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ON AN APPLICATION OF ALMOST INCREASING SEQUENCES

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ABSTRACT. Using an almost increasing sequence, a result of Mazhar (1977) on $|C,1|_k$ summability factors has been generalized for $|C,\alpha;\beta|_k$ and $|\bar{N},p_n;\beta|_k$ summability factors under weaker conditions.

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1. Introduction. A sequence of (b_n) of positive numbers is said to be δ-quasi-monotone, if $b_n \to 0$, $b_n > 0$ ultimately and $\Delta b_n \geq -\delta_n$, where (δ_n) is a sequence of positive numbers (see [2]). Let $\sum a_n$ be a given infinite series with (s_n) as the sequence of its nth partial sums. Let σ_n^α and t_n^α denote the nth (C,α) means of the sequences (s_n) and (na_n), respectively, that is,

$$\sigma_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \tag{1.1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1.2}$$

where

$$A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1, \quad A_{-n}^\alpha = 0, \quad \text{for } n > 0. \tag{1.3}$$

The series $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \geq 1$ and $\alpha > -1$, if (see [6])

$$\sum_{n=1}^{\infty} n^{k-1}\left|\sigma_n^\alpha - \sigma_{n-1}^\alpha\right|^k = \sum_{n=1}^{\infty}\frac{1}{n}\left|t_n^\alpha\right|^k < \infty, \tag{1.4}$$

and it is said to be summable $|C,\alpha;\beta|_k$, $k \geq 1$, $\alpha > -1$ and $\beta \geq 0$, if (see [7])

$$\sum_{n=1}^{\infty} n^{\beta k+k-1}\left|\sigma_n^\alpha - \sigma_{n-1}^\alpha\right|^k = \sum_{n=1}^{\infty} n^{\beta k-1}\left|t_n^\alpha\right|^k < \infty. \tag{1.5}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \longrightarrow \infty \quad \text{as } n \longrightarrow \infty,\ P_{-i} = p_{-i} = 0,\ i \geq 1. \tag{1.6}$$

The sequence-to-sequence transformation

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{1.7}$$

defines the sequence (T_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [8]).

The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|\Delta T_{n-1}\right|^k < \infty, \tag{1.8}$$

and it is said to be summable $|\bar{N}, p_n; \beta|_k$, $k \geq 1$, and $\beta \geq 0$, if (see [4])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\beta k+k-1} \left|\Delta T_{n-1}\right|^k < \infty, \tag{1.9}$$

where

$$\Delta T_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \tag{1.10}$$

In the special case when $\beta = 0$ (resp., $p_n = 1$ for all values of n), $|\bar{N}, p_n; \beta|_k$ summability is the same as $|\bar{N}, p_n|_k$ (resp., $|C, 1; \beta|_k$) summability.

Also it is known that $|C, \alpha; \beta|_k$ and $|\bar{N}, p_n; \beta|_k$ summabilities are, in general, independent of each other.

Mazhar [9] has proved the following theorem for $|C, 1|_k$ summability factors of infinite series.

Theorem 1.1 (see [9]). *Let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi-monotone with $\sum n\delta_n \log n < \infty$, $\sum B_n \log n$ is convergent and $|\Delta\lambda_n| \leq |B_n|$ for all n. If*

$$\sum_{n=1}^{m} \frac{1}{n} \left|t_n\right|^k = O(\log m) \quad \text{as } m \longrightarrow \infty, \tag{1.11}$$

where (t_n) is the nth $(C, 1)$ mean of the sequence (na_n), then the series $\sum a_n \lambda_n$ is summable $|C, 1|_k$, $k \geq 1$.

Remark 1.2. It should be noted that the condition "$\sum nB_n \log n$ is convergent" is enough to prove Theorem 1.1 rather than the conditions "$\sum n\delta_n \log n < \infty$ and $\sum B_n \log n$ is convergent."

