IJMMS 27:1 (2001) 53–63
PII. S0161171201010882
http://ijmms.hindawi.com

FUZZY r-CONTINUOUS AND FUZZY r-SEMICONTINUOUS MAPS

SEOK JONG LEE and EUN PYO LEE

(Received 7 September 2000)

ABSTRACT. We introduce a new notion of fuzzy r-interior which is an extension of Chang's fuzzy interior. Using fuzzy r-interior, we define fuzzy r-semiopen sets and fuzzy r-semicontinuous maps which are generalizations of fuzzy semiopen sets and fuzzy semicontinuous maps in Chang's fuzzy topology, respectively. Some basic properties of fuzzy r-semiopen sets and fuzzy r-semicontinuous maps are investigated.

2000 Mathematics Subject Classification. 54A40.

1. Introduction. Chang [2] introduced fuzzy topological spaces. Some authors [3, 5, 6, 7, 8] introduced other definitions of fuzzy topology as generalizations of Chang's fuzzy topology.

In this note, we introduce a new notion of fuzzy r-interior in a similar method by which Chattopadhyay and Samanta [4] introduced the notion of fuzzy closure. It determines a fuzzy topology and it is an extension of Chang's fuzzy interior.

Using fuzzy r-interior, we define fuzzy r-semiopen sets and fuzzy r-semicontinuous maps which are generalizations of fuzzy semiopen sets and fuzzy semicontinuous maps in Chang's fuzzy topology, respectively. Some basic properties of fuzzy r-semiopen sets and fuzzy r-semicontinuous maps are investigated.

2. Preliminaries. In this note, let I denote the unit interval $[0,1]$ of the real line and $I_0 = (0,1]$. A member μ of I^X is called a fuzzy subset of X. For any $\mu \in I^X$, μ^c denotes the complement $1-\mu$. By $\tilde{0}$ and $\tilde{1}$ we denote constant maps on X with value 0 and 1, respectively. All other notation are standard notation of fuzzy set theory.

Recall that a *Chang's fuzzy topology* (see [2]) on X is a family T of fuzzy sets in X which satisfies the following properties:

(1) $\tilde{0}, \tilde{1} \in T$;
(2) if $\mu_1, \mu_2 \in T$, then $\mu_1 \wedge \mu_2 \in T$;
(3) if $\mu_i \in T$ for each i, then $\bigvee \mu_i \in T$.

The pair (X,T) is called a *Chang's fuzzy topological space*.

Hence a Chang's fuzzy topology on X can be regarded as a map $T : I^X \to \{0,1\}$ which satisfies the following three conditions:

(1) $T(\tilde{0}) = T(\tilde{1}) = 1$;
(2) if $T(\mu_1) = T(\mu_2) = 1$, then $T(\mu_1 \wedge \mu_2) = 1$;
(3) if $T(\mu_i) = 1$ for each i, then $T(\bigvee \mu_i) = 1$.

But fuzziness in the concept of openness of a fuzzy subset is absent in the above Chang's definition of fuzzy topology. So for fuzzifying the openness of a fuzzy subset, some authors [3, 5, 6] gave other definitions of fuzzy topology.

Definition 2.1 (see [3, 7, 8]). A *fuzzy topology* on X is a map $\mathcal{T}: I^X \to I$ which satisfies the following properties:

(1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$,

(2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$,

(3) $\mathcal{T}(\bigvee \mu_i) \geq \bigwedge \mathcal{T}(\mu_i)$.

The pair $(X, \mathcal{T})$ is called a *fuzzy topological space*.

Definition 2.2 (see [3]). A *family of closed sets* in X is a map $\mathcal{F}: I^X \to I$ satisfying the following properties:

(1) $\mathcal{F}(\tilde{0}) = \mathcal{F}(\tilde{1}) = 1$,

(2) $\mathcal{F}(\mu_1 \vee \mu_2) \geq \mathcal{F}(\mu_1) \wedge \mathcal{F}(\mu_2)$,

(3) $\mathcal{F}(\bigwedge \mu_i) \geq \bigwedge \mathcal{F}(\mu_i)$.

Let $\mathcal{T}$ be a fuzzy topology on X and $\mathcal{F}_{\mathcal{T}}: I^X \to I$ a map defined by $\mathcal{F}_{\mathcal{T}}(\mu) = \mathcal{T}(\mu^c)$. Then $\mathcal{F}_{\mathcal{T}}$ is a family of closed sets in X. Also, let $\mathcal{F}$ be a family of closed sets in X and $\mathcal{T}_{\mathcal{F}}: I^X \to I$ a map defined by $\mathcal{T}_{\mathcal{F}}(\mu) = \mathcal{F}(\mu^c)$. Then $\mathcal{T}_{\mathcal{F}}$ is a fuzzy topology on X.

The notions of fuzzy semiopen, semiclosed sets and the weaker forms of fuzzy continuity which are related to our discussion, can be found in [1, 9].

Definition 2.3 (see [4]). Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy r-closure* is defined by

$$\operatorname{cl}(\mu, r) = \bigwedge \{\rho \in I^X \mid \mu \leq \rho,\ \mathcal{F}_{\mathcal{T}}(\rho) \geq r\}. \tag{2.1}$$

From now on, for $r \in I_0$ we will call μ a *fuzzy r-open set* of X if $\mathcal{T}(\mu) \geq r$, μ a *fuzzy r-closed set* of X if $\mathcal{F}(\mu) \geq r$. Note that μ is fuzzy r-closed if and only if $\mu = \operatorname{cl}(\mu, r)$.

Let $(X, \mathcal{T})$ be a fuzzy topological space. For an r-cut $\mathcal{T}_r = \{\mu \in I^X \mid \mathcal{T}(\mu) \geq r\}$, it is obvious that $(X, \mathcal{T}_r)$ is a Chang's fuzzy topological space for all $r \in I_0$.

