{\mathscr{L},\delta}$.

The Severi varieties of plane curves have been classically extensively studied. It is natural to study the Severi varieties on other surfaces. The Severi varieties of surfaces in $\mathbb{P}^3$ were studied in [2]. Let S be a smooth surface of degree d in $\mathbb{P}^3$ and let $|\mathbb{O}_S(n)|$ be the linear series on S cut out by the degree n surfaces in $|P^3|$. We use the notation $V^0_{n,\delta}(S)$ to denote the Severi variety of curves with δ nodes in $|\mathbb{O}_S(n)|$. It was proved in [2] that for S general, $n \ge d$ and all δ with $0 \le \delta \le \dim(|\mathbb{O}_S(n)|)$, $V^0_{n,\delta}(S)$ has at least one component which is reduced of the expected dimension. That, combined with a construction of irregular components of $V^0_{n,\delta}(S)$ for $n \gg d \ge 8$ and some δ, gives examples of reducible Severi varieties on surfaces in $\mathbb{P}^3$.

The purpose of this paper is to prove the main theorem in [2] in the case $n < d$, which was not covered in that paper. Namely, we show the following theorem.

Theorem 1.1. *For a general surface S of degree d in $\mathbb{P}^3$, $n < d$ and all δ with $0 \le \delta \le \dim(|\mathbb{O}_S(n)|) = \binom{n+3}{3} - 1$, the Severi variety $V^0_{n,\delta}(S)$ has at least one regular component.*

2. Proof of Theorem 1.1

2.1. Sketch of the proof. Our construction of the regular component of $V^0_{n,\delta}(S)$ for $n < d$ is very close to that of Chiantini and Ciliberto in spirit: we take a degeneration of degree d surfaces in $\mathbb{P}^3$ and try to locate a subscheme of $|\mathbb{O}_{S_0}(n)|$ on the "degenerated" degree d surface S_0 which is the limit of some components of the Severi varieties on the general fibers; we are done as long as the subscheme we find has the expected dimension of the Severi varieties on the general fibers. However, both the way we degenerate the surfaces and the way we locate those "limiting nodal curves" are different from those used in [2].

Let $X \subset \mathbb{P}^3 \times \Delta$ be a pencil of surfaces of degree d in $\mathbb{P}^3$ whose central fiber $X_0 = H_1 \cup H_2 \cup \cdots \cup H_d$ is a union of d hyperplanes. Suppose that $H_1, H_2, \ldots, H_d$ are in relatively general position, so no four of them meet at a point. Suppose that the pencil X is chosen to be general so that it has $d\binom{d}{2}$ distinct base points and none of them is the intersection of three planes among $H_1, H_2, \ldots, H_d$. Then the threefold X has exactly $d\binom{d}{2}$ rational double points, which are the base points of the pencil, and no other singularities. These double points lie on the lines $L_{ij} = H_i \cap H_j$ for $i \neq j$ and $1 \leq i, j \leq d$.

There are various ways we may take the limit of the linear series $|\mathbb{O}_{X_t}(n)|$ as $t \to 0$. Here we just let $\lim_{t\to 0} |\mathbb{O}_{X_t}(n)| = |\mathbb{O}_{X_0}(n)|$. Let $C = \cup_{i=1}^d C_i$ be a curve in $|\mathbb{O}_{X_0}(n)|$, where $C_i \subset H_i$ for $i = 1, 2, \ldots, d$. If C passes through a double point p of X on the line L_{ij}, and C_i and C_j meet L_{ij} transversely at p on H_i and H_j, respectively, then C can be deformed to a curve $C_t \in |\mathbb{O}_{X_t}(n)|$ with p deforming to a node of C_t (cf. [1, Theorem 2.2]). So, to produce δ nodes of C_t, it suffices to let C pass through δ double points of X. Notice that we have enough double points to play with since $\delta \leq \binom{n+3}{3} - 1 \leq d\binom{d}{2}$ for $d \geq 4$ and $d > n$. So pick a collection Z of δ double points of X and there always exists $C \in |\mathbb{O}_{X_0}(n)|$ passing through Z. But the catch here is that we have to make sure of the following things.

(1) Most important of all, we have to choose Z such that Z imposes independent conditions on $|\mathbb{O}_{X_0}(n)|$, or equivalently, on $|\mathbb{O}_{\mathbb{P}^3}(n)|$.

(2) A general member $C \in |\mathbb{O}_{X_0}(n)|$ that passes through Z must be cut out by an irreducible surface S of degree n in $\mathbb{P}^3$.

(3) For a general member $C = \cup_{i=1}^d C_i \in |\mathbb{O}_{X_0}(n)|$ that passes through Z, each C_i must meet L_{ij} transversely at points in $Z \cap L_{ij}$.

We are not sure whether part (3) is essential. It is conceivable that even if C_i fails to meet L_{ij} transversely at a point $p \in Z \cap L_{ij}$, it is still possible to deform C to C_t with p becoming a node of C_t. However, we will verify (3) anyway since it is quite trivial by our construction.

