IJMMS 26:3 (2001) 179–182
PII. S0161171201005245
http://ijmms.hindawi.com


# COMMON COINCIDENCE POINTS OF $R$-WEAKLY COMMUTING MAPS

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(Received 22 May 2000)

Abstract. A common coincidence point theorem for $R$-weakly commuting mappings is obtained. Our result extend several ones existing in literature.

2000 Mathematics Subject Classification. 54H25.

**1. Introduction.** Throughout this paper, $X$ denotes a metric space with metric $d$. For $x \in X$ and $A \subseteq X$, $d(x,A) = \inf\{d(x,y) : y \in A\}$. We denote by $CB(X)$ the class of all nonempty bounded closed subsets of $X$. Let $H$ be the Hausdorff metric with respect to $d$, that is,

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\} \tag{1.1}$$

for every $A,B \in CB(X)$. The mappings $T : X \to CB(X)$, $f : X \to X$ are said to be commuting if, $fTX \subseteq TfX$. A point $p \in X$ is said to be a fixed point of $T : X \to CB(X)$ if $p \in Tp$. The point $p$ is called a coincidence point of $f$ and $T$ if $fp \in Tp$. The mappings $f : X \to X$ and $T : X \to CB(X)$ are called weakly commuting if, for all $x \in X$, $fTx \in CB(X)$ and $H(fTx,Tfx) \le d(fx,Tx)$.

Recently Daffer and Kaneko [2] reaffirmed the positive answer [5] to the conjecture of Reich [8] by giving an alternative proof to Theorem 5 of Mizoguchi and Takahashi [5]. We state Theorem 2.1 of Daffer and Kaneko [2] for convenience.

**Theorem 1.1.** *Let $X$ be a complete metric space and $T : X \to CB(X)$. If $\alpha$ is a function of $(0,\infty)$ to $(0,1]$ such that $\limsup_{r\to t^+} \alpha(r) < 1$ for each $t \in [0,\infty)$ and if*

$$H(Tx,Ty) \le \alpha(d(x,y))\, d(x,y) \tag{1.2}$$

*for each $x,y \in X$, then $T$ has a fixed point in $X$.*

The purpose of this paper is to obtain a coincidence point theorem for $R$-weakly commuting multivalued mappings analogous to Theorem 1.1. We follow the same technique used in [2]. The notion of $R$-weak commutativity for single-valued mappings was defined by Pant [7] to generalize the concept of commuting and weakly commuting mappings [9]. Recently, Shahzad and Kamran [10] extended this concept to the setting of single and multivalued mappings, and studied the structure of common fixed points.

**Definition 1.2** (see [10]). The mappings $f : X \to X$ and $T : X \to CB(X)$ are called $R$-weakly commuting if for all $x \in X$, $fTx \in CB(X)$ and there exists a positive real number $R$ such that

$$H(Tfx, Tfx) \le R\,d(fx, Tx). \tag{1.3}$$

**2. Main result.** Before giving our main result, we state the following lemmas which are noted in Nadler [6], and Assad and Kirk [1].

**Lemma 2.1.** *If $A, B \in CB(X)$ and $a \in A$, then for each $\varepsilon > 0$, there exists $b \in B$ such that*

$$d(a, b) \le H(A, B) + \varepsilon. \tag{2.1}$$

**Lemma 2.2.** *If $\{A_n\}$ is a sequence in $CB(X)$ and $\lim_{n\to\infty} H(A_n, A) = 0$ for $A \in CB(X)$. If $x_n \in A_n$ and $\lim_{n\to\infty} d(x_n, x) = 0$, then $x \in A$.*

Now, we prove our main result.

**Theorem 2.3.** *Let $X$ be a complete metric space, $f, g : X \to X$ and $S, T : X \to CB(X)$ are continuous mappings such that $SX \subseteq gX$ and $TX \subseteq fX$. Let $\alpha : (0, \infty) \to (0, 1]$ be such that $\limsup_{r\to t^+} \alpha(r) < 1$ for each $t \in [0, \infty)$ and*

$$H(Sx, Ty) \le \alpha(d(gx, fy))\,d(gx, fy) \tag{2.2}$$

*for each $x, y \in X$. If the pairs $(g, T)$ and $(f, S)$ are $R$-weakly commuting, then $g, S$ and $f, T$ have a common coincidence point.*

**Proof.** Our method is constructive. We construct sequences $\{x_n\}, \{y_n\}$, and $\{A_n\}$ in $X$ and $CB(X)$, respectively as follows. Let $x_0$ be an arbitrary point of $X$ and $y_0 = fx_0$. Since $Sx_0 \subseteq gX$, there exists a point $x_1 \in X$ such that $y_1 = gx_1 \in Sx_0 = A_0$. Choose a positive integer $n_1$ such that

$$\alpha^{n_1}(d(y_0, y_1)) < \{1 - \alpha(d(y_0, y_1))\}\,d(y_0, y_1). \tag{2.3}$$