3. Fuzzy r-interior.

Now, we are going to define the fuzzy interior operator in $(X, \mathcal{T})$.

Definition 3.1. Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $\mu \in I^X$ and each $r \in I_0$, the *fuzzy r-interior of μ* is defined as follows:

$$\operatorname{int}(\mu, r) = \bigvee \{\rho \in I^X \mid \mu \geq \rho,\ \mathcal{T}(\rho) \geq r\}. \tag{3.1}$$

The operator $\operatorname{int}: I^X \times I_0 \to I^X$ is called the *fuzzy interior operator* in $(X, \mathcal{T})$.

Obviously, $\operatorname{int}(\mu, r)$ is the greatest fuzzy r-open set which is contained in μ and $\operatorname{int}(\mu, r) = \mu$ for any fuzzy r-open set μ. Moreover, we have the following results.

Theorem 3.2. *Let $(X, \mathcal{T})$ be a fuzzy topological space and $\operatorname{int}: I^X \times I_0 \to I^X$ the fuzzy interior operator in $(X, \mathcal{T})$. Then for $\mu, \rho \in I^X$ and $r, s \in I_0$,*

(1) $\operatorname{int}(\tilde{0}, r) = \tilde{0}$, $\operatorname{int}(\tilde{1}, r) = \tilde{1}$,

(2) $\operatorname{int}(\mu, r) \leq \mu$,

(3) $\operatorname{int}(\mu, r) \geq \operatorname{int}(\mu, s)$ *if* $r \leq s$,

(4) $\operatorname{int}(\mu \wedge \rho, r) = \operatorname{int}(\mu, r) \wedge \operatorname{int}(\rho, r)$,

(5) $\operatorname{int}(\operatorname{int}(\mu, r), r) = \operatorname{int}(\mu, r)$,

(6) *if* $r = \bigvee \{s \in I_0 \mid \operatorname{int}(\mu, s) = \mu\}$, *then* $\operatorname{int}(\mu, r) = \mu$.

Proof. (1), (2), and (5) are obvious. (3) Let $r \leq s$. Then every fuzzy s-open set is also fuzzy r-open. Hence we have

$$\begin{aligned} \operatorname{int}(\mu,r) &= \bigvee \{\rho \in I^X \mid \mu \geq \rho,\ \mathcal{T}(\rho) \geq r\} \\ &\geq \bigvee \{\rho \in I^X \mid \mu \geq \rho,\ \mathcal{T}(\rho) \geq s\} \\ &= \operatorname{int}(\mu,s). \end{aligned} \tag{3.2}$$

(4) Since $\mu \wedge \rho \leq \mu$ and $\mu \wedge \rho \leq \rho$, $\operatorname{int}(\mu\wedge\rho,r) \leq \operatorname{int}(\mu,r)$ and $\operatorname{int}(\mu\wedge\rho,r) \leq \operatorname{int}(\rho,r)$. Thus $\operatorname{int}(\mu\wedge\rho,r) \leq \operatorname{int}(\mu,r)\wedge\operatorname{int}(\rho,r)$. Conversely, it is clear that $\mu\wedge\rho \geq \operatorname{int}(\mu,r)\wedge\operatorname{int}(\rho,r)$. Also,

$$\mathcal{T}\big(\operatorname{int}(\mu,r)\wedge\operatorname{int}(\rho,r)\big) \geq \mathcal{T}\big(\operatorname{int}(\mu,r)\big)\wedge\mathcal{T}\big(\operatorname{int}(\rho,r)\big) \geq r\wedge r = r. \tag{3.3}$$

So, by the definition of fuzzy r-interior, $\operatorname{int}(\mu\wedge\rho,r) \geq \operatorname{int}(\mu,r)\wedge\operatorname{int}(\rho,r)$. Hence $\operatorname{int}(\mu\wedge\rho,r) = \operatorname{int}(\mu,r)\wedge\operatorname{int}(\rho,r)$.

(6) Note that $\mathcal{T}(\mu) \geq r$ if and only if $\operatorname{int}(\mu,r) = \mu$. Suppose that $\operatorname{int}(\mu,r) \neq \mu$. Then $\mathcal{T}(\mu) < r$ and hence there is an $\alpha \in I$ such that $\mathcal{T}(\mu) < \alpha < r$. Since $r = \bigvee\{s \in I_0 \mid \operatorname{int}(\mu,s) = \mu\}$, there is an $s \in I$ such that $\mathcal{T}(\mu) < \alpha < s \leq r$ and $\operatorname{int}(\mu,s) = \mu$. Since $\mathcal{T}(\mu) < s$, $\operatorname{int}(\mu,s) \neq \mu$. This is a contradiction. □

Theorem 3.3. *Let* $\operatorname{int} : I^X \times I_0 \to I^X$ *be a map satisfying (1), (2), (3), (4), (5), and (6) of Theorem 3.2. Let* $\mathcal{T} : I^X \to I$ *be a map defined by*

$$\mathcal{T}(\mu) = \bigvee \{r \in I_0 \mid \operatorname{int}(\mu,r) = \mu\}. \tag{3.4}$$

Then $\mathcal{T}$ *is a fuzzy topology on* X *such that* $\operatorname{int} = \operatorname{int}_{\mathcal{T}}$.

Proof. (i) By (1), $\mathcal{T}(\tilde{0}) = 1 = \mathcal{T}(\tilde{1})$.

