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ON A FAMILY OF DENDRITES

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ABSTRACT. We study the open images of members of a countable family $\mathcal{F}$ of dendrites. We show that only two members of $\mathcal{F}$ are minimal and only one of them is unique minimal with respect to open mappings.

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1. Introduction. Let $\mathcal{S}$ be a family of topological spaces and $\mathbb{F}$ a class of mappings between members of $\mathcal{S}$. Then $\mathcal{S}$ can be *quasi-ordered with respect to* $\mathbb{F}$, writing for any $X, Y \in \mathcal{S}$

$$\begin{gathered}(Y \leq_{\mathbb{F}} X) \iff (\text{there exists a surjection } f \in \mathbb{F} \text{ of } X \text{ onto } Y), \\ (X =_{\mathbb{F}} Y) \iff (Y \leq_{\mathbb{F}} X \text{ and } X \leq_{\mathbb{F}} Y).\end{gathered} \tag{1.1}$$

A member X_0 of $\mathcal{S}$ is said to be

- *minimal* in $\mathcal{S}$ with respect to $\mathbb{F}$ provided that, for each Y in $\mathcal{S}$ the condition $Y \leq_{\mathbb{F}} X_0$ implies $Y =_{\mathbb{F}} X_0$;
- *unique minimal* in $\mathcal{S}$ with respect to $\mathbb{F}$ provided that for each Y in $\mathcal{S}$ if $Y \leq_{\mathbb{F}} X_0$ then Y is homeomorphic to X_0.

Thus, in particular, all spaces in $\mathcal{S}$ which are homeomorphic to all its images under mappings belonging to $\mathbb{F}$ are unique minimal in $\mathcal{S}$ with respect to $\mathbb{F}$. (See [5, Chapter 3, page 7] for more information.)

In this paper, we take as $\mathcal{S}$ the family $\mathcal{D}$ of dendrites (i.e., locally connected continua containing no simple closed curves) and as $\mathbb{F}$ the class $\mathbb{O}$ of open mappings (i.e., ones which map open subsets of the domain onto open subsets of the range). Various properties of the relation $\leq_{\mathbb{O}}$ on the family $\mathcal{D}$ are discussed in [5, Chapter 6, pages 22–51]. Examples of dendrites which are homeomorphic to all its open images can be found, for example, in [2, Corollary, page 493 and the paragraph following it] and in [5, Theorem 6.45, page 30].

Answering a question in [5, Q2($\mathbb{O}$), page 51] (see also [3, Section 6, 2, page 245]) a dendrite C is constructed in [9, Section 2] which is minimal with respect to $\mathbb{O}$ and which has two topologically distinct open images, thus is not unique minimal with respect to $\mathbb{O}$ (see [9, Proposition 3.5(α)]). The quoted paper contains also a construction of a countable family $\mathcal{F}$ of dendrites, with $C \in \mathcal{F}$. Since each member of $\mathcal{F}$ has a similar structure as the one of C, it is natural to ask about open mapping properties of other members of $\mathcal{F}$, especially properties which are related to the minimality of members of $\mathcal{F}$ with respect to the class $\mathbb{O}$. This is a subject of the present paper.

All spaces considered in the paper are assumed to be metric and all mappings are continuous. A *continuum* means a compact connected space. Given a space X and its subset S, we denote by $\mathrm{cl}\,S$ the closure of S and by $\mathrm{int}\,S$ its interior in X. As usual $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{R}$ stands for the space of real numbers.

