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VECTOR-VALUED SEQUENCE SPACES GENERATED BY INFINITE MATRICES

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ABSTRACT. Let $A = (a_{nk})$ be an infinite matrix with all $a_{nk} \geq 0$ and P a bounded, positive real sequence. For normed spaces E and E_k the matrix A generates paranormed sequence spaces such as $[A,P]_\infty((E_k))$, $[A,P]_0((E_k))$, and $[A,P](E)$ which generalize almost all the existing sequence spaces, such as l_∞, c_0, c, l_p, w_p, and several others. In this paper, conditions under which these three paranormed spaces are separable, complete, and r-convex, are established.

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1. Introduction. The study of sequence spaces is generally initiated by problems in summability theory, Fourier series and power series. During the third decade of the present century, sequence spaces were studied with more insight and vision through the application of functional analysis. Rich settings for the analytic approach to the study of sequence spaces have already been provided by pioneers like Banach (see [1]), Köthe and Toeplitz (see [6, 7, 8]). Consequently, the study of sequence spaces is now generally regarded as a branch of linear topological spaces. But it is not hard to realize that it has more intimate relation with the summability theory and matrix transformation than any other area.

One of the classic problems in the theory of sequence spaces is to transform one sequence space into another and study the behavior pattern of the transformed sequences related to the original space. A decisive break with the classical approach is made in this paper by introducing vector-valued sequences in place of sequences of numbers. We study the sequence spaces which are generated by infinite matrices. These spaces are linear topological spaces, the topologies of which are induced by paranorms. The paranormed sequence spaces were introduced by Borwien (see [2, 3]), Bourgin (see [4]), Simons (see [16, 17]), and later, Maddox (see [9, 10, 11, 12, 13, 14]) developed it in considerable details. We study certain topological properties like separability, completeness, and r-convexity of these generalized paranormed sequence spaces. In fact, all the corresponding results related to the sequence spaces c_0, l_p, c, l_∞, and w_p follow as special cases of the theorems established here. Stated otherwise, sequence spaces are studied in this paper with a new approach and insight.

2. Preliminaries. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{C}$ the set of complex numbers. Throughout this paper, we assume that E and E_k, for all $k \in \mathbb{N}$, are normed linear spaces. The zero element of a normed linear space is denoted by θ and the unit matrix is denoted by I. Also, we assume that for the sequence $P = (p_k)$, $p_k \geq 0$, for all

$k \in \mathbb{N}$ and for the matrix $A = (a_{nk})$, $a_{nk} \geq 0$ for all $n, k \in \mathbb{N}$. If P is a bounded sequence, we write $p_k = O(1)$ and $M = \max(1, \sup_k p_k)$. It should be noted that all the results remain valid, if we assume that E and E_k are seminormed or p-normed spaces, while Theorem 4.10 is established for seminormed spaces. The conditions under which the vector-valued sequence spaces are r-convex, when each E_k is p-normed, are yet to be explored.

The following spaces are frequently used:

$$\begin{aligned}
[A,P]_\infty((E_k)) &= \Big\{x = (x_k) \in \prod E_k \;\Big|\; \sup_n \sum_k a_{nk} \|x_k\|^{p_k} < \infty\Big\}, \\
[A,P]_0((E_k)) &= \Big\{x = (x_k) \in \prod E_k \;\Big|\; \sum_k a_{nk} \|x_k\|^{p_k} \text{ exists, } \forall n \in \mathbb{N} \text{ and } \longrightarrow 0 \text{ as } n \longrightarrow \infty\Big\}, \\
[A,P](E) &= \Big\{x = (x_k) \mid x_k \in E,\ \forall k \text{ and } \exists l \in E \text{ such that} \\
&\qquad \sum_k a_{nk} \|x_k - l\|^{p_k} \text{ exists } \forall n \in \mathbb{N} \text{ and } \longrightarrow 0 \text{ as } n \longrightarrow \infty\Big\}.
\end{aligned} \tag{2.1}$$

These spaces are called *vector-valued sequence spaces generated by infinite matrices*. In the special case, when $A = I$ (resp., the Cesaro matrix $(C,1)$), $[A,P]_\infty((E_k))$ reduces to $l_\infty(P,E_k)$ (resp., $w_\infty(P,E_k)$). The space $l(P,E_k)$ is obtained by considering the matrix $A = (a_{nk})$, where $a_{nk} = 1$, for all k, $1 \leq k \leq n$ and $a_{nk} = 0$, for all $k > n$. Similarly, we have the following spaces:

$$\begin{aligned}
[I,P](E) = c(P,E), &\qquad [I,P]_0((E_k)) = c_o(P,(E_k)), \\
[(C,1),P]_0((E_k)) = w_0(P,(E_k)), &\qquad [(C,1),P](E) = w(P,E).
\end{aligned} \tag{2.2}$$

These representations are not unique, because we can also write $[B,P]_0((E_k)) = l(P,(E_k))$, where $B = (b_{nk})$ is the matrix defined as $b_{nk} = 1$, if $k > n$ and $b_{nk} = 0$, if $1 \leq k \leq n$. Similarly, the spaces $w(P,E)$, $w_0(P,E)$, and $w_\infty(P,(E_k))$ can be represented as $[D,P](E)$, $[D,P]_0((E_k))$, and $[D,P]_\infty((E_k))$, respectively, where $D = (d_{nk})$ is the matrix defined by $d_{nk} = 2^{-(n-1)}$, if $2^{n-1} \leq k < 2^n$ and $d_{nk} = 0$, otherwise. A few decades ago, Maddox introduced $[A,P]_0$ (resp., $[A,P]$, $[A,P]_\infty$) which is a special case of $[A,P]_0((E_k))$ (resp., $[A,P](E)$, $[A,P]_\infty((E_k))$), when $E_k = C$, for all $k \geq 1$. The conditions under which the three spaces $[A,P]_0$, $[A,P]$, and $[A,P]_\infty$ are linear paranormed can be found in [12]. These results hold without any substantial change for the vector-valued sequence spaces $[A,P]_0((E_k))$, $[A,P](E)$, and $[A,P]_\infty((E_k))$. If $p_k = O(1)$, then each of these three spaces are linear topological spaces, the topology being induced by a paranorm g defined by