(ii) Suppose that $\mathcal{T}(\mu_1\wedge\mu_2) < \mathcal{T}(\mu_1)\wedge\mathcal{T}(\mu_2)$. Then there is an $\alpha \in I$ such that $\mathcal{T}(\mu_1\wedge\mu_2) < \alpha < \mathcal{T}(\mu_1)\wedge\mathcal{T}(\mu_2)$. So, there are $s_1, s_2 \in I$ such that $\alpha < s_i \leq \mathcal{T}(\mu_i)$ and $\operatorname{int}(\mu_i,s_i) = \mu_i$ for each $i = 1,2$. Let $s = s_1\wedge s_2$. Then $\operatorname{int}(\mu_i,s) \geq \operatorname{int}(\mu_i,s_i) = \mu_i$ and hence $\operatorname{int}(\mu_i,s) = \mu_i$ for each $i = 1,2$. By (4), $\operatorname{int}(\mu_1\wedge\mu_2,s) = \operatorname{int}(\mu_1,s)\wedge\operatorname{int}(\mu_2,s) = \mu_1\wedge\mu_2$. Thus

$$\alpha > \mathcal{T}(\mu_1\wedge\mu_2) = \bigvee \{r \in I_0 \mid \operatorname{int}(\mu_1\wedge\mu_2,r) = \mu_1\wedge\mu_2\} \geq s = s_1\wedge s_2 > \alpha. \tag{3.5}$$

This is a contradiction. Therefore $\mathcal{T}(\mu_1\wedge\mu_2) \geq \mathcal{T}(\mu_1)\wedge\mathcal{T}(\mu_2)$.

(iii) Suppose $\mathcal{T}(\bigvee\mu_i) < \bigwedge\mathcal{T}(\mu_i)$. Then there is an $\alpha \in I$ such that $\mathcal{T}(\bigvee\mu_i) < \alpha < \bigwedge\mathcal{T}(\mu_i)$. So for each i, there is an $s_i \in I$ such that $\alpha < s_i \leq \mathcal{T}(\mu_i)$ and $\operatorname{int}(\mu_i,s_i) = \mu_i$. Let $s = \bigwedge s_i$. Then $\operatorname{int}(\mu_i,s) \geq \operatorname{int}(\mu_i,s_i) = \mu_i$ and hence $\operatorname{int}(\bigvee\mu_i,s) \geq \operatorname{int}(\mu_i,s) = \mu_i$ for each i. Thus $\operatorname{int}(\bigvee\mu_i,s) \geq \bigvee\mu_i$ and hence $\operatorname{int}(\bigvee\mu_i,s) = \bigvee\mu_i$. Hence

$$\alpha > \mathcal{T}\Big(\bigvee\mu_i\Big) \geq s \geq \alpha. \tag{3.6}$$

This is a contradiction. Therefore $\mathcal{T}(\bigvee\mu_i) \geq \bigwedge\mathcal{T}(\mu_i)$.

Next we will show that $\operatorname{int} = \operatorname{int}_{\mathcal{T}}$. Note that for $s \leq r$,

$$\operatorname{int}(\mu,r) = \operatorname{int}\big(\operatorname{int}(\mu,r),r\big) \leq \operatorname{int}\big(\operatorname{int}(\mu,r),s\big) \leq \operatorname{int}(\mu,r). \tag{3.7}$$

So $\operatorname{int}(\mu,r) = \operatorname{int}(\operatorname{int}(\mu,r),s)$ for $s \le r$ and $\operatorname{int}(\mu,r) \le \mu$. Thus

$$\begin{aligned}\operatorname{int}_{\mathcal{T}}(\mu,r) &= \bigvee\{\rho \in I^X \mid \rho \le \mu,\ \mathcal{T}(\rho) \ge r\}\\ &= \bigvee\Big\{\rho \in I^X \mid \rho \le \mu,\ \bigvee\{s \in I_0 \mid \operatorname{int}(\rho,s) = \rho\} \ge r\Big\}\\ &= \bigvee\{\rho \in I^X \mid \rho \le \mu,\ \operatorname{int}(\rho,s) = \rho \text{ for } s \le r\}\\ &\ge \operatorname{int}(\mu,r).\end{aligned} \tag{3.8}$$

On the other hand, let $\rho \le \mu$ and $\operatorname{int}(\rho,s) = \rho$ for $s \le r$. Then by (6), $\rho = \operatorname{int}(\rho,r) \le \operatorname{int}(\mu,r)$. Thus

$$\operatorname{int}_{\mathcal{T}}(\mu,r) = \bigvee\{\rho \in I^X \mid \rho \le \mu,\ \operatorname{int}(\rho,s) = \rho \text{ for } s \le r\} \le \operatorname{int}(\mu,r). \tag{3.9}$$

Therefore, $\operatorname{int}_{\mathcal{T}}(\mu,r) = \operatorname{int}(\mu,r)$. Hence the theorem follows. □

If $\operatorname{int}: I^X \times I_0 \to I^X$ is a fuzzy interior operator on X, then for each $r \in I_0$, $\operatorname{int}_r: I^X \to I^X$ defined by

$$\operatorname{int}_r(\mu) = \operatorname{int}(\mu,r) \tag{3.10}$$

is a Chang's fuzzy interior operator on X.

Fuzzy r-interior is an extension of the Chang's fuzzy interior.

Theorem 3.4. *An operator* $\operatorname{int}: I^X \times I_0 \to I^X$ *is a fuzzy interior for the fuzzy topological space* $(X,\mathcal{T})$ *if and only if for any* $r \in I_0$, $\operatorname{int}_r: I^X \to I^X$ *is a Chang's fuzzy interior for the Chang's fuzzy topological space* $(X,\mathcal{T}_r)$.

Proof. (⇒). This direction (⇒) is obvious.

(⇐). (1), (2), (4), and (5) are obvious.

(3) Let $r \le s$. Then $\mathcal{T}_r \supseteq \mathcal{T}_s$ and hence $\operatorname{int}(\mu,r) = \operatorname{int}_r(\mu) = \bigvee\{\rho \in I^X \mid \rho \le \mu,\ \rho \in \mathcal{T}_r\} \ge \bigvee\{\rho \in I^X \mid \rho \le \mu,\ \rho \in \mathcal{T}_s\} = \operatorname{int}_s(\mu) = \operatorname{int}(\mu,s)$.

