IJMMS 26:10 (2001) 643–648
PII. S0161171201006317
http://ijmms.hindawi.com

STABILITY IN E-CONVEX PROGRAMMING

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(Received 12 April 2000)

ABSTRACT. We define and analyze two kinds of stability in E-convex programming problem in which the feasible domain is affected by an operator E. The first kind of this stability is that the set of all operators E that make an optimal set stable while the other kind is that the set of all operators E that make certain side of the feasible domain still active.

2000 Mathematics Subject Classification. 90Cxx.

1. Introduction. The stability notion plays an important role in the mathematical programming field, it is important for solver or for the decision maker to preserve effort and time.

Stability in mathematical programming has many types, one of these types depends on making perturbation to the decision space or to the objective space or to both by a parameter. This type is called stability in parametric programming problems (see [3, 4, 5]). Other types are called internal and external stability. This kind of stability depends on the set of efficient solution for multi-objective programming problems, namely, if for each efficient solution there exist some points in decision space dominated by it (this is external stability). On the other hand, internal stability means that any efficient solution is not preferred to another efficient solution, see [6].

In this paper, the author presents a new type of stability in mathematical programming. This type is very important in composite programming [2, 8] and in E-convex programming which was presented by the author in [9]. This kind of stability is called E-stability which is discussed in this paper.

Now, we give some examples to show that there are many operators E that transform an optimal point to another optimal point.

EXAMPLE 1.1. Consider the following problem:

$$\min (x-1)^2+(y-1)^2 \tag{1.1}$$

subject to $x,y \geq 0$. It is clear that the optimal solution is $(\bar{x},\bar{y}) = (1,1)$. Operators $E:\mathbb{R}^2 \to \mathbb{R}^2$, defined as

$$E(x,y) = (2x-1,2y-1), \qquad E(x,y) = (x^2,y^2), \qquad E(x,y) = (xy,y), \tag{1.2}$$

map (1,1) to (1,1).

EXAMPLE 1.2. Consider the problem

$$\max\{x+y\} \tag{1.3}$$

subject to $M = \{(x,y) \in \mathbb{R}^2 : x+y \leq 8,\ x \leq 6,\ y \leq 5,\ x+y \geq 1\}$.

The set of optimal solutions is

$$S = \{(x,y) \in \mathbb{R}^2 : (x,y) = \lambda(3,5) + (1-\lambda)(6,2),\ 0 \le \lambda \le 1\}. \tag{1.4}$$

It is clear that a transformation $E : \mathbb{R}^2 \to \mathbb{R}^2$ which maps (x,y) to $(9-x, 7-y)$ assigns any point in S again to a point in S.

Also, for the solution (3,5) we can find more than one $E : \mathbb{R}^2 \to \mathbb{R}^2$ such that $E(3,5) \in S$, for example

$$\begin{aligned} E(x,y) &= \left(xy - x - y - 1, \frac{1}{4}(x+y)\right), \\ E(x,y) &= \left(\frac{1}{2}x^2,\ y - \frac{3}{2}\right), \\ E(x,y) &= \left(x^2 y - 41,\ \frac{12y^2}{25x}\right). \end{aligned} \tag{1.5}$$

The above two examples lead to the discussion of the set of all these operators qualitatively.

2. E-stability of an optimal set. Consider the following mathematical programming problem:

$$\min f(x) \quad \text{subject to} \quad M = \{x \in \mathbb{R}^n : g_r(x) \le 0,\ r = 1,2,\dots,m\}. \tag{2.1}$$

Denote S the set of optimal solutions for problem (2.1).

Definition 2.1. An E-stability of S is denoted by $D(\bar{x})$, $\bar{x} \in S$ and is defined as

$$D(\bar{x}) = \{E : \mathbb{R}^n \longrightarrow \mathbb{R}^n : E(\bar{x}) \in S\}. \tag{2.2}$$

Let ξ be the space of all operators from $\mathbb{R}^n$ to $\mathbb{R}^n$ then $D(\bar{x})$ is a point-to-set map from S to ξ.