$$g(x) = \sup_n \left[\sum_k a_{nk} \|x_k\|^{p_k}\right]^{1/M}, \tag{2.3}$$

where $M = \max(1, \sup_k p_k)$. In this paper, we limit our attention to paranormed sequence spaces generated by infinite matrices and consequently, we assume for the remainder of the paper that $p_k = O(1)$, unless otherwise indicated.

3. Lemmas. The following lemmas are used in proving the theorems of this paper.

Lemma 3.1 [9]. *A linear topological space is r-convex for some $r > 1$ if and only if X is the only neighborhood of the origin.*

Lemma 3.2 [5]. *Let x, y, λ, μ be complex numbers. Then*

$$\begin{aligned} |x+y|^p &\le |x|^p + |y|^p, \quad \text{if } 0 < p \le 1, \\ (|\lambda x| + |\mu y|)^p &\le |\lambda||x|^p + |\mu||y|^p, \quad \text{if } p \ge 1,\ |\lambda| + |\mu| \le 1. \end{aligned} \tag{3.1}$$

Lemma 3.3 [15]. *If x is a complex number with $0 < |x| \le 1$, $0 < p \le r$, and $a > 1$, then*

$$|x|^p < |x|^r(1 + a\log a) + a^\pi, \tag{3.2}$$

where $1/\pi + r/p = 1$.

Lemma 3.4 [11]. *The space $w_\infty(p)$ is paranormed by $g(x) = \sup_n[\sum_k a_{nk}|x_k|^{p_k}]^{1/M}$, where $M = \max(1, \sup_k p_k)$ if and only if $0 < \inf_k p_k \le \sup_k p_k < \infty$.*

Lemma 3.5. *Let E be a nontrivial space. Then $[A,P](E) \subseteq [A,P]_\infty(E)$ if and only if $\alpha = \sup_n \sum_k a_{nk} < \infty$.*

Lemma 3.6. *Let $B = (b_{nk})$ be any matrix of zeros and ones and let r be any positive number. If B is any column finite matrix and*

$$[B,(r)]_\infty((E_k)) \subseteq [B,P]_\infty((E_k)), \tag{3.3}$$

where $(r) = (r,r,r,\dots)$, then there exists an integer $i > 1$ such that $\sup_n \sum_{s(n)} a_i^{\pi_k} < \infty$, where $1/\pi_k + r/p_k = 1$, for $k \in \mathbb{N}$ and $s(n) = \{k \mid b_{nk} = 1,\ p_k < r\}$, for each $n \in \mathbb{N}$.

Lemma 3.7. *The space $l_\infty(P,(E_k))$ is a linear topological space if and only if $\inf_k p_k > 0$.*

The analogue of Lemmas 3.5, 3.6, and 3.7 for the spaces $[A,P]_\infty$, $[B,P]_\infty$, and $l_\infty(P)$ can be found in [9, 10, 12], respectively.

4. Main results. This section deals with the results established in this paper. The necessary and sufficient conditions for separability, completeness, and r-convexity of the vector-valued sequence spaces $[A,P]_0((E_k))$, $[A,P](E)$, and $[A,P]_\infty((E_k))$ are obtained in Sections 4.1, 4.2, and 4.3, respectively.

4.1. A topological space is said to be *separable*, if it has a countable dense subset. In this subsection, we obtain necessary and sufficient conditions for separability of the spaces $[A,P]_0((E_k))$ and $[A,P](E)$. In general, the space $[A,P]_\infty((E_k))$ is not separable, since as a special case of this space l_∞ is not separable.

Theorem 4.1. *Let $\lim_{n\to\infty} a_{nk} = 0$ and $L_k = \sup_n a_{nk} > 0$, for each fixed $k \in \mathbb{N}$. Then $[A,P]_0((E_k))$ is separable if and only if each E_k is separable.*

Proof. ($\Leftarrow$). Suppose that each E_k is separable. Let $\mathcal{B}$ be the set of all finite sequences in $\prod_k E_k$. Then it can be easily shown that $\mathcal{B}$ is dense in $[A,P]_0((E_k))$. Next, we show that $\mathcal{B}$ has a countable dense subset. Since each E_k is separable, we can find a countable dense subset $F_k \subseteq E_k$, for each $k \in \mathbb{N}$. Let F denote the set of all finite sequences in $\prod_k F_k$. Clearly, F is a countable subset of $\mathcal{B}$. Also, if $y = (y_1, y_2, \dots, y_r, 0, 0, 0, \dots) \in \mathcal{B}$, we choose $z_k \in F_k$ such that

$$\|y_k - z_k\|^{p_k} < \epsilon, \tag{4.1}$$

for each $1 \le k \le r$.

Let $z = (z_1, z_2, \dots, z_r, 0, 0, 0, \dots) \in F$. Since $\lim_{n\to\infty} a_{nk} = 0$, for each fixed $k \in \mathbb{N}$,

$$[g(y-z)]^M = \sup_n \sum_{k=1}^{r} a_{nk} \|y_k - z_k\|^{p_k} < \epsilon \mu, \tag{4.2}$$

where $\mu = \sup_n \sum_{k=1}^{r} a_{nk}$ is a finite constant (depending on the sequence y). Hence, it follows that $[A,P]_0((E_k))$ is separable.