(6) Suppose that $\operatorname{int}(\mu,r) \ne \mu$. Then $\operatorname{int}_r(\mu) = \operatorname{int}(\mu,r) \ne \mu$. So $\mu \notin \mathcal{T}_r$ and hence $\mathcal{T}(\mu) < r$. Thus there is an $\alpha \in I$ such that $\mathcal{T}(\mu) < \alpha < r$. Since $r = \bigvee\{s \in I_0 \mid \operatorname{int}(\mu,s) = \mu\}$, there is an $s \in I_0$ such that $\mathcal{T}(\mu) < \alpha < s \le r$ and $\operatorname{int}(\mu,s) = \operatorname{int}_s(\mu) = \mu$. Since $\mathcal{T}(\mu) < s$, $\mu \notin \mathcal{T}_s$ and hence $\operatorname{int}_s(\mu) \ne \mu$. It is a contradiction. □

For a family $\{\mu_i\}_{i\in\Gamma}$ of fuzzy sets in a fuzzy topological space X and $r \in I_0$, $\bigvee \operatorname{cl}(\mu_i,r) \le \operatorname{cl}(\bigvee \mu_i,r)$, and the equality holds when Γ is a finite set. Similarly $\bigwedge \operatorname{int}(\mu_i,r) \ge \operatorname{int}(\bigwedge \mu_i,r)$ and $\bigwedge \operatorname{int}(\mu_i,r) = \operatorname{int}(\bigwedge \mu_i,r)$ for a finite set Γ.

Theorem 3.5. *For a fuzzy set* μ *in a fuzzy topological space* X *and* $r \in I_0$,

(1) $\operatorname{int}(\mu,r)^c = \operatorname{cl}(\mu^c,r)$.

(2) $\operatorname{cl}(\mu,r)^c = \operatorname{int}(\mu^c,r)$.

Proof.

$$\begin{aligned}\operatorname{int}(\mu,r)^c &= \Big(\bigvee\{\rho \in I^X \mid \rho \le \mu,\ \mathcal{T}(\rho) \ge r\}\Big)^c\\ &= \bigwedge\{\rho^c \in I^X \mid \rho^c \ge \mu^c,\ \mathcal{F}_{\mathcal{T}}(\rho^c) \ge r\}\\ &= \operatorname{cl}(\mu^c,r).\end{aligned} \tag{3.11}$$

Similarly we can show (2). □

4. Fuzzy r-semiopen sets

Definition 4.1. Let μ be a fuzzy set in a fuzzy topological space $(X,\mathcal{T})$ and $r \in I_0$. Then μ is said to be

(1) *fuzzy r-semiopen* if there is a fuzzy r-open set ρ in X such that $\rho \le \mu \le \mathrm{cl}(\rho,r)$,

(2) *fuzzy r-semiclosed* if there is a fuzzy r-closed set ρ in X such that $\mathrm{int}(\rho,r) \le \mu \le \rho$.

Theorem 4.2. *Let μ be a fuzzy set in a fuzzy topological space $(X,\mathcal{T})$ and $r \in I_0$. Then the following statements are equivalent:*

(1) *μ is a fuzzy r-semiopen set.*

(2) *μ^c is a fuzzy r-semiclosed set.*

(3) $\mathrm{cl}(\mathrm{int}(\mu,r),r) \ge \mu$.

(4) $\mathrm{int}(\mathrm{cl}(\mu^c,r),r) \le \mu^c$.

Proof. (1)⇔(2). The proof follows from Theorem 3.5.

(1)⇒(3). Let μ be a fuzzy r-semiopen set of X. Then there is a fuzzy r-open set ρ in X such that $\rho \le \mu \le \mathrm{cl}(\rho,r)$. Since $\mathcal{T}(\rho) \ge r$ and $\mu \ge \rho$, $\mathrm{int}(\mu,r) \ge \rho$. Hence $\mathrm{cl}(\mathrm{int}(\mu,r),r) \ge \mathrm{cl}(\rho,r) \ge \mu$.

(3)⇒(1). Let $\mathrm{cl}(\mathrm{int}(\mu,r),r) \ge \mu$ and take $\rho = \mathrm{int}(\mu,r)$. Since $\mathcal{T}(\mathrm{int}(\mu,r)) \ge r$, ρ is a fuzzy r-open set. Also, $\rho = \mathrm{int}(\mu,r) \le \mu \le \mathrm{cl}(\mathrm{int}(\mu,r),r) = \mathrm{cl}(\rho,r)$. Hence μ is a fuzzy r-semiopen set.

(2)⇔(4). The proof is similar to the proof of (1)⇔(3). □

Theorem 4.3. (1) *Any union of fuzzy r-semiopen sets is fuzzy r-semiopen.*

(2) *Any intersection of fuzzy r-semiclosed sets is fuzzy r-semiclosed.*

Proof. (1) Let $\{\mu_i\}$ be a collection of fuzzy r-semiopen sets. Then for each i, there is a fuzzy r-open set ρ_i such that $\rho_i \le \mu_i \le \mathrm{cl}(\rho_i,r)$. Since $\mathcal{T}(\bigvee \rho_i) \ge \bigwedge \mathcal{T}(\rho_i) \ge r$, $\bigvee \rho_i$ is a fuzzy r-open set. Moreover,

$$\bigvee \rho_i \le \bigvee \mu_i \le \bigvee \mathrm{cl}(\rho_i,r) \le \mathrm{cl}\Big(\bigvee \rho_i, r\Big). \tag{4.1}$$

Hence $\bigvee \mu_i$ is a fuzzy r-semiopen set.

