IJMMS 27:11 (2001) 695–699
PII. S0161171201010547
http://ijmms.hindawi.com

STRICTLY WEBBED SPACES AND REGULARITY PROPERTIES OF INDUCTIVE LIMITS

ARMANDO GARCÍA-MARTÍNEZ

(Received 9 January 2001)

ABSTRACT. Sequentially complete, locally complete, locally Baire, and bornivorously webbed are equivalent for strictly webbed spaces. For inductive limits of strictly webbed spaces these properties are equivalent. Moreover, they imply regularity.

2000 Mathematics Subject Classification. 46A13, 46A17.

1. Introduction. Throughout this note E is a locally convex space and $E_1 \subset E_2 \subset \cdots$ is a sequence of Hausdorff locally convex spaces with continuous identity maps $\mathrm{id} : (E_n, \tau_n) \to (E_{n+1}, \tau_{n+1})$, $n \in \mathbb{N}$ where τ_n is the topology of E_n. Their locally convex inductive limit is denoted by $\operatorname{ind} E_n$. A *web* W in a locally convex space E is a countable family of absolutely convex subsets of E, arranged in *layers*. The first layer of the web consists of a sequence $(A_p : p = 1, 2, \ldots)$ whose union absorbs each point of E. For each set A_p of the first layer there is a sequence $(A_{pq} : q = 1, 2, \ldots)$ of sets, called the sequence determined by A_p, such that

$$\begin{gathered} A_{pq} + A_{pq} \subset A_p \quad \text{for each } q, \\ \bigcup \{A_{pq} : q = 1, 2, \ldots\} \quad \text{absorbs each point of } A_p. \end{gathered} \tag{1.1}$$

Further layers are made up in a corresponding way so that each set of the kth layer is indexed by a finite row of k integers and at each step the above mentioned two conditions are satisfied. Suppose that we choose a set A_p from the first layer, then a set A_{pq} of the sequence determined by A_p and so on. The resulting sequence $S = (A_p, A_{pq}, A_{pqr}, \ldots)$ is called a strand. Whenever we are dealing with only one strand we can simplify the notation by writing $W_1 = A_p$, $W_2 = A_{pq}$, and so forth, thus $S = (W_k)$ is a strand where for each k, W_k is a set of the kth layer.

Let $S = (W_k)$ be a strand. Consider $x_k \in W_k$ and the series $\sum_{k=1}^{\infty} x_k$. The space E is *webbed* if the series $\sum_{k=1}^{\infty} x_k$ is convergent for any choice of $x_k \in W_k$; E is *strictly webbed* if $\sum_{k=n+1}^{\infty} x_k$ converges to some element in W_n for every $n \in \mathbb{N}$ and for any choice of $x_k \in W_k$; E is *bornivorously webbed* if it is strictly webbed and for every bounded set $A \subset E$, there exist a strand $(W_k)_k$ and a sequence $(\alpha_k)_k \subset \mathbb{C}$ such that $A \subset \alpha_k W_k$, for every $k \in \mathbb{N}$ [2, 6, 7, 9].

A *disk* $A \subset E$ is an absolutely convex, bounded and closed set. Let E_A denote the linear span of A endowed with the normed topology generated by the Minkowski functional ρ of A. This topology is finer than the topology inherited from E. If (E_A, ρ_A) is a Banach (Baire) space, A is a *Banach* (*Baire*) *disk*. A locally convex space is *locally* *complete* (*locally Baire*) if every bounded subset is contained in a Banach (Baire) disk. E is a *quasi-locally complete space* if for each bounded subset B in (E,τ) there exists a weaker locally convex topology $\varsigma=\varsigma(B)$ on E and a Banach disk A in (E,ς) such that $B\subset A$ [8]. Note that locally complete implies quasi-locally complete.

E satisfies the *Mackey convergence condition* if for every null sequence $(x_n)_n\subset E$, there exists a disk A such that $(x_n)_n$ is a ρ_A-null sequence. Finally, E satisfies *property* K if each null sequence has a series convergent subsequence.

2. Bornivorously webbed

Lemma 2.1. *Let (E,τ) be a bornivorously webbed space. Then for every bounded set $A\subset E$ there exists a Fréchet space (F,γ) such that A is contained and bounded in F.*