($\Rightarrow$). Conversely, let D be a countable dense subset in $[A,P]_0((E_k))$. For each fixed $r \in \mathbb{N}$, let $D_r = \{y_r \mid y = (y_k) \in D\}$. We show that D_r is dense in E_r. Let $x \in E_r$. Define a sequence x^r by

$$x_k^r = \begin{cases} x, & \text{if } k = r, \\ 0, & \text{if } k \neq r. \end{cases} \tag{4.3}$$

Then, $x^r \in [A,P]_0((E_k))$, since $\lim_{n\to\infty} a_{nr} = 0$. For a given $\epsilon > 0$, we can choose $y = (y_k) \in F$ such that

$$g(y-x)^r = \sup_n \left[\sum_k a_{nk} \|y_k - x_k\|^{p_k} \right]^{1/M} < \left[\epsilon^{p_r} L_r\right]^{1/M}. \tag{4.4}$$

Therefore,

$$\sup_n \left[a_{nr} \|y_r - x\|^{p_r}\right] < \epsilon^{p_r} L_r, \tag{4.5}$$

which implies that $\|y_r - x\| < \epsilon$. This completes the proof of Theorem 4.1. □

Köthe [6, 7] obtained the necessary and sufficient condition for the separability of $l_p(E)$, for $1 < p < \infty$. This can be deduced from the following corollary which is a direct consequence of Theorem 4.1.

Corollary 4.2. *The space $l_p((E_k))$ (also, $c_0(P,(E_k)), w_0(P,(E_k))$ is separable if and only if each E_k is separable. In particular, each of $l_p(E), c_p(E)$, and $w^p(E)$) is separable if and only if E is separable.*

Theorem 4.3. *Let $L = \inf_k p_k > 0$, $H = \sup_n \sum_k a_{nk} < \infty$, $\lim_{n\to\infty} a_{nk} = 0$, for each fixed $k \in \mathbb{N}$ and $\mathcal{L}_r = \sup_n a_{nr} \ge 0$, for at least one $r \in \mathbb{N}$. Then, $[A,P](E)$ is separable if and only if E is separable.*

Proof. ($\Leftarrow$). Let $B = (x_i)$ be a countable dense subset of E and let F denote the set of all ultimately constant sequences of elements of B, that is, all sequences of the type $y^r = (x_{i_1}, x_{i_2}, \dots, x_{i_r}, u, u, u, \dots)$, where $x_{i_1}, \dots, x_{i_r}$ and u are in B and $i_1, \dots, i_r, r$ are in $\mathbb{N}$. It is clear that $F \subseteq [A,P](E)$ is countable. Hence, it suffices to show that F is dense in $[A,P](E)$.

Let $y = (y_k) \in [A,P](E)$ and let $0 < \epsilon < 1$. For each $k \in \mathbb{N}$, choose $x_k^i \in B$ such that

$$||y_k - x_k^i||^{p_k} \leq \epsilon^M. \tag{4.6}$$

Then, there exists $l \in E$ such that

$$\lim_{n\to\infty} \sum_k a_{nk} ||y_l^k||^{p_k} = 0. \tag{4.7}$$

So, we can find $k_0 \in \mathbb{N}$, for which

$$\sup_{n\in\mathbb{N}} \sum_{k=k_0+1}^{\infty} a_{nk} \|y_k - l\|^{p_k} \leq \epsilon^M, \tag{4.8}$$

and $b \in B$ such that

$$\|l - b\|^L \leq \epsilon^{M/(M-1)}. \tag{4.9}$$

Define $z = (z_k)$ by

$$z_k = \begin{cases} x_k^i, & \text{if } 1 \leq k \leq k_0, \\ b, & \text{if } k > k_0. \end{cases} \tag{4.10}$$

Clearly, $z \in F$. Also, it follows from (4.6) and Minkowski's inequality that

$$\begin{aligned} &\sup_{n\in\mathbb{N}} \left[\sum_k a_{nk} \|y_k - z_k\|^{p_k} \right]^{1/M} \\ &\quad \leq \sup_{n\in\mathbb{N}} \left[\sum_{k=1} k_0 a_{nk} ||y_k - x_k^i||^{p_k} \right]^{1/M} + \sup_{n\in\mathbb{N}} \left[\sum_{k=k_0+1}^{\infty} a_{nk} \|y_k - b\|^{p_k} \right]^{1/M} \\ &\quad \leq \mu^{1/M} \epsilon + \sup_{n\in\mathbb{N}} \left[\sum_{k=k_0+1}^{\infty} a_{nk} \|y_k - l\|^{p_k} \right]^{1/M} + \sup_{n\in\mathbb{N}} \left[\sum_{k=k_0+1}^{\infty} a_{nk} \|l - b\|^{p_k} \right]^{1-(1/M)}, \end{aligned} \tag{4.11}$$

since $p_k \leq M$ and $M \geq 1$.

Now, considering the inequalities in (4.8), (4.9), and since $l \leq p_k$ we can have

$$g(y - z) \leq \mu^{1/M} \epsilon + \epsilon + \mu^{1/M} \sup_k \epsilon^{p_k/L}. \tag{4.12}$$

This proves that $[A,P](E)$ is separable.

($\Rightarrow$). Conversely, let $[A,P](E)$ be separable with a countable dense subset $D = (y^i)_{i\in\mathbb{N}}$, where $y^i = (y_k^i)_{k\in\mathbb{N}}$ for each $i \in \mathbb{N}$.