(2) It follows from (1) using Theorem 4.2. □

Definition 4.4. Let $(X,\mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy r-semiclosure* is defined by

$$\mathrm{scl}(\mu,r) = \bigwedge \{\rho \in I^X \mid \mu \le \rho,\ \rho \text{ is fuzzy } r\text{-semiclosed}\} \tag{4.2}$$

and the *fuzzy r-semi-interior* is defined by

$$\mathrm{sint}(\mu,r) = \bigvee \{\rho \in I^X \mid \mu \ge \rho,\ \rho \text{ is fuzzy } r\text{-semiopen}\}. \tag{4.3}$$

Obviously $\mathrm{scl}(\mu,r)$ is the smallest fuzzy r-semiclosed set which contains μ and $\mathrm{sint}(\mu,r)$ is the greatest fuzzy r-semiopen set which is contained in μ. Also, $\mathrm{scl}(\mu,r)=\mu$ for any fuzzy r-semiclosed set μ and $\mathrm{sint}(\mu,r) = \mu$ for any fuzzy r-semiopen set μ. Moreover, we have

$$\mathrm{int}(\mu,r) \le \mathrm{sint}(\mu,r) \le \mu \le \mathrm{scl}(\mu,r) \le \mathrm{cl}(\mu,r). \tag{4.4}$$

Also, we have the following results:

(1) $\operatorname{scl}(\tilde{0},r)=\tilde{0}$, $\operatorname{scl}(\tilde{1},r)=\tilde{1}$, $\operatorname{sint}(\tilde{0},r)=\tilde{0}$, $\operatorname{sint}(\tilde{1},r)=\tilde{1}$.

(2) $\operatorname{scl}(\mu,r)\geq\mu$, $\operatorname{sint}(\mu,r)\leq\mu$.

(3) $\operatorname{scl}(\mu\vee\rho,r)\geq\operatorname{scl}(\mu,r)\vee\operatorname{scl}(\rho,r)$, $\operatorname{sint}(\mu\wedge\rho,r)\leq\operatorname{sint}(\mu,r)\wedge\operatorname{sint}(\rho,r)$.

(4) $\operatorname{scl}(\operatorname{scl}(\mu,r),r)=\operatorname{scl}(\mu,r)$, $\operatorname{sint}(\operatorname{sint}(\mu,r),r)=\operatorname{sint}(\mu,r)$.

Remark 4.5. It is obvious that every fuzzy r-open (r-closed) set is fuzzy r-semiopen (r-semiclosed). The converse does not hold as in Example 4.6. It also shows that the intersection (union) of any two fuzzy r-semiopen (r-semiclosed) sets need not be fuzzy r-semiopen (r-semiclosed). Even the intersection (union) of a fuzzy r-semiopen (r-semiclosed) set with a fuzzy r-open (r-closed) set may fail to be fuzzy r-semiopen (r-semiclosed).

Example 4.6. Let $X=I$ and μ_1,μ_2 and μ_3 be fuzzy sets of X defined as

$$\mu_1(x)=\begin{cases}0, & \text{if } 0\leq x\leq\frac{1}{2},\\ 2x-1, & \text{if } \frac{1}{2}\leq x\leq 1;\end{cases}$$

$$\mu_2(x)=\begin{cases}1, & \text{if } 0\leq x\leq\frac{1}{4},\\ -4x+2, & \text{if } \frac{1}{4}\leq x\leq\frac{1}{2},\\ 0, & \text{if } \frac{1}{2}\leq x\leq 1;\end{cases} \tag{4.5}$$

$$\mu_3(x)=\begin{cases}0, & \text{if } 0\leq x\leq\frac{1}{4},\\ \frac{1}{3}(4x-1), & \text{if } \frac{1}{4}\leq x\leq 1.\end{cases}$$

Define $\mathcal{T}:I^X\to I$ by

$$\mathcal{T}(\mu)=\begin{cases}1 & \text{if } \mu=\tilde{0},\tilde{1},\\ \frac{1}{2} & \text{if } \mu=\mu_1,\mu_2,\ \mu_1\vee\mu_2,\\ 0 & \text{otherwise.}\end{cases} \tag{4.6}$$

Then clearly $\mathcal{T}$ is a fuzzy topology on X.

(1) Note that $\operatorname{cl}(\mu_1,1/2)=\mu_2^c$. Since $\mu_1\leq\mu_3\leq\operatorname{cl}(\mu_1,1/2)$ and μ_1 is a fuzzy 1/2-open set, μ_3 is a fuzzy 1/2-semiopen set. But μ_3 is not a fuzzy 1/2-open set, because $\mathcal{T}(\mu_3)=0$.

(2) In view of Theorem 4.2, μ_3^c is a fuzzy 1/2-semiclosed set which is not a fuzzy 1/2-closed set.

(3) Note that μ_2 is fuzzy 1/2-open and hence fuzzy 1/2-semiopen. Since $\tilde{0}$ is the only fuzzy 1/2-open set contained in $\mu_2\wedge\mu_3$ and $\operatorname{cl}(\tilde{0},1/2)=\tilde{0}$, $\mu_2\wedge\mu_3$ is not a fuzzy 1/2-semiopen set.

(4) Clearly μ_2^c and μ_3^c are fuzzy 1/2-semiclosed sets, but $\mu_2^c\vee\mu_3^c=(\mu_2\wedge\mu_3)^c$ is not a fuzzy 1/2-semiclosed set.

The next two theorems show the relation between r-semiopenness and semiopenness.