Definition 2.2 (see [1]). A point-to-set map $F : X \to 2^Y$ is said to be closed at a point $x^0 \in X$; if for each pair of sequences $\{x^n\} \subset X$ and $\{y^n\} \subset Y$, $n = 1,2,\dots$ with the properties $x^n \to x^0$, $y^n \in F(x^n)$ and $y^n \to y^0$ it follows that $y^0 \in F(x^0)$.

Proposition 2.3. *If S is closed, then a point-to-set map $D : S \to \xi$ is closed.*

Proof. Let $\{x^t\} \subset S$, $x^t \to x^0$ as $t \to \infty$ and let $\{E^t\} \subset \xi$, $E^t \to E^0$ as $t \to \infty$ such that $E^t \in D(x^t)$, then we have a sequence $f(E^n x^n)$ with $f(E^n x^n) \le f(x)$ for each $x \in M$. Since S is closed, then $f(E^0 x^0) \le f(x)$ for all $x \in M$. Hence $E^0 \in D(x^0)$ and D is closed. □

Proposition 2.4. *If f is lower-semicontinuous, then the set $D(x^0)$, $x^0 \in S$, is closed.*

Proof. Let E^n be a sequence of $D(x^0)$ and $E^n \to E^0$ as $n \to \infty$, then $f(E^n x^0) \le f(x)$ for each $x \in M$. Since f is lower-semicontinuous, then

$$f(E^0 x^0) = f\left(\lim_{n\to\infty} E^n x^0\right) \le \liminf_{n\to\infty} f(E^n x^0) \le f(x) \tag{2.3}$$

for each $x \in M$, that is, $E^0 x^0 \in S$ and $E^0 \in D(x^0)$. Hence $D(x^0)$ is closed. □

Theorem 2.5. *If f is lower-semicontinuous and D is upper-semicontinuous at $x^0 \in S$, then D is closed.*

Proof. From Proposition 2.3, the set $D(x^0)$ is closed and from [1, Lemma 2.2.1] the mapping D is closed. □

Theorem 2.6. *If S is compact, then D is upper-semicontinuous.*

Proof. Since S is compact, then S is closed and, from Proposition 2.3, D is closed. From [1, Lemma 2.2.3] the mapping D is upper-semicontinuous. □

Theorem 2.7. *Let f be linear, S convex, and M^* a dual space of M. Then a point-to-set map $D : S \to M^*$ is convex.*

Proof. Assume that $\bar{x}, \tilde{x} \in S$ and $E \in D(\lambda\bar{x} + (1-\lambda)\tilde{x})$, $0 \le \lambda \le 1$, then $\lambda E\bar{x} + (1-\lambda)E\tilde{x} \in S$. Let $E \notin \lambda D(\bar{x}) + (1-\lambda)D(\tilde{x})$ for any $0 \le \lambda \le 1$, then $E \notin D(\bar{x}) \bigcup D(\tilde{x})$ and hence $E\bar{x}, E\tilde{x} \notin S$. Therefore, there exist $\hat{x}, x^* \in M$ such that

$$f(\lambda\hat{x} + (1-\lambda)x^*) < f(\lambda E\bar{x} + (1-\lambda)E\tilde{x}), \tag{2.4}$$

which is a contradiction. Then

$$D(\lambda\bar{x} + (1-\lambda)\tilde{x}) \subset \lambda D(\bar{x}) + (1-\lambda)D(\tilde{x}) \tag{2.5}$$

and hence D is convex. □

Definition 2.8. Let S be a subset of $\mathbb{R}^n$ and $E : \mathbb{R}^n \to \mathbb{R}^n$ be an operator. The set S is called an E-convex set if and only if

$$\lambda Ex + (1-\lambda)Ey \in S, \quad \forall x, y \in S,\ 0 \le \lambda \le 1. \tag{2.6}$$

(For more details about E-convex sets see [9].)