Now Lemma 2.1 and the fact $TX \subseteq fX$ guarantee that there is a point $y_2 = fx_2 \in Tx_1 = A_1$ such that

$$d(y_2, y_1) \le H(A_1, A_0) + \alpha^{n_1}(d(y_0, y_1)). \tag{2.4}$$

The above inequality in view of (2.2) and (2.3) implies that $d(y_2, y_1) < d(y_0, y_1)$. Now choose a positive integer $n_2 > n_1$ such that

$$\alpha^{n_2}(d(y_2, y_1)) < \{1 - \alpha(d(y_2, y_1))\}\,d(y_2, y_1). \tag{2.5}$$

Again using Lemma 2.1 and the fact $SX \subseteq gX$, we get a point $y_3 = gx_3 \in Sx_2 = A_2$ such that

$$d(y_3, y_2) \le H(A_2, A_1) + \alpha^{n_2}(d(y_2, y_1)). \tag{2.6}$$

Now (2.2) and (2.5) further imply that $d(y_3, y_2) < d(y_2, y_1)$.

By induction we obtain sequences $\{x_n\},\{y_n\}$, and $\{A_n\}$ in $X$ and $CB(X)$, respectively, such that

$$y_{2k+1} = gx_{2k+1} \in Sx_{2k} = A_{2k}, \qquad y_{2k} = fx_{2k} \in Tx_{2k-1} = A_{2k-1}, \tag{2.7}$$

$$d(y_{2k+1},y_{2k}) \le H(A_{2k},A_{2k-1}) + \alpha^{n_k}(d(y_{2k},y_{2k-1})), \tag{2.8}$$

where

$$\alpha^{n_{2k}}(d(y_{2k},y_{2k-1})) < \{1-\alpha(d(y_{2k},y_{2k-1}))\}d(y_{2k},y_{2k-1}) \tag{2.9}$$

for each $k$. So we have $d(y_{2k+1},y_{2k}) < d(y_{2k},y_{2k-1})$. Therefore, the sequence $\{d(y_{2k+1},y_{2k})\}$ is monotone nonincreasing. Then, as in the proof of Theorem 2.1 in [2], $\{y_n\}$ is a Cauchy sequence in $X$. Further, equation (2.2) ensures that $\{A_n\}$ is a Cauchy sequence in $CB(X)$. It is well known that if $X$ is complete, then so is $CB(X)$. Therefore, there exist $z \in X$ and $A \in CB(X)$ such that $y_n \to z$ and $A_n \to A$. Moreover, $gx_{2k+1} \to z$ and $fx_{2k} \to z$. Since

$$d(z,A) = \lim_{n\to\infty} d(y_n,A_n) \le \lim_{n\to\infty} H(A_{n-1},A_n) = 0, \tag{2.10}$$

it follows from Lemma 2.2 that $z \in A$. Also

$$\lim_{k\to\infty} fx_{2k} = z \in A = \lim_{k\to\infty} Sx_{2k}, \qquad \lim_{k\to\infty} gx_{2k+1} = z \in A = \lim_{k\to\infty} Tx_{2k-1}. \tag{2.11}$$

Using (2.7) and $R$-weak commutativity of the pairs $(g,T)$ and $(f,S)$, we have

$$\begin{aligned} d(gfx_{2k+2},Tgx_{2k+1}) &\le H(gTx_{2k+1},Tgx_{2k+1}) \le R\,d(gx_{2k+1},Tx_{2k+1}), \\ d(fgx_{2k+1},Sfx_{2k}) &\le H(fSx_{2k},Sfx_{2k}) \le R\,d(fx_{2k},Sx_{2k}). \end{aligned} \tag{2.12}$$

Now it follows from the continuity of $f$, $g$, $T$, and $S$ that $gz \in Tz$ and $fz \in Sz$. □

If we put $T = S$ and $f = g$ in Theorem 2.3, we get the following corollary.

**Corollary 2.4.** Let $X$ be a complete metric space, and let $f : X \to X$ be a continuous mapping and $T : X \to CB(X)$ be a mapping such that $TX \subseteq fX$. Let $\alpha : (0,\infty) \to (0,1]$ be such that $\limsup_{r\to t^+} \alpha(r) < 1$ for each $t \in [0,\infty)$ and

$$H(Tx,Ty) \le \alpha(d(fx,fy))\,d(fx,fy) \tag{2.13}$$

for each $x,y \in X$. If the mappings $f$ and $T$ are $R$-weakly commuting, then $f$ and $T$ have coincidence point.

**Remark 2.5.** (1) Theorem 2.3 improves and extends some known results of Hu [3], Kaneko [4], Mizoguchi and Takahashi [5], and Nadler [6].

(2) In Corollary 2.4, $T$ is not assumed to be continuous. In fact the continuity of $T$ follows from the continuity of $f$.

(3) If we put $f = I$, the identity map, in Corollary 2.4, we obtain Theorem 1.1.

**Acknowledgement.** The author is grateful to Dr. Naseer Shahzad for his useful suggestions.

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