2. The main result.

In view of Remark 1.2, the aim of this paper is to generalize Theorem 1.1 for $|C, \alpha; \beta|_k$ and $|\bar{N}, p_n; \beta|_k$ summabilities under weaker conditions. For this we need the concept of almost increasing sequence. A positive sequence (d_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq d_n \leq Bc_n$ (see [1]). Obviously, every increasing sequence is almost increasing but the converse need not be true as can be seen from the example $d_n = ne^{(-1)^n}$. Since $\log n$ is increasing, we are weakening the hypotheses of the theorem replacing the increasing sequence by an almost increasing sequence.

Now, we prove the following theorems.

Theorem 2.1. *Let (X_n) be an almost increasing sequence and $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi-monotone with $\sum nB_nX_n$ convergent and $|\Delta\lambda_n| \le |B_n|$ for all n. If the sequence (u_n^α), defined by (see [10])*

$$u_n^\alpha = \begin{cases} |t_n^\alpha|, & \alpha = 1, \\ \max\limits_{1\le v\le n} |t_v^\alpha|, & 0 < \alpha < 1, \end{cases} \tag{2.1}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{\beta k-1}(u_n^\alpha)^k = O(X_m) \quad \text{as } m \longrightarrow \infty, \tag{2.2}$$

then the series $\sum a_n\lambda_n$ is summable $|C,\alpha;\beta|_k$, $k \ge 1$ and $0 \le \beta < \alpha \le 1$.

Theorem 2.2. *Let (X_n) be an almost increasing sequence and $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi-monotone with $\sum nB_nX_n$ convergent and $|\Delta\lambda_n| \le |B_n|$ for all n. If (p_n) is a sequence such that*

$$\begin{gathered} \sum_{n=v+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\beta k-1} \frac{1}{P_{n-1}} = O\left\{\left(\frac{P_v}{p_v}\right)^{\beta k} \frac{1}{P_v}\right\}, \\ \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\beta k-1} |t_n|^k = O(X_m) \quad \text{as } m \longrightarrow \infty, \\ \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\beta k} \frac{1}{n}|t_n|^k = O(X_m) \quad \text{as } m \longrightarrow \infty, \\ \sum_{n=1}^{m} \frac{|\lambda_n|}{n} = O(1) \quad \text{as } m \longrightarrow \infty, \end{gathered} \tag{2.3}$$

then the series $\sum a_n\lambda_n$ is summable $|\bar{N},p_n;\beta|_k$ for $k \ge 1$ and $0 \le \beta < 1/k$.

We need the following lemmas for the proof of our theorems.

Lemma 2.3 (see [5]). *If $0 < \alpha \le 1$ and $1 \le v \le n$, then*

$$\left|\sum_{p=0}^{v} A_{n-p}^{\alpha-1}a_p\right| \le \max_{1\le m\le v}\left|\sum_{p=0}^{m} A_{m-p}^{\alpha-1}a_p\right|. \tag{2.4}$$

Under the conditions of Theorem 2.2 we obtain the following result.

Lemma 2.4. *The following equation holds:*

$$|\lambda_n|X_n = O(1) \quad \text{as } n \longrightarrow \infty. \tag{2.5}$$

Proof. Since $\lambda_n \to 0$ as $n \to \infty$, we have

$$|\lambda_n| X_n = X_n \left| \sum_{v=n}^{\infty} \Delta\lambda_v \right| \le X_n \sum_{v=n}^{\infty} |\Delta\lambda_v| \le \sum_{v=0}^{\infty} X_v |\Delta\lambda_v| \le \sum_{v=0}^{\infty} X_v |B_v| < \infty. \quad (2.6)$$