We will use the notion of *order of a point* in the sense of Menger-Urysohn (cf. [7, Section 51, I, page 274]), and we denote by $\mathrm{ord}(p,X)$ the order of the space X at a point $p \in X$. It is well known (cf. [7, Section 51, pages 274–307]) that the function ord takes its values from the set

$$S = \{0,1,2,\dots,n,\dots,\omega,\aleph_0,2^{\aleph_0}\}. \tag{1.2}$$

Points of order 1 in a space X are called *end points* of X; the set of all end points of X is denoted by $E(X)$. Points of order 2 are called *ordinary points* of X. It is known that the set of all ordinary points is a dense subset of a dendrite. And for each $r \in \{3,4,\dots,\omega,\aleph_0,2^{\aleph_0}\}$ points of order r are called *ramification points* of X; the set of all ramification points is denoted by $R(X)$. It is known that for each dendrite X the set $R(X)$ is at most countable, and that the points of order $\aleph_0$ and $2^{\aleph_0}$ do not occur in any dendrite.

A space X is said to be *universal in a class of spaces* if it belongs to the class and it contains a homeomorphic copy of every element of that class.

2. The construction. It should be stressed that the construction below is modeled onto the one described in [9, Section 2], and also the proofs of the properties of the dendrites $D(r,s)$ are patterned after the corresponding ones presented in [9, Sections 2 and 3].

To construct the mentioned family $\mathcal{F}$ of dendrites, we fix some notation and terminology. For $n \in \mathbb{N}$ let F_n denote the *simple n-od*, that is, a continuum homeomorphic to the cone over a (discrete) set of n points. The vertex of the cone is called the *vertex* of F_n. In the Cartesian coordinates in the plane $\mathbb{R}^2$ put $v = (0,0)$, and for each $n \in \mathbb{N}$ let $e_n = (1/n, 1/n^2)$. Denoting by pq the straight line segment in the plane with end points p and q, define

$$F_\omega = \bigcup \{ve_n : n \in \mathbb{N}\}. \tag{2.1}$$

The continua F_n and F_ω are called *fans of order n and ω*, respectively. Any fan of order $n \in \mathbb{N}$ (thus having the set $E(F_n)$ of its end points finite) is also named a *finite fan*, and F_ω is also termed an *infinite locally connected fan*. Obviously fans F_n and F_ω are dendrites.

An arc pq with end points p and q in a continuum X is called a *free arc* provided that $pq \setminus \{p,q\}$ is an open subset of X. If a free arc is not properly contained in another one, it is called a *maximal free arc*. Then three cases are possible: either both p and q are ramification points (and then it is called an *interior free arc*), or one of them is a ramification point and the other is an end point of X (and then pq is called an *end free arc*), or finally both p and q are end points of X. Note that the third possibility holds only in a trivial case when $X = pq$.

Theorem 2.1. *For every $r \in \{3,4,\dots,\aleph_0\}$ and $s \in \{0,1,2,\dots,\aleph_0\}$ there exists a dendrite $D(r,s)$ such that:*

(2.1.1) *each ramification point of $D(r,s)$ belongs to exactly r interior free arcs in $D(r,s)$;*

(2.1.2) *each ramification point of $D(r,s)$ belongs to exactly s end free arcs in $D(r,s)$;*

(2.1.3) *any two ramification points of $D(r,s)$ are contained in an arc in $D(r,s)$ containing only finitely many ramification points of $D(r,s)$.*

Moreover, conditions (2.1.1), (2.1.2), and (2.1.3) determine the dendrite $D(r,s)$ up to a homeomorphism.

Proof. Let $X_1 = F_r^1 \cup F_s^1$ be the one-point union of the fans F_r^1 and F_s^1 such that $F_r^1 \cap F_s^1 = \{v\}$, where v is the common vertex of the two fans. If $s = 0$, we take $F_s = \{v\}$, and if $s = 1$ or $s = 2$ we understood F_s^1 as the union of one or two arcs, respectively, emanating from the point v and disjoint out of this point. Thus X_1 is a fan with the vertex v, either finite or homeomorphic to F_ω. In the set $E(X_1)$ we distinguish a subset $E_1 = E(F_r^1)$.