Proof. Let $A\subset E$ be a bounded set. Then there exist a strand $(W_k)_k\subset W$ and a sequence $(\alpha_k)_k\subset\mathbb{C}$ such that $A\subset\alpha_kW_k$, for every $k\in\mathbb{N}$. Consider $E_{W_k}=\operatorname{span}(W_k)$ and $F=\bigcap_{k\in\mathbb{N}}E_{W_k}$. Let $\{F\cap(1/k)W_k : k\in\mathbb{N}\}$ be a fundamental system of neighborhoods of zero in F. This topology is metrizable and finer than τ. We will see that it is complete. Let $(x_k)_k\subset F$ be a Cauchy sequence, and take $(y_k)_k\subset(x_k)_k$ such that $(y_{k+1}-y_k)\in W_k/k$. Then $\sum_{k=1}^{\infty}(y_{k+1}-y_k)\xrightarrow{\tau}u$, for some u in E. $\sum_{k=p+1}^{\infty}(y_{k+1}-y_k)\in W_p/p$, for every $p\in\mathbb{N}$, so $\sum_{k=1}^{\infty}(y_{k+1}-y_k)\in F$. Hence $\sum_{k=1}^{\infty}(y_{k+1}-y_k)\xrightarrow{F}u$ and if $x=u+y_1$, we have $y_k\xrightarrow{F}x$ and $x_k\xrightarrow{F}x$. □

If $F=E$, and $\{(1/k)W_k : k\in\mathbb{N}\}$ is a fundamental system of neighborhoods, then E with the topology γ generated by this family is a Fréchet space. This topology is finer than the original one.

Theorem 2.2. *Let (E,τ) be a locally convex space. If E is strictly webbed, then the following properties are equivalent:*

(a) *E is sequentially complete.*
(b) *E is locally complete.*
(c) *E is locally Baire.*
(d) *E is bornivorously webbed.*

Proof. (a)⇒(b)⇒(c). The proof is obvious. (c)⇒(d). Let A be a bounded subset of E and $B\subset E$ be a Baire disk such that A is contained and bounded in B. By [6, Theorem 5.6.3] for $\mathrm{id}: E_B\to E$, there exists a strand $(W_k)_k$ such that $\mathrm{id}^{-1}(W_k)\in N_0(E_B)$. Hence, for every $k\in\mathbb{N}$ there exists $\alpha_k\in\mathbb{C}$ such that $A\subset\alpha_k\,\mathrm{id}^{-1}(W_k)\subset E_B$ and $A\subset\alpha_kW_k\subset E$.

(d)⇒(a). The argument of the proof is taken from [1, Theorem 1]: let $(x_n)_n$ be a Cauchy sequence in E, and $B_n=\mathrm{cl}_E\operatorname{co}\bigcup\{x_n : m\geq n\}$, $n\in\mathbb{N}$. The set B_1 is bounded in E which is bornivorously webbed, hence there exists a strand (W_k) in E and a sequence $(\alpha_k)_k\subset\mathbb{C}$ such that $B_1\subset\alpha_kW_k$ for each $k\in\mathbb{N}$. Denote by γ the topology on E generated by the subbasis $\{W_k : k\in\mathbb{N}\}$ and, for brevity, by F the space (E,γ).

The set $B_1\subset E$ is closed in E, and by the preceding lemma, it is closed in the locally convex space F. Since B_1 is convex, it is also weakly closed in F.

By lemma, F is a Fréchet space. Hence the canonical imbedding $F\to F''$, where F'' is the second dual of F equipped with the strong topology, is a topological isomorphism into F''. Since F is complete, it is closed in F'' and each functional from the strong dual F' of F can be continuously extended to F''. Thus the $\sigma(F,F')$-closed set B_1 is also $\sigma(F'',F')$-closed in F''.

Further, since B_1 is bounded in F'', it is equicontinuous in F'. Hence by Alaoglu theorem, the set B_1 is relatively $\sigma(F'',F')$-compact. This, together with the $\sigma(F'',F')$-closedness, implies that B_1 is $\sigma(F'',F')$-compact in F''.

Similarly, all sets B_n, $n \in \mathbb{N}$, are $\sigma(F'',F')$-compact. Every finite intersection $\bigcap\{B_n : 1 \le n \le m\} = B_m$, $m \in \mathbb{N}$, is nonempty. Hence there exists $x_0 \in \bigcap\{B_n : n \in \mathbb{N}\} \subset B_1 \subset E$. This implies the existence of an upper triangular matrix $\Lambda = (\lambda_{nm})$ with all entries $\lambda_{nm} \ge 0$, only finite number of nonzeros in each row, and the sum of all entries in each row is equal to 1, such that the sequence $\{y_n = \sum_{m=n}^{\infty} \lambda_{nm} x_m\}_n$ converges to x_0 in the topology γ. Then the continuity of the identity map $F \to E$ implies the convergence $y_k \to x_0$ in E.

Take a balanced, convex, zero neighborhood V in E. Then there exist $p,q \in \mathbb{N}$ such that $y_n - x_0 \in V$ for $n \ge p$ and $x_m - x_n \in V$ for $m \ge n \ge q$. Then for $n \ge \max(p,q)$, we have

$$x_0 - x_n = (x_0 - y_n) + (y_n - x_n) = (x_0 - y_n) + \sum_{m=n}^{\infty} \lambda_{nm}(x_m - x_n) \in V + V. \tag{2.1}$$

This implies $x_n \to x_0$ in the space E. □

Since property K implies locally Baire (see [4, Theorem 2]), this theorem proves that for strictly webbed spaces, property K implies local completeness. This answers Gilsdorf's question 3.2 in [4] in a negative way. Moreover, these different additional properties for strictly webbed spaces, which appear in [1, 4, 5], are proved to be all equivalent.