2.2. Proof of Theorem 1.1 when $n < d - 1$. First, we look at when $Z \cap H_i$ imposes independent conditions on $|\mathbb{O}_{H_i}(n)|$. In general, we may ask the following question.

QUESTION 2.1. Let $L_1, L_2, \ldots, L_p$ be p lines on $\mathbb{P}^2$ and let Z be a zero-dimensional subscheme of $\mathbb{P}^2$ which consists of a_1 points on L_1, a_2 points on $L_2, \ldots$, and a_p points on L_p with none of these points being the intersection of two lines among $L_1, L_2, \ldots, L_p$. Fix $n < p$, we ask that under what conditions does Z impose independent conditions on the linear series $|\mathbb{O}(n)|$?

To be more specific, if Z imposes independent conditions on $|\mathbb{O}(n)|$, then

(1) What kind of numerical conditions should be satisfied by $a_1, a_2, \ldots, a_p$?

(2) Should $L_1, L_2, \ldots, L_p$ be in general position or could they be arbitrary?

(3) Should the points $Z \cap L_k$ be in general position on L_k or could they be arbitrary for $k = 1, 2, \ldots, p$?

For example, it is not hard to obtain a necessary numerical condition on $a_1, a_2, \dots, a_p$ by noticing that the linear series cut out by $|\mathcal{O}(n)|$ on a union of k lines has dimension $nk - (k^2 - 3k)/2 - 1$ for $k \le n$. So $a_1, a_2, \dots, a_p$ have to satisfy the following condition. For each $k \le n$ and each $I \subset \{1, 2, \dots, p\}$ with $\#I = k$,

$$\sum_{i \in I} a_i \le nk - \frac{k^2 - 3k}{2} - 1. \tag{2.1}$$

Although we do not have the answer for Question 2.1 in general, we can prove a very special case which suffices for our purpose.

Lemma 2.2. *Let $L_1, L_2, \dots, L_{n+1}$ be $n+1$ distinct lines on $\mathbb{P}^2$ and let Z be a zero-dimensional subscheme of $\mathbb{P}^2$ which consists of a_1 points on L_1, a_2 points on $L_2, \dots,$ and a_{n+1} points on L_{n+1} with none of these points being the intersection of two lines among $L_1, L_2, \dots, L_{n+1}$. Suppose that $a_k \le n + 2 - k$ for $k = 1, 2, \dots, n+1$. Then Z imposes independent conditions on $|\mathcal{O}(n)|$, that is, $H^1(I_Z(n)) = 0$.*

Proof. It suffices to show that there is no curve in $|\mathcal{O}(n)|$ passing through Z if $a_k = n + 2 - k$ for $k = 1, 2, \dots, n+1$.

Suppose that $a_k = n + 2 - k$ for $k = 1, 2, \dots, n+1$ and there exists a curve C of degree n passing through Z. Since C passes through $n+1$ points on L_1 and $\deg C = n$, $L_1 \subset C$. Let $C = L_1 \cup C'$. Now C' passes through n points on L_2 and $\deg C' = n - 1$. So $L_2 \subset C$. So inductively we have $L_k \subset C$ for $k = 1, 2, \dots, n+1$. Obviously, this is impossible because $\deg C = n$. □

Lemma 2.3. *Let $H_1, H_2, \dots, H_{n+2}$ be $n+2$ distinct planes in $\mathbb{P}^3$ with no three of them meeting along a line and let $L_{ij} = H_i \cap H_j$ for $1 \le i < j \le n+2$. Let Z be a zero-dimensional subscheme of $\mathbb{P}^3$ which consists of a_{ij} points on L_{ij} for $1 \le i < j \le n+2$. Suppose that $a_{ij} \le n + 3 - j$ for $1 \le i < j \le n+2$. Then Z imposes independent conditions on $|\mathcal{O}(n)|$.*

Proof. It suffices to show that there are no surfaces in $|\mathcal{O}(n)|$ passing through Z if $a_{ij} = n + 3 - j$ for $1 \le i < j \le n+2$.

Suppose that $a_{ij} = n + 3 - j$ for $1 \le i < j \le n+2$ and there is a degree n surface S in $\mathbb{P}^3$ passing through Z. First, we claim that $H_1 \subset S$. If not, $C = S \cap H_1$ is a degree n curve in H_1 passing through $Z \cap H_1$. Notice that $Z \cap H_1$ is a zero-dimensional subscheme of H_1 consisting of $n+1$ points on L_{12}, n points on $L_{13}, \dots,$ and 1 point on $L_{1,n+2}$. So by Lemma 2.2, there is no degree n curve passing through $Z \cap H_1$. Contradiction. So $H_1 \subset S$ and let $S = H_1 \cup S'$. Next, we claim that $H_2 \subset S'$. If not, $C' = S' \cap H_2$ is a degree $n-1$ curve passing through $(Z \cap H_2) \backslash L_{12}$. Notice that $(Z \cap H_2) \backslash L_{12}$ is a zero-dimensional subscheme of H_2 consisting of n points on L_{23}, $n-1$ points on $L_{24}, \dots,$ and 1 point on $L_{2,n+2}$. So by Lemma 2.2, there is no degree $n-1$ curve passing through $(Z \cap H_2) \backslash L_{12}$. Contradiction. So we may carry on this line of argument and finally conclude that $H_1 \cup H_2 \cup \cdots \cup H_{n+1} \subset S$. This is impossible since $\deg S = n$. □