For $x \in E$, let x^r denote the sequence

$$x_k^r = \begin{cases} x, & \text{if } k = r, \\ 0, & \text{otherwise.} \end{cases} \tag{4.13}$$

Clearly, $x^r \in [A,P](E)$. Then, as in Theorem 4.1, we can show that the set $G = \{y_r^i \mid r \in \mathbb{N}\}$ is dense in E. This completes the proof of Theorem 4.3. □

Corollary 4.4. *Let p_k be as in Theorem 4.3 and $A = (a_{nk})$ a nonnegative, nonzero, and regular matrix. Then, $[A,P](E)$ is separable if and only if E is separable. In particular, $c(P,E)$ and $w(P,E)$ are separable if and only if E is separable.*

4.2. A paranormed space is said to be *complete*, if every Cauchy sequence converges. This subsection deals with the completeness of the generalized sequence spaces $[A,P]_\infty((E_k))$, $[A,P]_0((E_k))$, and $[A,P](E)$. In Theorem 4.5(i), we show that completeness of each space E_k implies completeness of $[A,P]_0((E_k))$ and $[A,P]_\infty((E_k))$, while in part (ii) (resp., (iii)), conditions for which completeness of $[A,P]_\infty((E_k))$ (resp., $[A,P]_0((E_k))$) implies completeness of each E_k are established. Finally, in Theorem 4.8, completeness of $[A,P](E)$ is discussed.

Theorem 4.5. *Let $L_k = \sup_n a_{nk} \geq 0$, for each $k \in \mathbb{N}$. Then the following statements are true:*

(i) *The spaces $[A,P]_0((E_k))$ and $[A,P]_\infty((E_k))$ are complete, whenever the spaces E_k are complete for each $k \in \mathbb{N}$.*

(ii) *The spaces E_k, for each $k \in \mathbb{N}$ are complete, whenever $[A,P]_\infty((E_k))$ is complete and $L_k = \sup_n a_{nk} < \infty$, for each $k \in \mathbb{N}$.*

(iii) *The spaces E_k, for each $k \in \mathbb{N}$ are complete, whenever $[A,P]_0((E_k))$ is complete and $\lim_{n\to\infty} a_{nk} = 0$.*

Proof. (i) Let $x^i = (x_k^i)_{k\in\mathbb{N}}$ be a Cauchy sequence in $[A,P]_\infty((E_k))$. For a given $\epsilon \geq 0$, let i_0 be such that

$$\sup_n \sum_k a_{nk}||x_k^i - x_k^j||^{p_k} \leq \epsilon^M, \quad (i,j \geq i_0). \tag{4.14}$$

If k is such that $L_k \geq 0$, then

$$L_k||x_k^i - x_k^j||^{p_k} \leq \epsilon^M, \quad \forall i,j > i_0, \tag{4.15}$$

which shows that $(x_k^i)_{i\in\mathbb{N}}$ is a Cauchy sequence in E_k, for each k. Let

$$y_k = \begin{cases} \lim_{i\to\infty} x_k^i, & \text{if } L_k > 0, \\ \text{any element of } E_k, & \text{if } L_k = 0. \end{cases} \tag{4.16}$$

Then, it follows from (4.11) that

$$\sup_n \sum_k a_{nk}||x_k^i - y_k||^{p_k} \leq \epsilon^M, \quad \forall i > i_0. \tag{4.17}$$

Hence, $y = (y_k) \in [A,P]_\infty((E_k))$ and $x^i \to y$ in $[A,P]_\infty((E_k))$, as $i \to \infty$. This proves the completeness of $[A,P]_\infty((E_k))$. The completeness of $[A,P]_\infty((E_k))$ can be proved by using a similar argument.

(ii) Let $[A,P]_\infty((E_k))$ be complete and $k \in \mathbb{N}$ be fixed such that $0 < L_k < \infty$. If $x = (x_k)$ is a Cauchy sequence in E_k and for each $r \in \mathbb{N}$, y_r denotes the sequence whose rth term is x_r and all other terms are zero, then $y_r \in [A,P]_\infty((E_k))$. Moreover, the sequence y^i whose ith term is y_i for each $i \in \mathbb{N}$ is a Cauchy sequence in $[A,P]_\infty((E_k))$, because

$$g(y^i - y^j) = \sup_n [a_{nk}\|x_i - x_j\|^{p_k}]^{1/M} = [L_k\|x_i - x_j\|^{p_k}]^{1/M}, \tag{4.18}$$

which converges to zero as $i,j \to \infty$. Since $[A,P]_\infty((E_k))$ is complete, there exists $z = (z_k) \in [A,P]_\infty((E_k))$ such that

$$g(y^i - z) \longrightarrow 0, \quad \text{as } i \longrightarrow \infty. \tag{4.19}$$

This implies that

$$L_k\|x_i - z_k\|^{p_k} \longrightarrow 0, \quad \text{as } i \longrightarrow \infty. \tag{4.20}$$

Hence, $x_i \to z_k$, as $i \to \infty$. This establishes the completeness of E_k. The proof of (iii) is similar to the proof of (ii). This completes the proof of Theorem 4.5. □

Corollary 4.6. *Each E_k is complete if and only if any of the spaces $l(P,(E_k))$, $c_0(P,(E_k))$, $w_0(P,(E_k))$, and $l_\infty(P,(E_k))$ is complete. In particular, c_0, l_p, l_∞, and w_0^p are complete.*

Corollary 4.7. *Let $\lim_{n\to\infty} a_{nk} = 0$ and $L_k = \sup_n a_{nk} > 0$, for some $k \in \mathbb{N}$. Then, E_k is complete, whenever $[A,P](E)$ is complete.*

Proof. This follows from part (iii) of Theorem 4.5 upon noting that $[A,P]_0(E)$ is complete, whenever $[A,P](E)$ is complete. □

The next theorem establishes the conditions for the completeness of $[A,P](E)$, whenever E is complete.