3. Inductive limits. Let $(E_n,\tau_n)_n$ be an inductive sequence of locally convex spaces, and let $(E,\tau) = \operatorname{ind}(E_n,\tau_n)$ be its inductive limit. The space (E,τ) is *regular* if for each bounded subset B in (E,τ), there exists $n = n(B) \in \mathbb{N}$ such that B is contained and bounded in (E_n,τ_n). (E,τ) is *sequentially retractive* if for each convergent sequence $(x_k)_k$ in (E,τ) there exists $n = n((x_k)_k) \in \mathbb{N}$ such that the sequence converges to the same limit in (E_n,τ_n). Equivalently, each null sequence in (E,τ) is a null sequence in some (E_n,τ_n).

Sequentially retractive inductive limits were introduced and studied by Floret [3, 7]. He proved that sequential retractivity implies regularity. In order to get more information about the relation between regularity and sequential retractivity, we will prove the following proposition.

Proposition 3.1. *Let $(E,\tau) = \operatorname{ind}(E_n,\tau_n)$ be a regular inductive limit. If E satisfies the Mackey convergence condition, then it is sequentially retractive.*

Proof. Let $(x_k)_k$ be a null sequence in E. Since the Mackey convergence condition holds, there exists a bounded disk $B \subset E$ such that $(x_k)_k$ is a ρ_B-null sequence. Now E is regular, so B is contained and bounded in some E_n. So, the topology ρ_B in $E_B \subset E_n$, is finer than that inherited from E_n. Hence $(x_k)_k$ is an E_n-null sequence. □

In the next propositions, we present other relations between these properties and regularity for strictly webbed spaces.

Following Floret [3] and the proof of Theorem 1 in [1], if $(E,\tau) = \operatorname{ind}(E_n,\tau_n)$ is an inductive limit of an inductive sequence of bornivorously webbed spaces note that we have:

E sequentially retractive it follows that E is regular and implies that if $x_k \xrightarrow{\tau} x_0$, then there exists $n_0 \in \mathbb{N}$ and a sequence $\{y_k : y_k \in \operatorname{conv}\{x_m\}_{m=k}^{\infty}\}_{k=1}^{\infty}$ such that $y_k \xrightarrow{\tau_{n_0}} x_0$.

Theorem 3.2. *Let $(E,\tau) = \operatorname{ind}(E_n,\tau_n)$ be the inductive limit of an inductive sequence of strictly webbed locally convex spaces. Consider the conditions:*

(a) *E satisfies property K*
(b) *E is locally Baire*
(c) *E is bornivorously webbed*
(d) *E is sequentially complete*
(e) *E is locally complete*
(f) *E is quasi-locally complete*
(g) *E is regular.*

Then (a)⇒(b)⇔(c)⇔(d)⇔(e) *and* (e)⇒(f)⇒(g).

Proof. (a)⇒(b). [4, Theorem 2].

(b)⇔(c)⇔(d)⇔(e). By Theorem 2.2, since the inductive limit of strictly webbed spaces is strictly webbed.

(e)⇒(f). It is clear.

(f)⇒(g). [8, Theorem 1]. □

Proposition 3.3. *Let $(E,\tau) = \operatorname{ind}(E_n,\tau_n)$ be the inductive limit of an inductive sequence of strictly webbed locally convex spaces such that every (E_n,τ_n) satisfies property K. If E is sequentially retractive then E satisfies property K.*

Proof. Let $(x_m)_m$ be a null sequence in E. Then there exist $n \in \mathbb{N}$, with $x_m \xrightarrow{E_n} 0$ and a subsequence $(x_{m_k})_k \subset (x_m)_m$ such that $\sum_{k=1}^{\infty} x_{m_k} \xrightarrow{E_n} x$. Therefore $\sum_{k=1}^{\infty} x_{m_k} \xrightarrow{E} x$. □

Note that combining the results of this section, and under the hypothesis of Proposition 3.3 we have (a)⇒(b)⇔(c)⇔(d)⇔(e)⇒(f)⇒(g). Moreover if E satisfies the Mackey convergence condition they are all equivalent.

Acknowledgement. I would like to thank Dr. C. Bosch for his valuable suggestions.

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Armando García-Martínez: Instituto de Matemáticas, U.N.A.M. Area de la Investigación Científica, Circuito Exterior, Ciudad Universitaria, México, D.F. 04510, Mexico

E-mail address: agarcia@matem.unam.mx