Assume that a dendrite X_n is defined for some $n \in \mathbb{N}$ and that in the set $E(X_n)$ of its end points a nonempty subset E_n is distinguished. Consider the one point union $U = F_{r-1} \cup F_s$ where the vertices of the fans F_{r-1} and F_s are identified to a point $v(U)$. Then X_{n+1} is obtained from X_n by attaching to each end point $e \in E_n \subset X_n$ a properly diminished copy $U(e) = F_{r-1}^{n+1}(e) \cup F_s^{n+1}(e)$ of U with the points $e \in X_n$ and $v(U(e)) \in U(e)$ identified, in such a way that $X_n \cap U(e) = \{e\}$, where $F_{r-1}^{n+1}(e)$ and $F_s^{n+1}(e)$ denote the corresponding copies of the fans F_{r-1} and F_s, respectively. Thus X_{n+1} is a dendrite by its definition. Further, define $E_{n+1} = \bigcup\{E(F_{r-1}(e)) : e \in E_n\}$.

Note that $X_n \subset X_{n+1}$ for each $n \in \mathbb{N}$. We assume that the diameters of the components of $X_{n+1} \setminus X_n$ tend to 0 if n tends to infinity. Let $f_n : X_{n+1} \to X_n$ be a monotone retraction. Thus f_n shrinks each of the attached fans $U(e)$ back to its vertex $v(U(e))$ which is identified with the corresponding end point $e \in E_n \subset E(X_n)$.

Consider the inverse sequence $\{X_n, f_n : n \in \mathbb{N}\}$ of dendrites X_n with monotone bonding mappings f_n, and define

$$D(r,s) = \varprojlim\{X_n, f_n : n \in \mathbb{N}\}. \tag{2.2}$$

By [8, Theorem 10.36, page 180 and Theorem 2.10, page 23] the defined inverse limit $D(r,s)$ is a dendrite which is homeomorphic to $\mathrm{cl}(\bigcup\{X_n : n \in \mathbb{N}\})$. Neglecting the homeomorphism we can simply write

$$D(r,s) = \mathrm{cl}\Big(\bigcup\{X_n : n \in \mathbb{N}\}\Big). \tag{2.3}$$

It is evident from the construction that $D(r,s)$ has properties (2.1.1), (2.1.2), and (2.1.3). In [9, Proposition 3.3] it is proved that these properties uniquely determine $D(r,s)$. The proof is then complete. □

Finally we put

$$\mathcal{F} = \{D(r,s) : r \in \{3,4,\dots,\aleph_0\},\ s \in \{0,1,2,\dots,\aleph_0\}\}. \tag{2.4}$$

Properties (2.1.1) and (2.1.2) imply the following.

Statement 2.2. The dendrite $D(r,s)$ is composed exclusively of points of order 1, 2, and $r+s$, with a convention that, in the case when one of r or s is $\aleph_0$, points of order $r+s$ are understood as ones of order ω.

The next statement is a consequence of property (2.1.3).

Statement 2.3. Let an integer $r \geq 3$ and $s \in \{0,1,2,\dots,\aleph_0\}$ be fixed. If $\{p_m : m \in \mathbb{N}\}$ is a convergent sequence of distinct ramification points of $D(r,s)$, then $\lim p_m$ is an end point.

As a consequence of (2.3) and Statement 2.3, we get the following inclusion.

$$W = \operatorname{cl}\Big(\bigcup\{X_n : n \in \mathbb{N}\}\Big) \setminus \bigcup\{X_n : n \in \mathbb{N}\} \subset E(D(r,s)). \tag{2.5}$$

The next inclusion is obvious.

$$\begin{gathered} D(r_1,s_1) \subset D(r_2,s_2) \quad \text{for every } r_1,r_2 \in \{3,4,\dots,\aleph_0\}, \\ s_1,s_2 \in \{0,1,2,\dots,\aleph_0\} \quad \text{with } r_1 \leq r_2,\ s_1 \leq s_2. \end{gathered} \tag{2.6}$$

In particular, we have the following:

$$D(r,0) \subset D(r,1) \subset D(r,2) \subset \cdots \subset D(r,\aleph_0) \quad \text{for each } r \in \{3,4,\dots,\aleph_0\}, \tag{2.7}$$

$$D(3,s) \subset D(4,s) \subset D(5,s) \subset \cdots \subset D(\aleph_0,s) \quad \text{for each } s \in \{0,1,2,\dots,\aleph_0\}. \tag{2.8}$$

Note that $D(\aleph_0,1)$ is C_ω^1 of [9].