Hence $|\lambda_n| X_n = O(1)$ as $n \to \infty$. □

3. Proof of Theorem 2.1. Let (T_n^{α}) be the nth (C,α), with $0 < \alpha \le 1$, mean of the sequence $(na_n\lambda_n)$. Then, by (1.1), we have

$$T_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v. \quad (3.1)$$

Applying Abel's transformation, we get

$$T_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \quad (3.2)$$

so that making use of Lemma 2.3, we have

$$\begin{aligned} |T_n^{\alpha}| &\le \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} |\Delta\lambda_v| \left| \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p \right| + \frac{|\lambda_n|}{A_n^{\alpha}} \left| \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \right| \\ &\le \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} A_v^{\alpha} u_v^{\alpha} |\Delta\lambda_v| + |\lambda_n| u_n^{\alpha} \\ &= T_{n,1}^{\alpha} + T_{n,2}^{\alpha}. \end{aligned} \quad (3.3)$$

Since

$$|T_{n,1}^{\alpha} + T_{n,2}^{\alpha}|^k \le 2^k \left(|T_{n,1}^{\alpha}|^k + |T_{n,2}^{\alpha}|^k \right), \quad (3.4)$$

to complete the proof of Theorem 2.1, it is enough to show that

$$\sum_{n=1}^{\infty} n^{\beta k-1} |T_{n,r}^{\alpha}|^k < \infty \quad \text{for } r = 1,2. \quad (3.5)$$

Now, when $k > 1$, applying Hölder's inequality with indices k and k', where $1/k + 1/k' = 1$, we get

$$\begin{aligned} &\sum_{n=2}^{m+1} n^{\beta k-1} |T_{n,1}^{\alpha}|^k \\ &\quad \le \sum_{n=2}^{m+1} n^{\beta k-1} (A_n^{\alpha})^{-k} \left\{ \sum_{v=1}^{n-} A_v^{\alpha} u_v^{\alpha} |B_v| \right\}^k \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n=2}^{m+1} n^{\beta k-1}(A_n^{\alpha})^{-k}\left\{\sum_{v=1}^{n-1}(A_v^{\alpha})^k(u_v^{\alpha})^k|B_v|\right\}\left\{\sum_{v=1}^{n-1}|B_v|\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1} n^{\beta k-\alpha k-1}\left\{\sum_{v=1}^{n-1}v^{\alpha k}(u_v^{\alpha})^k|B_v|\right\}\\
&= O(1)\sum_{v=1}^{m}v^{\alpha k}(u_v^{\alpha})^k|B_v|\sum_{n=v+1}^{m+1}\frac{1}{n^{1+\alpha k-\beta k}}\\
&= O(1)\sum_{v=1}^{m}v^{\alpha k}(u_v^{\alpha})^k|B_v|\int_v^{\infty}\frac{dx}{x^{1+\alpha k-\beta k}}\\
&= O(1)\sum_{v=1}^{m}v^{\beta k}(u_v^{\alpha})^k|B_v| = O(1)\sum_{v=1}^{m}v|B_v|v^{\beta k-1}(u_v^{\alpha})^k\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v|B_v|)\sum_{r=1}^{v}r^{\beta k-1}(u_r^{\alpha})^k+O(1)m|B_m|\sum_{v=1}^{m}v^{\beta k-1}(u_v^{\alpha})^k\\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v|B_v|)|X_v+O(1)m|B_m|X_m\\
&= O(1)\sum_{v=1}^{m-1}v|B_v|X_v+O(1)\sum_{v=1}^{m-1}(v+1)|B_{v+1}|X_{v+1}+O(1)m|B_m|X_m\\
&= O(1)\quad \text{as } m\longrightarrow\infty,
\end{aligned}
\tag{3.6}
$$

by virtue of the hypotheses of Theorem 2.1.