Now, we go back to the proof of Theorem 1.1 in the case of $n < d - 1$. It suffices to prove Theorem 1.1 when $\delta = \binom{n+3}{3} - 1$. As in the sketch of the proof, we need to find a collection Z of δ double points of the family X such that Z imposes independent conditions on $|\mathcal{O}_{\mathbb{P}^3}(n)|$.

The family X has $d\binom{d}{2}$ double points with d points on each L_{ij}. Let Z' be a collection of $\binom{n+3}{3}-1$ double points of X with exactly $n+3-j$ on each L_{ij} for $1\le i<j\le n+2$, and let Z be a subset of Z' with one point removed on L_{12}.

By Lemma 2.3, Z imposes δ independent conditions on $|\mathcal{O}_{\mathbb{P}^3}(n)|$. So $\dim|I_Z(n)|=0$. Of course, $|I_Z(n)|$ consists of exactly one surface S.

Next, we need to verify that S is irreducible for a general choice of X. Choose an irreducible surface $S\in|\mathcal{O}_{\mathbb{P}^3}(n)|$ and a surface $S'\in|\mathcal{O}_{\mathbb{P}^3}(d-n)|$. Let X be the pencil of degree d surfaces given by

$$F_1F_2\cdots F_d+tGG'=0, \tag{2.2}$$

where $F_1,F_2,\dots,F_d$ are the defining equations of $H_1,H_2,\dots,H_d$ and G and G' are the defining equations of S and S'. Obviously, we may choose Z to consist of points in $S\cap L_{ij}$ for $1\le i<j\le n+2$. Then $S\in|I_Z(n)|$ and it is irreducible.

Finally, we need to verify that S meets L_{ij} transversely at points in $(Z\cap L_{ij})$ for a general choice of X, which is quite obvious with the construction of X in (2.2). This concludes the proof of Theorem 1.1 in the case of $n<d-1$.

2.3. Proof of Theorem 1.1 when $n=d-1$. The proof of Theorem 1.1 in the case of $n=d-1$ is a little more involved. Again it suffices to prove the theorem when $\delta=\binom{d+2}{3}-1$.

Let Z' be a collection of $\binom{d+2}{3}-d$ double points of X with exactly $d+2-j$ on each L_{ij} for $1\le i<j\le d$ and let Z be a subset of Z' with one point removed on L_{12}.

By Lemma 2.3, Z imposes independent conditions on $|\mathcal{O}_{\mathbb{P}^3}(d-1)|$. So $\dim|I_Z(d-1)|=d$. Actually, we can very explicitly write down a basis for $H^0(I_Z(d-1))$ as follows.

Since Z' also imposes independent conditions on $|\mathcal{O}_{\mathbb{P}^3}(d-1)|$, there exists $G\in H^0(I_Z(d-1))$ such that the surface defined by $G=0$ does not contain the line L_{12}. On the other hand, for each $1\le i\le d$, $\prod_{j\ne i}F_j$ belongs to $H^0(I_Z(d-1))$, where $F_1,F_2,\dots,F_d$ are the defining equations of $H_1,H_2,\dots,H_d$, respectively. It is easy to see that G and $\prod_{j\ne i}F_j$ $(i=1,2,\dots,d)$ are linearly independent and hence they span the vector space $H^0(I_Z(d-1))$.

So the restriction $|I_{Z\cap H_i}\otimes\mathcal{O}_{H_i}(d-1)|$ of $|I_Z(d-1)|$ to H_i is a pencil of degree $d-1$ curves generated by $G|_{H_i}$ and $(\prod_{j\ne i}F_j)|_{H_i}$. By the same argument as before, we can show that G defines a general surface of degree $d-1$ for a general choice of X. So the pencil $|I_{Z\cap H_i}\otimes\mathcal{O}_{H_i}(d-1)|$ contains an isolated irreducible curve C_i with exactly one node. So $C=\cup_{i=1}^d C_i\in|\mathcal{O}_{X_0}(d-1)|$ has d nodes, passes through $\binom{d+1}{3}-d-1$ double points of X, and is isolated and cut out by an irreducible surface of degree $d-1$. Therefore, C is the flat limit of an isolated irreducible curve on X_t with exactly $\delta=\binom{d+2}{3}-1$ nodes. This concludes the proof of Theorem 1.1 when $n=d-1$.

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