For each integer $n \geq 3$, a dendrite G^n is constructed in [1, Chapter 4] which is universal in the class of all dendrites with a closed set of end points and of orders of their ramification points not greater than n (see [1, Theorems 4.1 and 4.2]). Comparing the two constructions, namely, of $D(r,s)$ and of G^n, it is evident that

$$D(r,0) \text{ is homeomorphic to } G^r \quad \text{for each integer } r \geq 3, \tag{2.9}$$

whence it follows from (2.7) that for every $r \in \{3,4,\dots\}$ and $s \in \{0,1,2,\dots,\aleph_0\}$ the dendrite G^r is contained in $D(r,s)$ even in such a way that $E(G^r) \subset E(D(r,s))$.

The next result follows from [1, Theorem 3.3] which gives a characterization of dendrites with a closed set of end points. But it is also a direct consequence of the definition of $D(r,s)$.

Proposition 2.4. *If $r \in \{3,4,5,\dots\}$ and $s \in \{0,1,2,\dots\}$, then the dendrite $D(r,s)$ has a closed set of end points.*

Therefore Proposition 2.4, (2.9), and the above mentioned universality of dendrites G^n imply the following corollary.

Corollary 2.5. *If $r \in \{3,4,5,\dots\}$ and $s \in \{0,1,2,\dots\}$, then $D(r,s)$ can be embedded in $D(r+s,0)$.*

3. Open images. In this section, we intend to study open images of members of the family $\mathcal{F}$. We start with recalling some theorems which are useful tools for the study of open mapping properties of some continua.

Proposition 3.1. *Let a mapping $f : X \to Y$ be a nonconstant open surjection between continua. Then*

- (3.1.1) *the order of a point is not increased, that is,* $\operatorname{ord}(f(x), Y) \le \operatorname{ord}(x, X)$*; in particular $f(E(X)) \subset E(Y)$;*
- (3.1.2) *if X is an arc, or a dendrite, or the infinite locally connected fan F_ω, then so is Y;*
- (3.1.3) *if X is a dendrite, then*
  - (a) *$f^{-1}(y)$ is finite for each $y \in Y \setminus E(Y)$;*
  - (b) *$f^{-1}(E(Y)) \setminus E(X)$ is finite;*
  - (c) *if* $\operatorname{ord}(x, X) = \omega$*, then* $\operatorname{ord}(f(x), Y) = \omega$*;*
  - (d) *the image under f of a free arc in X is a free arc in Y.*

Proof. The proof of (3.1.1) follows from [10, Chapter 8, (7.31), page 147].

(3.1.2): for an arc and a dendrite see [10, Chapter 10, (1.3), page 184], [10, Chapter 8, (7.7), page 148, and Chapter 10, page 185]; compare [5, (6.1), page 22]; for F_ω see [4, Proposition 9.4, page 42].

(3.1.3): see [5, Propositions 6.16, page 25, 6.5 and Corollary 6.7, page 23]. □

Using the above facts concerning open mappings, the following proposition is shown in [9, Proposition 3.5(α)].