Finally, since $|\lambda_n| = O(1)$, by hypothesis

$$
\begin{aligned}
\sum_{n=1}^{m} n^{\beta k-1}|T_{n,2}^{\alpha}|^k &= \sum_{n=1}^{m}|\lambda_n|^{k-1}n^{\beta k-1}(u_n^{\alpha})^k\\
&= O(1)\sum_{n=1}^{m}|\lambda_n|n^{\beta k-1}(u_n^{\alpha})^k\sum_{v=n}^{\infty}|\Delta\lambda_v|\\
&= O(1)\sum_{v=1}^{\infty}|\Delta\lambda_v|\sum_{n=1}^{v}n^{\beta k-1}(u_v^{\alpha})^k\\
&= O(1)\sum_{v=1}^{\infty}|B_v|X_v<\infty,
\end{aligned}
\tag{3.7}
$$

by virtue of the hypotheses of Theorem 2.1.

Therefore, we get

$$
\sum_{n=1}^{m} n^{\beta k-1}|T_{n,r}^{\alpha}|^k = O(1)\quad \text{as } m\longrightarrow\infty,\ \text{for } r=1,2. \tag{3.8}
$$

This completes the proof of Theorem 2.1. □

Remark 3.1. It is natural to ask whether our theorem is true with $\alpha > 1$. All we can say with certainty is that our proof fails if $\alpha > 1$, for our estimate of $T_{n,1}^{\alpha}$ depends upon Lemma 2.3, and Lemma 2.3 is known to be false when $\alpha > 1$ (see [5] for details).

Proof of Theorem 2.2 . Let (T_n) denotes the $(\bar{N}, p_n)$ mean of the series $\sum a_n \lambda_n$. Then, by definition and changing the order of summation, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{i=0}^{v} a_i \lambda_i = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v \lambda_v. \tag{3.9}$$

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} \lambda_v}{v} v a_v. \tag{3.10}$$

By Abel's transformation, we have

$$\begin{aligned} T_n - T_{n-1} = {} & \frac{n+1}{n P_n} p_n t_n \lambda_n - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v t_v \lambda_v \frac{v+1}{v} \\ & + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta \lambda_v t_v \frac{v+1}{v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \lambda_{v+1} \frac{1}{v} \\ = {} & T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}. \end{aligned} \tag{3.11}$$

Since

$$|T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}|^k \leq 4^k (|T_{n,1}|^k + |T_{n,2}|^k + |T_{n,3}|^k + |T_{n,4}|^k), \tag{3.12}$$

to complete the proof of Theorem 2.2, it is enough to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\beta k + k - 1} |T_{n,r}|^k < \infty \quad \text{for } r = 1, 2, 3, 4. \tag{3.13}$$

Since $(\lambda_n) \to 0$ as $n \to \infty$ by the hypothesis of Theorem 2.2, we have

$$\begin{aligned} \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\beta k + k - 1} |T_{n,1}|^k &= O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\beta k - 1} |\lambda_n|^{k-1} |\lambda_n| \, |t_n|^k \\ &= O(1) \sum_{n=1}^{m} |\lambda_n| \left(\frac{P_n}{p_n}\right)^{\beta k - 1} |t_n|^k \\ &= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^{\beta k - 1} |t_v|^k \\ &\quad + O(1) |\lambda_m| \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\beta k - 1} |t_n|^k \\ &= O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| X_n + O(1) |\lambda_m| X_m \\ &= O(1) \sum_{n=1}^{m-1} |B_n| X_n + O(1) |\lambda_m| X_m = O(1) \quad \text{as } m \longrightarrow \infty, \end{aligned} \tag{3.14}$$

by virtue of the hypotheses of Theorem 2.2 and in view of Lemma 2.4.