Theorem 2.9. *If S is an E-convex set with respect to $E \in D(\bar{x})$, $\bar{x} \in S$, then $D(\bar{x}) = \bigcap_n D(E^n\bar{x})$.*

Proof. Let $E \in D(\bar{x})$, then $E(\bar{x}) \in S$. Let $E \notin D(E\bar{x})$, then $E(E\bar{x}) \notin S$ which contradicts the E-convexity of S. Hence $E \in D(E\bar{x})$, that is, $D(\bar{x}) \subset D(E\bar{x})$. Similarly, $D(E\bar{x}) \subset D(E^2\bar{x}) \subset \cdots \subset D(E^n\bar{x})$. Thus $D(\bar{x}) = \bigcap_n D(E^n\bar{x})$. □

Theorem 2.10. *Let f be a strictly convex function. If $E \in D(\bar{x})$ then $E^{-1} \in D(\bar{x})$.*

Proof. Let $E \in D(\bar{x})$, then $E(\bar{x}) \in S$. If $E^{-1} \notin D(\bar{x})$, then $E^{-1}(\bar{x}) \notin S$ and $\bar{x} \notin E(S)$. Since f is strictly convex, then $\bar{x}$ is the unique minimal point, that is, $s = \{\bar{x}\}$ and hence $E(\bar{x}) = \bar{x}$. Thus $\bar{x} \notin E(S)$ contradicts $E(\bar{x}) = \bar{x}$. Hence $E^{-1} \in D(\bar{x})$. □

Theorem 2.11. *Let S be a convex cone with vertex at the origin. If $E \in D(\bar{x})$, then $aE + b \in D(\bar{x})$ for each real numbers $a, b \ge 0$, $(a, b) \ne 0$.*

Proof. Since $E \in D(\bar{x})$, then $E(\bar{x}) \in S$. So $(aE + b)\bar{x} = aE(\bar{x}) + bI(\bar{x})$, where I is the identity map. Since S is convex cone, then for $a, b \ge 0$, $(a, b) \ne 0$, $aE(\bar{x})$ and $bI(\bar{x})$ belong to S. Thus $aE(\bar{x}) + bI(\bar{x}) \in S$ and hence $aE + b \in D(\bar{x})$. □

Theorem 2.12. *Let S be a convex cone with vertex at the origin. If $D(x)-\{I\}$, $x\in S$, is compact, then there exists $E\in D(x)-\{I\}$ with $E(x)=x$.*

Proof. Suppose that $\varphi(x)=\min_{E\in D(x)-\{I\}}\|Ex-x\|$. Since $D(x)-\{I\}$ is compact, then the minimal of φ exists. Let this minimal be $\bar{E}$. Therefore

$$||\bar{E}x-x||\le\|Ex-x\|,\quad\forall E\in D(x)-\{I\}.\tag{2.7}$$

Since S is a convex cone with vertex at the origin, then $a\bar{E}+b\in D(x)-\{I\}$ and hence

$$||\bar{E}x-x||\le||a\bar{E}x+bx-x||,\quad\forall a,b\ge 0,\ (a,b)\ne 0.\tag{2.8}$$

Then for $a=b$ we have

$$||\bar{E}x-x||\le||a(\bar{E}x-x)+(2a-1)x||\le a||\bar{E}x-x||+(2a-1)\|x\|\tag{2.9}$$

and hence

$$||\bar{E}x-x||\le\frac{2a-1}{1-a}\|x\|,\quad\forall a\ge 0,\tag{2.10}$$

which implies $\bar{E}x=x$ and hence the result. □

3. E-stability set of a side.

Denote $\rho(I)$ the side in the set M which is defined as

$$\rho(I)=\{x\in\mathbb{R}^n:g_i(x)=0,\ i\in I=\{1,2,\dots,r\},\ g_i(x)<0,\ 1\notin I\}.\tag{3.1}$$