Theorem 4.8. *Let $\sup_n \sum_k a_{nk} < \infty$. Then, completeness of E implies that $[A,P](E)$ is complete, whenever any of the following conditions hold:*

(i) $\lim_{n\to\infty} \sum_k a_{nk} = 0$,

(ii) $\limsup_n \sum_k a_{nk} > 0$ *and* $\inf_k p_k = L > 0$.

We omit the proof of Theorem 4.8, since the proof is exactly similar to that given by Maddox (see [12, Theorem 5]). However, the removal of the restriction $\inf_k p_k > 0$ is possible in some special cases such as $c(P,E)$ and $w(P,E)$. In fact, the next theorem shows that part (ii) of Theorem 4.8 holds for these two spaces without the restriction $\inf_k p_k > 0$. So it generalizes a result of Maddox (see [12, Theorem 6]).

Theorem 4.9. (i) *The space $c(P,E)$ is complete if and only if E is complete.*

(ii) *The space $w(P,E)$ is complete if and only if E is complete.*

Proof. (i) Let (x^i) be a Cauchy sequence in $c(P,E)$. So, for each x^i in $c(P,E)$, there exists $l^i \in E$ such that

$$\|x_k^i - l^i\|^{p_k} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty \text{ for each } i \in \mathbb{N}. \tag{4.21}$$

Then, as in Theorem 4.8(ii), we can find $x \in l_\infty(P,E)$ such that $g(x^i - x) \to 0$, as $i \to \infty$. So it suffices to show that $x \in c(P,E)$. The case $\inf_k p_k > 0$ can be deduced from Theorem 4.8 as a special case, when the matrix $A = I$.

Let $\inf_k p_k = 0$ and $q_k = p_k/M$, for each $k \in \mathbb{N}$. Choose an integer i_0 such that

$$g(x^i - x^{i_0}) < \frac{1}{6}, \quad \forall i > i_0. \tag{4.22}$$

Taking $i > i_0$ and k sufficiently large, we get

$$\|x_k^i - l^i\|^{q_k} < \frac{1}{6}, \qquad \|x_k^{i_0} - l^{i_0}\|^{q_k} < \frac{1}{6}. \tag{4.23}$$

So it follows that

$$s_k = \|l^i - l^{i_0}\|^{q_k} < \frac{1}{2}, \tag{4.24}$$

for each $k \in \mathbb{N}$. But, since $\inf_k q_k = 0$, it follows that $s_k > 1/2$, for infinitely many $k \in \mathbb{N}$, unless $l^i = l^{i_0}$, for all sufficiently large i. This implies that (l^i) is ultimately a constant sequence. Hence,

$$||x_k - l^{i_0}||^{p_k/M} \le ||x_k^i - x_k||^{p_k/M} + ||x_k^i - l^i||^{p_k/M} + ||l^i - l^{i_o}||^{p_k/M}, \tag{4.25}$$

which converges to zero as $k \to \infty$, implying that $x \in c(P,E)$.

Conversely, let $c(P,E)$ be complete. Let $x = (x_n)$ be a Cauchy sequence in E and let y^n denote the sequence whose first term is x_n and all other terms are zero. Clearly, $y^n \in c(P,E)$, for all $n \in \mathbb{N}$ and (y^i) is a Cauchy sequence in $c(P,E)$. Since $c(P,E)$ is complete, there exists $z = (z_k)$ in $c(P,E)$ such that $g(y-z) \to 0$, as $i \to \infty$. This implies that

$$\|x_i - z_1\| \longrightarrow 0, \quad \text{as } i \longrightarrow \infty, \tag{4.26}$$

and therefore, E is complete.

(ii) Let (x_i) be a Cauchy sequence in $w(P,E)$ with l^i as the strong Cesaro limit of x^i, for each $i \in \mathbb{N}$. Then by Theorem 4.8(ii), there exists $x \in w_\infty(P,E)$ such that $g(x^i - x) \to 0$, as $i \to \infty$. So it suffices to show that $x \in w(P,E)$. Choose i_0 for which

$$2^{-r} \sum_{2^r \le k < 2^{r+1}} ||x_k^i - x_k^j||^{p_k} < \epsilon, \quad \forall i,j > i_0. \tag{4.27}$$

Since for each fixed $i,j > i_0$, there exists r_0 such that

$$2^{-r} \sum_{2^r \le k < 2^{r+1}} ||x_k^i - l^i||^{p_k} < \epsilon, \qquad 2^{-r} \sum_{2^r \le k < 2^{r+1}} ||x_k^i - l^j||^{p_k} < \epsilon, \tag{4.28}$$

it follows that

$$2^{-r} \sum_{2^r \le k < 2^{r+1}} ||l^i - l^j||^{p_k} < 3^M \epsilon, \tag{4.29}$$

for each $i,j > i_0$ and all $r > r_0$. Also, for $3^M\epsilon < 1/2$,

$$||l^i - l^j|| < 1, \quad \forall i,j > i_0, \tag{4.30}$$

which implies that

$$||l^i - l^j||^\mu < \sum_{2^r \le k < 2^{r+1}} 2^{-r} ||l^i - l^j||^{p_k} < 3^M \epsilon, \tag{4.31}$$

for all $i,j > i_0$, where $\mu = \sup_k p_k$. So (l^i) is a Cauchy sequence in E. Since E is complete, there exists $l \in E$ such that $l^i \to l$, as $i \to \infty$, for some $l \in E$. It now remains to show that $x_k \to l[w(P,E)]$. If we denote by $N_r(\alpha)$ the number of k in the interval $[2^r, 2^{r+1})$ such that $p_k < \alpha$, then we have the following two possibilities:

$$\inf_{\alpha>0} \limsup_{r\to\infty} 2^{-r} N_r(\alpha) = 0 \tag{4.32}$$

or

$$\inf_{\alpha>0} \limsup_{r\to\infty} 2^{-r} N_r(\alpha) > 0. \tag{4.33}$$

In the case (4.32), for given $\epsilon > 0$, there exists $\alpha_0 > 0$ such that $\limsup_{r\to\infty} 2^{-r}N_r(\alpha_0) < \epsilon/2$. Therefore, $2^{-r}N_r(\alpha_0) < \epsilon$, for all sufficiently large values of r. Now choose i so large that

$$||l^i - l|| < \min(1, \epsilon^{1/\alpha_0}). \tag{4.34}$$

Hence, for all sufficiently large r,

$$\begin{aligned} 2^{-r}\sum_{2^r \le k < 2^{r+1}} ||l - l^i||^{p_k} &\le 2^{-r}\sum_{p_k<\alpha_0} ||l^i - l||^{p_k} + 2^{-r}\sum_{p_k \ge \alpha_0} ||l^i - l||^{p_k} \\ &< 2^{-r}\sum_{p_k<\alpha_0} 1 + 2^{-r}\sum_{p_k\ge\alpha_0} \epsilon < 2^{-r}N_r(\alpha_0) + \epsilon < 2\epsilon, \end{aligned} \tag{4.35}$$

which implies that

$$2^{-r}\sum_{2^r \le k < 2^{r+1}} ||l - l^i||^{p_k} \longrightarrow 0, \quad \text{as } r \longrightarrow \infty. \tag{4.36}$$

Since for any $i \in \mathbb{N}$,

$$\begin{aligned} \left[2^{-r}\sum_{2^r\le k<2^{r+1}} \|x_k - l\|^{p_k}\right]^{1/M} &\le g(x - x^i) + \left[2^{-r}\sum_{2^r\le k<2^{r+1}} ||x_k^i - l^i||^{p_k}\right]^{1/M} \\ &\quad + \left[2^{-r}\sum_{2^r\le k<2^{r+1}} ||l - l^i||^{p_k}\right]^{1/M}, \end{aligned} \tag{4.37}$$

by choosing i, r sufficiently large, it can be shown that $x_k \to l[w(P,E)]$.

In the case (4.33), if we choose $\beta = \inf_{\alpha>0}\limsup_{r\to\infty} 2^{-r}N_r(\alpha) > 0$, then there exists r_1 such that $2^{-r}N_r(1) > \beta$, for $r = r_1$ and there exists r_2 such that $2^{-r}N_r(1/2) > \beta$, for $r = r_2$ and so on. In fact, this determines a sequence of integers $r_1 < r_2 < r_3 < \cdots$, such that

$$2^{-r}N_r\left(\frac{1}{s}\right) > \beta, \tag{4.38}$$

for each $r = r_s$ and each $s \in \mathbb{N}$. By inequalities (4.29) and (4.30), there exists an integer $t = t(\beta)$ such that

$$2^{-r}\sum_{2^r\le k<2^{r+1}} ||l^i - l^j||^{p_k} < \frac{\beta}{2}, \tag{4.39}$$

for sufficiently large r and $\|l^i - l^j\| < 1$, for all $i > t$. Now we must have $l^i = l^t$, for all $i > t$. Because, otherwise we have $\|l^i - l^t\| > 1/2$, for some $i > t$, which implies that

$$\begin{aligned} 2^{-r}\sum_{2^r\le k<2^{r+1}} ||l^i - l^t||^{p_k} &\ge 2^{-r}\sum_{p_k<1/s} ||l^i - l^t||^{p_k} \\ &\ge 2^{-r}N_r\left(\frac{1}{s}\right)||l^i - l^t||^{1/S} > \beta||l^i - l^t|| > \frac{\beta}{2}, \end{aligned} \tag{4.40}$$

for sufficiently large i and $r = r_s$. This contradicts (4.39). Therefore, it follows from (4.37) that $x_k \to l[w(P,E)]$, that is, $w(P,E)$ is complete. The proof of the converse part of (ii) is similar to the converse part of (i). □

4.3. A subset V of a linear topological space X is said to be *absolutely r-convex* (we say *r-convex* for brevity), if $\lambda x+\mu y\in V$ whenever $x,y\in V$ and λ,μ are scalars such that $|\lambda|^r+|\mu|^r\le 1$. A linear topological space X is said to be r-convex, if the family of all r-convex neighborhoods of θ form a neighborhood base. This section establishes the results related to the r-convexity of the spaces $[A,P]_\infty((E_k))$ and $[A,P]_0((E_k))$. Maddox and Roles (see [15, Theorem 4]) have already obtained necessary and sufficient conditions for the r-convexity of $[A,P]_\infty$, which is a special case of Theorem 4.10. Note that by Lemma 3.1, we need to consider only the case $0<r\le 1$, while characterizing the r-convexity of the space $[A,P]_\infty((E_k))$.

Theorem 4.10. *Let A be a column finite matrix and suppose that there exists a constant $\alpha>0$ such that for each n and k with $0<\sup_n a_{nk}<\infty$ and $a_{nk}>0$, we have $a_{nk}\ge\alpha\sup_n a_{nk}$. If $0<r\le 1$, then the following are equivalent:*

(i) *The space $[A,P]_\infty((E_k))$ is r-convex.*

(ii) *$[\mu,(r)]_\infty((E_k))\subseteq[\mu,P]_\infty((E_k))$, where $\mu=(h_{nk})$ is the matrix defined by $h_{nk}=1$, if $0<\sup_n a_{nk}<\infty$, $a_{nk}>0$, and $h_{nk}=0$, otherwise.*

(iii) *There exists an integer $i>1$ such that*

$$\sup_n\sum_{s(n)} i^{\pi_k}<\infty, \tag{4.41}$$

where $s(n)=\{k\mid a_{nk}>0,\ p_k<r,\ \sup_n a_{nk}<\infty\}$, for each $n\in\mathbb{N}$ and $1/\pi_k+r/p_k=1$, for each $k\in\mathbb{N}$.