Now, when $k>1$, applying Hölder's inequality with indices k and k', where $1/k+1/k'=1$, as in $T_{n,1}$, we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k+k-1}\left|T_{n,2}\right|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k-1}\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1}p_v\left|\lambda_v\right|^k\left|t_v\right|^k\right\}\\
&\quad\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}\\
&= O(1)\sum_{v=1}^{m}p_v\left|\lambda_v\right|^{k-1}\left|\lambda_v\right|\left|t_v\right|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k-1}\frac{1}{P_{n-1}}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\beta k-1}\left|t_v\right|^k\left|\lambda_v\right| = O(1)\quad \text{as } m\longrightarrow\infty.
\end{aligned}\tag{3.15}$$

Again, we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k+k-1}\left|T_{n,3}\right|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k-1}\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1}P_v\left|B_v\right|\left|t_v\right|^k\right\}\\
&\quad\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_v\left|B_v\right|\right\}^{k-1}\\
&= O(1)\sum_{v=1}^{m}P_v\left|B_v\right|\left|t_v\right|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k-1}\frac{1}{P_{n-1}}\\
&= O(1)\sum_{v=1}^{m}\left|B_v\right|\left(\frac{P_v}{p_v}\right)^{\beta k}\left|t_v\right|^k\\
&= O(1)\sum_{v=1}^{m}v\left|B_v\right|\left(\frac{P_v}{p_v}\right)^{\beta k}\frac{1}{v}\left|t_v\right|^k\\
&= O(1)\sum_{v=1}^{m-1}\Delta\left(v\left|B_v\right|\right)\sum_{i=1}^{v}\left(\frac{P_i}{p_i}\right)^{\beta k}\frac{1}{i}\left|t_i\right|^k\\
&\quad+O(1)m\left|B_m\right|\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\beta k}\frac{1}{v}\left|t_v\right|^k\\
&= O(1)\sum_{v=1}^{m-1}\left|\Delta\left(v\left|B_v\right|\right)\right|X_v+O(1)m\left|B_m\right|X_m\\
&= O(1)\sum_{v=1}^{m-1}vX_v\left|B_v\right|+O(1)\sum_{v=1}^{m-1}(v+1)\left|B_{v+1}\right|X_{v+1}\\
&\quad+O(1)m\left|B_m\right|X_m\\
&= O(1)\quad \text{as } m\longrightarrow\infty,
\end{aligned}\tag{3.16}$$

by virtue of the hypotheses of Theorem 2.2.

Finally, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k+k-1}\left|T_{n,4}\right|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_v\frac{\left|\lambda_{v+1}\right|}{v}\left|t_v\right|^k \\ &\quad\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_v\frac{\left|\lambda_{v+1}\right|}{v}\right\}^{k-1} \\ &= O(1)\sum_{v=1}^{m}P_v\frac{\left|\lambda_{v+1}\right|}{v}\left|t_v\right|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta k-1}\frac{1}{P_{n-1}} \\ &= O(1)\sum_{v=1}^{m}\left|\lambda_{v+1}\right|\left(\frac{P_v}{p_v}\right)^{\beta k}\frac{1}{v}\left|t_v\right|^k \\ &= O(1)\sum_{v=1}^{m-1}\Delta\left|\lambda_{v+1}\right|\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\beta k}\frac{1}{r}\left|t_r\right|^k \qquad (3.17) \\ &\quad+O(1)\left|\lambda_{m+1}\right|\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\beta k}\frac{1}{v}\left|t_v\right|^k \\ &= O(1)\sum_{v=1}^{m-1}\left|\Delta\lambda_{v+1}\right|X_{v+1}+O(1)\left|\lambda_{m+1}\right|X_{m+1} \\ &= O(1)\sum_{v=1}^{m-1}\left|B_{v+1}\right|X_{v+1}+O(1)\left|\lambda_{m+1}\right|X_{m+1} \\ &= O(1)\quad\text{as } m\longrightarrow\infty,\end{aligned}$$

by virtue of the hypotheses of Theorem 2.2 and in view of Lemma 2.4.

Therefore, we get

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta k+k-1}\left|T_{n,r}\right|^k = O(1)\quad\text{as } m\longrightarrow\infty,\ \text{for } r=1,2,3,4. \qquad (3.18)$$

This completes the proof of Theorem 2.2. □

If we take $p_n = 1$ for all values of n in this theorem, then we get a result concerning the $|C,1;\beta|_k$ summability factors.

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