Definition 3.1. An E-stability set of a side $\rho(I)$ is denoted by $H(\bar{x},I)$, $\bar{x}\in S$, and is defined as

$$H(\bar{x},I)=\Big\{E\in D(\bar{x}):E(\bar{x})\in S\cap\rho(I)\Big\}.\tag{3.2}$$

Theorem 3.2. *If $I_1\ne I_2$, then $H(\bar{x},I_1)\cap H(\bar{x},I_2)=\varnothing$.*

Proof. Let $\hat{E}\in H(\bar{x},I_1)\cap H(\bar{x},I_2)$, then $\hat{E}(\bar{x})\in S\cap\rho(I_1)$ and $\hat{E}(\bar{x})\in S\cap\rho(I_2)$, that is,

$$\begin{aligned}&g_i(\hat{E}\bar{x})=0,\quad i\in I_1,\qquad g_i(\hat{E}\bar{x})<0,\quad i\notin I_1,\\&g_i(\hat{E}\bar{x})=0,\quad i\in I_2,\qquad g_i(\hat{E}\bar{x})<0,\quad i\notin I_2.\end{aligned}\tag{3.3}$$

Therefore $g_s(\hat{E}\bar{x})=0$, $g_s(\hat{E}\bar{x})<0$ for at least one $S\in\{1,2,\dots,r\}$ which is a contradiction. Hence the result. □

Theorem 3.3. *If S and $\rho(I)$ are closed, then the set $H(x,I)$ with $x \in S$ is closed.*

Proof. Let E^n be a sequence in $H(x,I)$ and E^n tends to E^0 as n tends to infinity, then $E^n(x) \in S \bigcap \rho(I)$. Since S and $\rho(I)$ are closed, then $E^0(x) \in S \bigcap \rho(I)$. So $E^0 \in H(x,I)$. Hence the result. $\square$

Theorem 3.4. *Let f, g_i, $i = 1,2,\ldots,m$ be convex functions and $\rho(I)$ be a convex set, then $H(x,I)$ is convex.*

Proof. Let $E_1, E_2 \in H(x,I)$, then $E_1, E_2 \in S \cap \rho(I)$. Since f and g_i, $i = 1,2,\ldots,m$, are convex, then S is convex and hence $S \cap \rho(I)$ is convex since $\rho(I)$ is convex. Thus $\lambda E_1 + (1-\lambda)E_2 \in S \cap \rho(I)$, $0 \le \lambda \le 1$. Therefore $H(x,1)$ is convex. $\square$

Definition 3.5. Let $M \subseteq \mathbb{R}^n$ be locally arcwise connected at $x^* \in \bar{M}$ (closure of M). A vector $e \in \mathbb{R}^n$ is said to be tangent to M at x^* if and only if there exists a positive number ν and a continuous function $\delta x(\cdot) : (0,\nu) \to \mathbb{R}^n$ such that

(i) $x^* + \alpha\delta x(\alpha) \in M$ for all $\alpha \in (0,\nu)$,

(ii) $\delta x(\alpha) \to e$ as $\alpha \to 0$.

Definition 3.6. For $M \subseteq \mathbb{R}^n$, with $x^* \in \bar{M}$, the set of points

$$T = \{e \in \mathbb{R}^n : e \text{ tangent to } M \text{ at } x^*\} \tag{3.4}$$

is called the tangent cone to M at x^*. (For more details see [7].)

Theorem 3.7. *Let T be a tangent cone of $\rho(I)$ and f convex and of class C^1. The sufficient condition to have $E \in H(x,I)$ with $x \in S$ is $(\partial f(Ex)/\partial x)e > 0$ for all nonzero vectors $e \in T$.*

Proof. Let $(\partial f(Ex)/\partial x)e > 0$ for all nonzero vectors $e \in T$, then $E(x)$ is a proper local minimum of f. Since f is convex, then $E(x)$ is a global minimum for f on $\rho(I)$, that is, $E(x) \in S \cap \rho(I)$. Hence $E \in H(x,I)$. $\square$

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