Proposition 3.2. *Each open image of the dendrite $D(\aleph_0, 1)$ is homeomorphic either to $D(\aleph_0, 1)$ or to the one-point union U of $D(\aleph_0, 1)$ with an end free arc pq, where $p \in R(D(\aleph_0, 1))$.*

Recall that an open mapping of $D(\aleph_0, 1)$ onto the union U is obtained as follows: locate $D(\aleph_0, 1)$ in the plane in such a way that

1. (1) all free arcs in $D(\aleph_0, 1)$ are straight line segments, and
2. (2) $D(\aleph_0, 1)$ is symmetric with respect to a straight line L which is perpendicular to an interior free segment S and passes through its mid point m so that $D(\aleph_0, 1) \cap L = \{m\}$ (see [9, Figure 3]).

Then the mentioned open mapping is the identity on one half of $D(\aleph_0, 1)$ (lying on one half-plane determined by L) and it is the symmetry on the other; equivalently, if $\sim$ denotes the symmetry with respect to L, then U is homeomorphic to $D(\aleph_0, 1)/\sim$, and pq is homeomorphic to $S/\sim$.

Exactly the same arguments as in the proof of [9, Proposition 3.5] can be applied to show the next two propositions. In particular, observe that if the above recalled open mapping is applied to $D(\aleph_0, \aleph_0)$, then the resulting union U is homeomorphic to $D(\aleph_0, \aleph_0)$. The details are left to the reader.

Proposition 3.3. *Each open image of the dendrite $D(\aleph_0, 0)$ is homeomorphic either to $D(\aleph_0, 0)$ or to the one-point union U of $D(\aleph_0, 0)$ with an end free arc pq, where $p \in R(D(\aleph_0, 0))$.*

Proposition 3.4. *Each open image of the dendrite $D(\aleph_0,\aleph_0)$ is homeomorphic to $D(\aleph_0,\aleph_0)$.*

Remark 3.5. According to (3.1.2), an arc and F_ω are examples of dendrites homeomorphic to their open images. Recall that such dendrites are said to be *unique minimal elements of the class* $\mathfrak{D}_{\leq\mathbb{O}}$ (see [5, Chapter 3, pages 7–8]). Among all universal dendrites D_S with $S \subset \{3,4,\dots,\omega\}$, only D_3, D_ω and $D_{\{3,\omega\}}$ have this property (see [2, Corollary, page 493] and [6, Corollary 6.10, page 232]). An uncountable family of some dendrites such that each member of the family is homeomorphic to any of its open images is constructed in [5, Theorem 6.45, page 30]. However, the internal structure of all dendrites having the considered property is not known (see [5, Chapter 7, problem Q1($\mathbb{O}$), page 51]). Proposition 3.4 gives a new example of a dendrite which is a unique minimal element of the class $\mathfrak{D}_{\leq\mathbb{O}}$.

As it is shown in [9, Proposition 3.5(γ)], each open image of $D(\aleph_0,1)$ can by mapped onto $D(\aleph_0,1)$ under an open mapping, that is, $D(\aleph_0,1)$ is a minimal (but not unique minimal, according to Proposition 3.2) element of the class $\mathfrak{D}_{\leq\mathbb{O}}$. Thus [5, Chapter 7, problem Q1($\mathbb{O}$), page 51] has a negative answer (this is the main result of [9]). Note that it is not the case for $D(\aleph_0,0)$ because (by (3.1.1) above) the union U of Proposition 3.3 cannot be openly mapped onto $D(\aleph_0,0)$. For further results in this direction see below.