Theorem 4.7. *Let μ be a fuzzy set in a fuzzy topological space $(X,\mathcal{T})$ and $r \in I_0$. Then μ is fuzzy r-semiopen (r-semiclosed) in $(X,\mathcal{T})$ if and only if μ is fuzzy semiopen (semiclosed) in $(X,\mathcal{T}_r)$.*

Proof. The proof is straightforward. □

Let (X,T) be a Chang's fuzzy topological space and $r \in I_0$. Recall [3] that a fuzzy topology $T^r : I^X \to I$ is defined by

$$T^r(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ r & \text{if } \mu \in T - \{\tilde{0}, \tilde{1}\}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

Theorem 4.8. *Let μ be a fuzzy set in a Chang's fuzzy topological space (X,T) and $r \in I_0$. Then μ is fuzzy semiopen (semiclosed) in (X,T) if and only if μ is fuzzy r-semiopen (r-semiclosed) in (X,T^r).*

Proof. The proof is straightforward. □

5. Fuzzy r-continuous and fuzzy r-semicontinuous maps

Definition 5.1. Let $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is called

(1) a *fuzzy r-continuous* map if $f^{-1}(\mu)$ is a fuzzy r-open set of X for each fuzzy r-open set μ of Y, or equivalently, $f^{-1}(\mu)$ is a fuzzy r-closed set of X for each fuzzy r-closed set μ of Y,

(2) a *fuzzy r-open* map if $f(\mu)$ is a fuzzy r-open set of Y for each fuzzy r-open set μ of X,

(3) a *fuzzy r-closed* map if $f(\mu)$ is a fuzzy r-closed set of Y for each fuzzy r-closed set μ of X,

(4) a *fuzzy r-homeomorphism* if f is bijective, fuzzy r-continuous and fuzzy r-open.

Theorem 5.2. *Let $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent:*

(1) *f is a fuzzy r-continuous map.*

(2) *$f(\mathrm{cl}(\rho,r)) \le \mathrm{cl}(f(\rho),r)$ for each fuzzy set ρ of X.*

(3) *$\mathrm{cl}(f^{-1}(\mu),r) \le f^{-1}(\mathrm{cl}(\mu,r))$ for each fuzzy set μ of Y.*

(4) *$f^{-1}(\mathrm{int}(\mu,r)) \le \mathrm{int}(f^{-1}(\mu),r)$ for each fuzzy set μ of Y.*

Proof. (1)⇒(2). Let f be fuzzy r-continuous and ρ any fuzzy set of X. Since $\mathrm{cl}(f(\rho),r)$ is fuzzy r-closed of Y, $f^{-1}(\mathrm{cl}(f(\rho),r))$ is fuzzy r-closed of X. Thus

$$\mathrm{cl}(\rho,r) \le \mathrm{cl}(f^{-1}f(\rho),r) \le \mathrm{cl}(f^{-1}(\mathrm{cl}(f(\rho),r)),r) = f^{-1}(\mathrm{cl}(f(\rho),r)). \tag{5.1}$$

Hence

$$f(\mathrm{cl}(\rho,r)) \le f f^{-1}(\mathrm{cl}(f(\rho),r)) \le \mathrm{cl}(f(\rho),r). \tag{5.2}$$

(2)⇒(3). Let μ be any fuzzy set of Y. By (2),

$$f(\mathrm{cl}(f^{-1}(\mu),r)) \le \mathrm{cl}(ff^{-1}(\mu),r) \le \mathrm{cl}(\mu,r). \tag{5.3}$$

Thus

$$\mathrm{cl}(f^{-1}(\mu),r) \le f^{-1}f(\mathrm{cl}(f^{-1}(\mu),r)) \le f^{-1}(\mathrm{cl}(\mu,r)). \tag{5.4}$$

(3)⇒(4). Let μ be any fuzzy set of Y. Then μ^c is a fuzzy set of Y. By (3),

$$\mathrm{cl}(f^{-1}(\mu)^c,r) = \mathrm{cl}(f^{-1}(\mu^c),r) \le f^{-1}(\mathrm{cl}(\mu^c,r)). \tag{5.5}$$

By Theorem 3.5,

$$f^{-1}(\mathrm{int}(\mu,r)) = f^{-1}(\mathrm{cl}(\mu^c,r))^c \le \mathrm{cl}(f^{-1}(\mu)^c,r)^c = \mathrm{int}(f^{-1}(\mu),r). \tag{5.6}$$

(4)⇒(1). Let μ be any fuzzy r-open set of Y. Then $\mathrm{int}(\mu,r) = \mu$. By (4),

$$f^{-1}(\mu) = f^{-1}(\mathrm{int}(\mu,r)) \le \mathrm{int}(f^{-1}(\mu),r) \le f^{-1}(\mu). \tag{5.7}$$

So $f^{-1}(\mu) = \mathrm{int}(f^{-1}(\mu),r)$ and hence $f^{-1}(\mu)$ is fuzzy r-open of X. Thus f is fuzzy r-continuous. □

Theorem 5.3. *Let $(X,\mathcal{T})$, $(Y,\mathcal{U})$ and $(Z,\mathcal{V})$ be three fuzzy topological spaces and $r \in I_0$. If $f:(X,\mathcal{T}) \to (Y,\mathcal{U})$ and $g:(Y,\mathcal{U}) \to (Z,\mathcal{V})$ are fuzzy r-continuous (r-open, r-closed) maps, then so is $g \circ f:(X,\mathcal{T}) \to (Z,\mathcal{V})$.*

Proof. The proof is straightforward. □

Definition 5.4. Let $f:(X,\mathcal{T}) \to (Y,\mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is called

(1) a *fuzzy r-semicontinuous* map if $f^{-1}(\mu)$ is a fuzzy r-semiopen set of X for each fuzzy r-open set μ of Y, or equivalently, $f^{-1}(\mu)$ is a fuzzy r-semiclosed set of X for each fuzzy r-closed set μ of Y,

(2) a *fuzzy r-semiopen* map if $f(\mu)$ is a fuzzy r-semiopen set of Y for each fuzzy r-open set μ of X,

(3) a *fuzzy r-semiclosed* map if $f(\mu)$ is a fuzzy r-semiclosed set of Y for each fuzzy r-closed set μ of X.