Proof. Since it follows immediately from Lemma 3.6 that (ii) implies (iii), we only show (i) implies (ii) and (iii) implies (i).

(i)⇒(ii). Let $x\in[\mu,(r)]_\infty((E_k))$. Then there exists $v\ge 1$ such that

$$\sup_n\sum_k h_{nk}\|x_k\|^r\le v. \tag{4.42}$$

Since $[A,P]_\infty((E_k))$ is r-convex, there exist an r-convex neighborhood U of the origin and a real number $d>0$ such that $s(d)\subseteq U\subseteq s(1)$. Let $k\in\mathbb{N}$ be such that $0<\sup_n a_{nk}<\infty$. Define the sequence (y^k) by $y^k=[d^M/\sup_n a_{nk}]^{1/p_k}(0,0,0,\dots,x_k/\|x_k\|,0,0,\dots)$, if $x_k\ne 0$ and $y^k=(0,0,0,\dots)$, if $x_k=0$. Clearly, $(y^k)\in s(d)\subseteq U$. Writing $\lambda_k=\|x_k\|v^{-1/r}$ for each $k\in\mathbb{N}$ and $\tilde{\sum}_k$ for any finite sum over k for which $h_{mk}=1$, $m\ge 1$ being a fixed integer, we see that

$$\tilde{\sum_k}|\lambda|^r=\tilde{\sum_k}h_{mk}\|x_k\|^r v^{-1}\le 1. \tag{4.43}$$

Then it follows from the r-convexity of U that $\sum_k\lambda_k y_k\in U$. Since $U\subseteq s(1)$, we have

$$\tilde{\sum_k}a_{mk}|\lambda|^{p_k}\left[\frac{d^M}{\sup_n a_{nk}}\right]\le 1, \tag{4.44}$$

which implies that

$$\tilde{\sum_k}h_{mk}|\lambda|^{p_k}\le\alpha^{-1}d^{-M}, \tag{4.45}$$

for each finite sum over k for which $h_{mk} = 1$. Now, since $v \geq 1$,

$$\tilde{\sum_k} h_{mk} \|x_k\|^{p_k} \leq v^{M/r} \alpha^{-1} d^{-M}. \tag{4.46}$$

This is true for every $m \in \mathbb{N}$. So it follows that $x \in [\mu, P]_\infty((E_k))$ and consequently, (ii) follows.

(iii)⇒(i). Suppose that (4.41) holds. To show the r-convexity of $[A,P]_\infty((E_k))$, we construct an r-convex neighborhood base at $\theta \in [A,P]_\infty((E_k))$. For each $k \in \mathbb{N}$, let $q_k = \max_k(r, p_k)$ and for each $0 < d < 1$, define

$$\begin{aligned} U_1(d) &= \left\{ x \in [A,P]_\infty((E_k)) \;\middle|\; \sup_n \sum_k \left(a_{nk} \|x_k\|^{p_k}\right)^{q_k/p_k} \leq d \right\}, \\ U_2(d) &= \left\{ x \in [A,P]_\infty((E_k)) \;\middle|\; \sup_{n,k} \left(a_{nk} \|x_k\|^{p_k}\right)^{q_k/p_k} \leq d \right\}, \end{aligned} \tag{4.47}$$

and $U(d) = U_1(d) \cap U_2(d)$. If $x, y \in U(d)$ and $|\lambda|^r + |\mu|^r \leq 1$, then by considering the cases $q_k < 1$ and $q_k \geq 1$ separately and applying Lemma 3.2, we obtain

$$\|\lambda x_k + \mu y_k\|^{q_k} \leq |\lambda|^r \|x_k\|^{q_k} + |\mu|^r \|y_k\|^{q_k}. \tag{4.48}$$

Therefore, we have

$$\sup_n \sum_k \left[a_{nk} \|\lambda x_k + \mu y_k\|^{p_k}\right]^{q_k/p_k} \leq \left(|\lambda|^r + |\mu|^r\right) d \leq d, \tag{4.49}$$

which implies that $\lambda x + \mu y \in U_1(d)$. Similarly, we show that $\lambda x + \mu y \in U_2(d)$. Hence $U(d)$ is r-convex. Since $s(d^{1/M}) \subseteq U(d)$ whenever $0 < d < 1$, it follows that $U(d)$ is a neighborhood of θ. To prove that the set of all the $U(d)$, for $0 < d < 1$, form a neighborhood base at θ, it suffices to show that for each $\epsilon > 0$, there exist $0 < d < 1$ such that $U(d) \subseteq s(\epsilon)$. Let $t(n) = \{k \in s(n) \mid p_k < r/2\}$. Since $-1 \leq \pi_k < 0$, for each $k \in t(n)$, it follows from (4.41) that $t(n)$ is a finite set for each n. Let $u(n)$ be the number of elements in $t(n)$. Since

$$\sum_{s(n)} i^{\pi_k} \geq \sum_{t(n)} i^{-1} = i^{-1} u(n), \tag{4.50}$$

for each n, it follows that $\mu_1 = \sup_n u(n) < \infty$. If $x \in U(d)$ for some $0 < d < 1$, observe that

$$\begin{aligned} \sum_{p_k \geq r} a_{nk} \|x_k\|^{p_k} &= \sum_{p_k \geq r} \left[a_{nk} \|x_k\|^{p_k}\right]^{q_k/p_k} \leq d, \\ \sum_{p_k < r/2} a_{nk} \|x_k\|^{p_k} &\leq \sup_{n,k} \|x_k\|^{p_k} \sum_{p_k < r/2} 1 \leq d \mu_1. \end{aligned} \tag{4.51}$$