Propositions 3.2, 3.3, and 3.4 describe open images of $D(\aleph_0,s)$ for $s \in \{0,1,\aleph_0\}$. For $s \in \{2,3,\dots\}$ some open images of $D(\aleph_0,s)$ can be obtained in the following way. Fix any nonempty subset $P \subset R(D(\aleph_0,s))$. For any ramification point $p \in P$ let $F_s(p) \subset D(\aleph_0,s)$ be the union of s end free arcs $pe_1^p, pes_2^p,\dots,pe_s^p$, every two of which have the singleton $\{p\}$ in common only. Further, for a fixed $t \in \{1,2,\dots,s\}$ let $F_t(p) \subset F_s(p)$ be the union of t end free arcs $pe_{i_1}^p, pe_{i_2}^p,\dots,pe_{i_t}^p$. Then there is an open surjective mapping $f^{(p)} : F_s(p) \to F_t(p)$ which is a homeomorphism on each free arc pe_j^p for each $j \in \{1,2,\dots,s\}$ with $f^{(p)}(p) = p$ and $f^{(p)}(e_j^p) = e_{i_j}^p$ for some $i_j \in \{i_1,i_2,\dots,i_t\}$. Then the mapping $f : D(\aleph_0,s) \to Y \subset D(\aleph_0,s)$ such that $f|F_s(p) = f^{(p)}$ for each $p \in P$ and defined as a homeomorphism otherwise is obviously open. In particular, if $P = R(D(\aleph_0,s))$ and if $t \in \{1,2,\dots,s\}$ is the same fixed number for all ramification points p, then $Y = D(\aleph_0,t)$, and the following proposition is obtained.

Proposition 3.6. *For each $s \in \{2,3,\dots\}$ and for each $t \in \{1,2,\dots,s\}$ there is an open mapping of $D(\aleph_0,s)$ onto $D(\aleph_0,t)$.*

Taking $t = 1$ in the above construction we conclude from Proposition 3.2 that for each $s \in \{2,3,\dots\}$ there is no open mapping from $D(\aleph_0,1)$ onto $D(\aleph_0,s)$. Therefore the next result follows.

Proposition 3.7. *For each $s \in \{2,3,\dots\}$ no dendrite $D(\aleph_0,s)$ is minimal in the class $\mathfrak{D}_{\leq\mathbb{O}}$.*

We now consider open images of other members of $\mathcal{F}$, namely of dendrites $D(r,s)$ for $r \in \{3,4,\dots\}$ and $s \in \{0,1,2,\dots,\aleph_0\}$. To see that no one of them is minimal in the class $\mathfrak{D}_{\leq\mathbb{O}}$ we need some facts about the structure of the set of end points of $D(r,s)$. To this aim represent $D(r,s)$ as in (2.2) and observe that if $r \neq \aleph_0$, then the set $(X_{n+1}\setminus X_n)\cap R(D(r,s))$ is finite. Putting

$$R_1=\{v\},\quad R_{n+1}=(X_{n+1}\setminus X_n)\cap R(D(r,s))\quad \forall n\in\mathbb{N}, \tag{3.1}$$

we see that $R(D(r,s))=\bigcup\{R_n : n\in\mathbb{N}\}$, and that the sets R_n are mutually disjoint. For each point $q\in R_n$ let $F_s^n(q)$ denote, as previously, the union of s end free arcs in $D(r,s)$ every two of which have the point q in common only. Since according to (2.5) the remainder W consists of end points of $D(r,s)$ only, we have

$$E(D(r,s))=W\cup\Big(\bigcup\Big\{\bigcup\Big\{E\Big(F_s^n(q)\Big):q\in R_n\Big\}:n\in\mathbb{N}\Big\}\Big). \tag{3.2}$$

Observe that, simply by the construction, each point of $E(F_s^n(q))$ is an isolated point of $E(D(r,s))$. Further, since G^r is homeomorphic to $D(r,0)\subset D(r,s)$ by (2.7) and (2.9) and since $E(G^r)$ is homeomorphic to the Cantor set according to [1, Theorem 4.1], it follows again from (2.9) that

$$W \text{ is homeomorphic to the Cantor set.} \tag{3.3}$$

Note further that if K is a component of the set

$$S=D(r,s)\setminus\Big(W\cup\bigcup\Big\{\bigcup\Big\{F_s^n(q):q\in R_n\Big\}:n\in\mathbb{N}\Big\}\Big), \tag{3.4}$$

then

$$\text{there is } n\in\mathbb{N} \text{ such that } K\subset X_n\setminus R_n, \tag{3.5}$$

$$\operatorname{cl}(K) \text{ is an interior free arc of } D(r,s) \text{ with one end point in } R_n \text{ and the other in } R_{n+1}. \tag{3.6}$$

Therefore $D(r,s)$ can be written as the following union:

$$\begin{aligned} D(r,s)=W&\cup\Big(\bigcup\Big\{K:K \text{ is a component of } S\Big\}\Big)\\ &\cup\Big(\bigcup\Big\{\bigcup\Big\{F_s^n(q):q\in R_n\Big\}:n\in\mathbb{N}\Big\}\Big). \end{aligned} \tag{3.7}$$

Now we are ready to show the next result.