Theorem 5.5. *Let $f:(X,\mathcal{T}) \to (Y,\mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent:*

(1) *f is a fuzzy r-semicontinuous map.*

(2) $f(\mathrm{scl}(\rho,r)) \le \mathrm{cl}(f(\rho),r)$ *for each fuzzy set ρ of X.*

(3) $\mathrm{scl}(f^{-1}(\mu),r) \le f^{-1}(\mathrm{cl}(\mu,r))$ *for each fuzzy set μ of Y.*

(4) $f^{-1}(\mathrm{int}(\mu,r)) \le \mathrm{sint}(f^{-1}(\mu),r)$ *for each fuzzy set μ of Y.*

Proof. The proof is similar to Theorem 5.2. □

Remark 5.6. Let $f:(X,\mathcal{T}) \to (Y,\mathcal{U})$ and $g:(Y,\mathcal{U}) \to (Z,\mathcal{V})$ be maps and $r \in I_0$. Then the following statements are true.

(1) If f is fuzzy r-semicontinuous and g is fuzzy r-continuous then $g \circ f$ is fuzzy r-semicontinuous.

(2) If f is fuzzy r-open and g is fuzzy r-semiopen then $g \circ f$ is fuzzy r-semiopen.

(3) If f is fuzzy r-closed and g is fuzzy r-semiclosed then $g \circ f$ is fuzzy r-semiclosed.

Remark 5.7. In view of Remark 4.5, a fuzzy r-continuous (r-open, r-closed, resp.) map is also a fuzzy r-semicontinuous (r-semiopen, r-semiclosed, resp.) map for each $r \in I_0$. The converse does not hold as in the following example.

Example 5.8. (1) A fuzzy r-semicontinuous map need not be a fuzzy r-continuous map.

Let $(X, \mathcal{T})$ be a fuzzy topological space as described in Example 4.6 and let $f : (X, \mathcal{T}) \to (X, \mathcal{T})$ be defined by $f(x) = x/2$. Note that $f^{-1}(\tilde{0}) = \tilde{0}$, $f^{-1}(\tilde{1}) = \tilde{1}$, $f^{-1}(\mu_1) = \tilde{0}$ and $f^{-1}(\mu_2) = \mu_1^c = f^{-1}(\mu_1 \vee \mu_2)$. Since $\operatorname{cl}(\mu_2, 1/2) = \mu_1^c$, μ_1^c is a fuzzy $1/2$-semiopen set and hence f is a fuzzy $1/2$-semicontinuous map. On the other hand, $\mathcal{T}(f^{-1}(\mu_2)) = \mathcal{T}(\mu_1^c) = 0 < 1/2$, and hence $f^{-1}(\mu_2)$ is not a fuzzy $1/2$-open set. Thus f is not a fuzzy $1/2$-continuous map.

(2) A fuzzy r-semiopen map need not be a fuzzy r-open map.

Let $(X, \mathcal{T})$ be as in (1). Define $\mathcal{T}_1 : I^X \to I$ by

$$\mathcal{T}_1(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \dfrac{1}{2} & \text{if } \mu = \mu_3, \\ 0 & \text{otherwise}. \end{cases} \tag{5.8}$$

Consider the map $f : (X, \mathcal{T}_1) \to (X, \mathcal{T})$ defined by $f(x) = x$. Then $f(\tilde{0}) = \tilde{0}$, $f(\tilde{1}) = \tilde{1}$ and $f(\mu_3) = \mu_3$ are fuzzy $1/2$-semiopen sets of $(X, \mathcal{T})$ and hence f is a fuzzy $1/2$-semiopen map. On the other hand, $\mathcal{T}(f(\mu_3)) = \mathcal{T}(\mu_3) = 0 < 1/2$, and hence $f(\mu_3)$ is not a fuzzy $1/2$-open set. Thus f is not a fuzzy $1/2$-open map.

(3) A fuzzy r-open (hence r-semiopen) map need not be a fuzzy r-semiclosed map.

Let $X = I$ and μ, ρ, and λ be fuzzy sets of X defined as

$$\begin{aligned} &\mu(x) = 1 - x; \\ &\rho(x) = \begin{cases} -2x + 1 & \text{if } 0 \le x \le \dfrac{1}{2}, \\ 0 & \text{if } \dfrac{1}{2} \le x \le 1; \end{cases} \\ &\lambda(x) = \begin{cases} 1 & \text{if } 0 \le x \le \dfrac{1}{2}, \\ 0 & \text{if } \dfrac{1}{2} < x \le 1. \end{cases} \end{aligned} \tag{5.9}$$

Define $\mathcal{T}_1 : I^X \to I$ and $\mathcal{T}_2 : I^X \to I$ by

$$\mathcal{T}_1(\nu) = \begin{cases} 1 & \text{if } \nu = \tilde{0}, \tilde{1}, \\ \dfrac{1}{2} & \text{if } \nu = \mu, \\ 0 & \text{otherwise}; \end{cases} \qquad \mathcal{T}_2(\nu) = \begin{cases} 1 & \text{if } \nu = \tilde{0}, \tilde{1}, \lambda, \\ \dfrac{1}{2} & \text{if } \nu = \rho, \\ 0 & \text{otherwise}. \end{cases} \tag{5.10}$$

Then clearly $\mathcal{T}_1$ and $\mathcal{T}_2$ are fuzzy topologies on X. Consider the map $f:(X,\mathcal{T}_1)\to(X,\mathcal{T}_2)$ defined by $f(x)=x/2$. It is easy to see that $f(\tilde{0})=\tilde{0}$, $f(\mu)=\rho$ and $f(\tilde{1})=\lambda$. Thus f is a fuzzy 1/2-open map and hence a fuzzy 1/2-semiopen map. On the other hand, because the only fuzzy 1/2-closed set containing λ is $\tilde{1}$, $\lambda=f(\tilde{1})$ is not a fuzzy 1/2-semiclosed set of $(X,\mathcal{T}_2)$. Thus f is not a fuzzy 1/2-semiclosed map.