Also, since $q_k = r$ for $r/2 \leq p_k < r$, it follows from Lemma 3.3 that

$$\begin{aligned} \sum_{r/2 \leq p_k < r} a_{nk} \|x_k\|^{p_k} &= \sum_{r/2 \leq p_k < r} \left(a_{nk}^{1/p_k} \|x_k\|\right)^{p_k} \\ &\leq \sum_{r/2 \leq p_k < r} \left(a_{nk}^{1/p_k} \|x_k\|^r\right)(1 + T \log T) + \sum_{r/2 \leq p_k < r} T^{p i_k}, \end{aligned} \tag{4.52}$$

for each $T>1$. If $r/2\le p_k<r$, then $\pi_k<-1$, so that

$$\sum_{r/2\le p_k<r}(jT)^{\pi_k}\le j^{-1}\sum_{s(n)}T^{\pi_k}, \tag{4.53}$$

for any positive integer j. Therefore, $\sup_n\sum_{r/2\le p_k<r}T^{\pi_k}$ can be made arbitrarily small by a suitable choice of T and by the fact that $\mu_1<\infty$. For a given $\epsilon>0$ choose $T>1$ such that

$$\sup_n\sum_{r/2\le p_k<r}i^{\pi_k}<\left(\frac{\epsilon}{2}\right)^M, \tag{4.54}$$

and choose $0<d<1$ such that

$$d(2+\mu_1+T\log T)<\left(\frac{\epsilon}{2}\right)^M. \tag{4.55}$$

Then,

$$\begin{aligned} g(x) &\le \left(\sum_{p_k\ge r}a_{nk}\|x_k\|^{p_k}+\sum_{p_k<r/2}a_{nk}\|x_k\|^{p_k}+\sum_{r/2\le p_k<r}a_{nk}\|x_k\|^{p_k}\right)^{1/M}\\ &\le\left[d+\mu_1\,d+(1+T\log T)\,d+\left(\frac{\epsilon}{2}\right)^M\right]^{1/M}<\epsilon, \end{aligned} \tag{4.56}$$

which shows that $x\in s(\epsilon)$.

This completes the proof of Theorem 4.10. □

Unlike other properties, r-convexity of the sequence space $[A,P]_\infty((E_k))$ does not depend on the r-convexity of the space E_k.

Remark 4.11. In Theorem 4.10 we may replace the sequence space $[A,P]_\infty((E_k))$, leaving the rest unchanged. The new result is still valid.

Corollary 4.12. *The following statements are equivalent:*

(i) $l(P,(E_k))$ *is* r*-convex;*

(ii) $0<r\le 1$ *and* $l((r),(E_k))\subseteq l(P,(E_k))$*;*

(iii) $0<r\le 1$ *and there exists an integer* $i>1$ *such that*

$$\sum_k i^{\pi_k}<\infty, \tag{4.57}$$

where $1/p_k+r/\pi_k=1$ *for each* k *and the summation is over* k *such that* $p_k<r$.

The proof is analogous to a result in [14, Theorem 1], which is special case of Theorem 4.10.

Corollary 4.13. *The following statements are equivalent:*

(i) $w_\infty(P,((E_k)))$ *is* r*-convex.*

(ii) $0<r\le 1$, $0<\inf_k p_k$, *and* $[B,(r)]_\infty((E_k))\subseteq[B,P]_\infty((E_k))$, *where* $b_{nk}=1$ *for* $2^{n-1}\le k<2^n$ *and* $b_{nk}=0$ *otherwise.*

(iii) $0<r\le 1$, $\inf_k p_k>0$, *and there exists an integer* $i>1$ *such that*

$$\sup_n\sum_{s(n)}i^{\pi_k}<\infty, \tag{4.58}$$

where $s(n)=\{k\mid 2^{n-1}\le k<2^n,\ p_k<r\}$ *and* $1/\pi_k+r/p_k=1$.

Proof. Let $w_\infty(P,((E_k)))$ be r-convex. Then by Lemma 3.4, $0<\inf_k p_k \le \sup_k p_k < \infty$. Since $w_\infty(p,(E_k))$ is not the only neighborhood of the origin, by Lemma 3.1, $0<r\le 1$. The rest of the proof of this corollary follows from Theorem 4.10 with $A = D$, where $D = (d_{nk})$ is the matrix defined by

$$d_{nk} = \begin{cases} 2^{n-1}, & \text{if } 2^{n-1} \le k < 2^n \text{ for each } n, \\ 0, & \text{otherwise.} \end{cases} \tag{4.59}$$

Here we make use of the fact that both the matrices D and the Cesaro matrix $(C,1)$ generate the same paranorm topology on $w_\infty(P,(E_k))$ (cf. [15, page 70]). □

Corollary 4.14. *The space $l_\infty(P,(E_k))$ is 1-convex if and only if $\inf_k p_k > 0$.*

Proof. By Lemma 3.7, $l_\infty(P,E_k)$ is a linear topological space if and only if $\inf_k p_k > 0$. The rest of the proof can be deduced from Theorem 4.10 by putting $A = I$ and $i = 2$ in (4.41). □

By Remark 4.11, we have the following result.

Corollary 4.15. *The sequence space $c_0(P,(E_k))$ is 1-convex.*

There are several other topological properties of the vector-valued sequence spaces $[A,P]_\infty((E_k))$, $[A,P]_0((E_k))$, and $[A,P](E)$, which still remain to be investigated. The construction of continuous duals and Köthe-Toeplitz duals of these spaces will also be interesting, since these spaces generalize the existing sequence spaces. Needless to say, there can be many applications of these three generalized sequence spaces in the study of topological and geometric properties of all our real and complex sequences.

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