Example 3.8. For every $r\in\{3,4,\dots\}$ and $s\in\{0,1,2,\dots,\aleph_0\}$ there are a subdendrite $Y\subset D(r,s)$ and an open retraction $g:D(r,s)\to Y$ such that $D(r,s)$ is not an open image of Y.

Proof. Fix r and s as assumed. Let $D(r,s)$ be defined as the inverse limit by (2.2) and let, as previously, v be the only ramification point of the fan X_1. For each $n\in\mathbb{N}$ choose a ramification point $p_n\in R_n\subset D(r,s)$. Thus $p_1=v$ and $p_n\in vp_{n+1}$ for each $n\in\mathbb{N}$. Then the sequence $\{p_n\}$ is convergent, and its limit $e_0=\lim p_n$ is, according to Statement 2.3, an end point of $D(r,s)$ lying in the set W. Thus all points p_n lie in the arc ve_0, and if $<$ is the natural ordering of ve_0 from v to e_0, then

$$v=p_1<p_2<\cdots<p_n<p_{n+1}<\cdots<e_0. \tag{3.8}$$

Further, for each $n\in\mathbb{N}$ take $F_s^n(p_n)\subset X_n\subset D(r,s)$ and note that $ve_0\cap F_s^n(p_n)=\{p_n\}$. Put

$$Y=ve_0\cup\bigcup\Big\{F_s^n(p_n):n\in\mathbb{N}\Big\}\subset D(r,s). \tag{3.9}$$

Define a mapping $g : D(r,s) \to Y$ such that

- $g \mid Y : Y \to Y$ is the identity;
- for each component $K = ab$ of S as in (3.4), if $a \in R_n$ and $b \in R_{n+1}$, where n is determined by (3.5), the restriction $g \mid K : K \to p_n p_{n+1} \subset ve_0$ is a homeomorphism such that $g(a) = p_n$ and $g(b) = p_{n+1}$;
- for each $F_s^n(q)$ with $q \in R_n$ for some $n \in \mathbb{N}$ the restriction $g \mid F_s^n(q) : F_s^n(q) \to F_s^n(p_n)$ is a homeomorphism;
- $g(W) = \{e_0\}$.

By (3.7) the mapping g is well defined. It can be verified that g is the needed open retraction.

To see that Y cannot be openly mapped onto $D(r,s)$, it is enough to observe that the set $E(Y)$ is countable, while $E(D(r,s))$ is not countable by (2.9). Hence the conclusion follows from (3.1.1) and (3.1.3)(b). □

Corollary 3.9. *For every $r \in \{3,4,\dots\}$ and $s \in \{0,1,2,\dots,\aleph_0\}$ the dendrite $D(r,s)$ is not minimal in the class $\mathfrak{D}_{\leq_{\mathbb{O}}}$.*

The above results can be summarized in the following theorem.

Theorem 3.10. *There are only two minimal elements of the class $\mathfrak{D}_{\leq_{\mathbb{O}}}$ among all members $D(r,s)$ of $\mathcal{F}$ for $r \in \{3,4,\dots,\aleph_0\}$ and $s \in \{0,1,2,\dots,\aleph_0\}$, namely, $D(\aleph_0,1)$ and $D(\aleph_0,\aleph_0)$. Only one of them, namely, $D(\aleph_0,\aleph_0)$, is unique minimal.*

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