(4) A fuzzy r-closed (hence r-semiclosed) map need not be a fuzzy r-semiopen map.

Let $X=I$ and μ, ρ, and λ be fuzzy sets of X defined as

$$\mu(x)=1-x;\qquad \rho(x)=\begin{cases}-2x+1 & \text{if } 0\le x\le \dfrac{1}{2},\\ 1 & \text{if } \dfrac{1}{2}<x\le 1;\end{cases}\qquad \lambda(x)=\begin{cases}0 & \text{if } 0\le x\le \dfrac{1}{2},\\ 1 & \text{if } \dfrac{1}{2}<x\le 1.\end{cases} \tag{5.11}$$

Define $\mathcal{T}_1:I^X\to I$ and $\mathcal{T}_2:I^X\to I$ by

$$\mathcal{T}_1(\nu)=\begin{cases}1 & \text{if } \nu=\tilde{0},\tilde{1},\\ \dfrac{1}{2} & \text{if } \nu=\mu,\\ 0 & \text{otherwise};\end{cases}\qquad \mathcal{T}_2(\nu)=\begin{cases}1 & \text{if } \nu=\tilde{0},\tilde{1},\lambda,\\ \dfrac{1}{2} & \text{if } \nu=\rho,\\ 0 & \text{otherwise}.\end{cases} \tag{5.12}$$

Then clearly $\mathcal{T}_1$ and $\mathcal{T}_2$ are fuzzy topologies on X. Consider the map $f:(X,\mathcal{T}_1)\to(X,\mathcal{T}_2)$ defined by $f(x)=x/2$. It is easy to see that $f(\tilde{0})=\tilde{0}$, $f(\mu^c)=\rho^c$ and $f(\tilde{1})=\lambda^c$. Thus f is a fuzzy 1/2-closed map and hence a fuzzy 1/2-semiclosed map. On the other hand, the only fuzzy 1/2-open set contained in λ^c is $\tilde{0}$, hence $\lambda^c=f(\tilde{1})$ is not a fuzzy 1/2-semiopen set of $(X,\mathcal{T}_2)$. Thus f is not a fuzzy 1/2-semiopen map.

The next two theorems show that a fuzzy continuous (open, closed, semicontinuous, semiopen, semiclosed, resp.) map is a special case of a fuzzy r-continuous (r-open, r-closed, r-semicontinuous, r-semiopen, r-semiclosed, resp.) map.

Theorem 5.9. *Let $f:(X,\mathcal{T})\to(Y,\mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r\in I_0$. Then f is fuzzy r-continuous (r-open, r-closed, r-semicontinuous, r-semiopen, r-semiclosed, resp.) if and only if $f:(X,\mathcal{T}_r)\to(Y,\mathcal{U}_r)$ is fuzzy continuous (open, closed, semicontinuous, semiopen, semiclosed, resp.).*

Proof. The proof is straightforward. □

Theorem 5.10. *Let $f:(X,T)\to(Y,U)$ be a map from a Chang's fuzzy topological space X to another Chang's fuzzy topological space Y and $r\in I_0$. Then f is fuzzy continuous (open, closed, semicontinuous, semiopen, semiclosed, resp.) if and only if $f:(X,T^r)\to(Y,U^r)$ is fuzzy r-continuous (r-open, r-closed, r-semicontinuous, r-semiopen, r-semiclosed, resp.).*

Proof. The proof is straightforward. □

References

[1] K. K. Azad, *On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity*, J. Math. Anal. Appl. **82** (1981), no. 1, 14–32. MR 82k:54006. Zbl 511.54006.

[2] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182–190. MR 38#5153. Zbl 167.51001.

[3] K. C. Chattopadhyay, R. N. Hazra, and S. K. Samanta, *Gradation of openness: fuzzy topology*, Fuzzy Sets and Systems **49** (1992), no. 2, 237–242. MR 93f:54004. Zbl 762.54004.

[4] K. C. Chattopadhyay and S. K. Samanta, *Fuzzy topology: fuzzy closure operator, fuzzy compactness and fuzzy connectedness*, Fuzzy Sets and Systems **54** (1993), no. 2, 207–212. MR 93k:54016. Zbl 809.54005.

[5] R. N. Hazra, S. K. Samanta, and K. C. Chattopadhyay, *Fuzzy topology redefined*, Fuzzy Sets and Systems **45** (1992), no. 1, 79–82. MR 92m:54013. Zbl 756.54002.

[6] A. A. Ramadan, *Smooth topological spaces*, Fuzzy Sets and Systems **48** (1992), no. 3, 371–375. MR 93e:54006. Zbl 783.54007.

[7] A. P. Šostak, *On a fuzzy topological structure*, Rend. Circ. Mat. Palermo (2) Suppl. (1985), no. 11, 89–103. MR 88h:54015. Zbl 638.54007.

[8] ———, *Two decades of fuzzy topology: the main ideas, concepts and results*, Russian Math. Surveys **44** (1989), no. 6, 125–186, [translated from Uspekhi Mat. Nauk **44** (1989) no. 6(270), 99–147 (Russian). MR 91a:54010]. Zbl 716.54004.

[9] T. H. Yalvaç, *Semi-interior and semiclosure of a fuzzy set*, J. Math. Anal. Appl. **132** (1988), no. 2, 356–364. MR 89f:54014. Zbl 645.54007.

Seok Jong Lee: Department of Mathematics, Chungbuk National University, Cheongju 361-763, Korea

E-mail address: sjlee@chungbuk.ac.kr

Eun Pyo Lee: Department of Mathematics, Seonam University, Namwon 590-711, Korea

E-mail address: eplee@tiger.seonam.ac.kr