IJMMS 27:2 (2001) 65–68
PII. S0161171201010742
http://ijmms.hindawi.com

REMARKS ON A PAPER BY SILVERMAN

VIKRAMADITYA SINGH

(Received 15 August 2000 and in revised form 23 November 2000)

Abstract. We improve a result in Silverman's paper (1999) and answer a question he posed. We also consider a similar problem and obtain sufficient conditions for starlikeness.

2000 Mathematics Subject Classification. 30C45.

1. Introduction. Let A be the class of analytic functions in the unit disc $U = \{z : |z| < 1\}$ having expansion of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots \tag{1.1}$$

and let $S \subset A$ be the set of univalent functions in U. A function $f \in S$ is said to be starlike of order α, $0 < \alpha < 1$, and is denoted by S^*_α if $\operatorname{Re} z(f'(z)/f(z)) > \alpha$, $z \in U$ and is said to be convex and is denoted by C if $\operatorname{Re}\{1 + z(f''(z)/f'(z))\} > 0$, $z \in U$. Silverman [2] investigated properties of the functions $f \in A$ and the class

$$G_b = \left\{ f \in A \;\middle|\; \left| \left(\frac{1 + z(f''(z)/f'(z))}{z(f'(z)/f(z))} \right) - 1 \right| < b,\ z \in U \right\}. \tag{1.2}$$

Some of the results established by him and relevant to us are given in the following theorem.

Theorem 1.1. *Let $f \in G_b$ then*

(i) *If $0 < b \le 1$, $G_b \subset S^*(2/(1+\sqrt{1+8b}))$ and in particular $G_1 \subset S^*(1/2)$.*

(ii) *$G_b \subset C$ for $0 < b \le 1/\sqrt{2}$ and $G_1 \not\subset C$.*

(iii) *For $b \ge 11.66$, $G_b \not\subset S^*(0)$ and for large enough b, $G_b \not\subset S$.*

His method did not extend to $b > 1$ and he expected the order of starlikeness of G_b to decrease from $1/2$ to 0 as b increases from 1 to some value b_0 after which functions in G_b need not be starlike.

In this paper we establish the following theorems.

Theorem 1.2. *Let $f \in G_b$, $0 < b \le 1$, then $G_b \subset S^*(1/(1+b))$ and this order of starlikeness is sharp. Furthermore, for $b > 1$ the elements of G_b need not be regular in U.*

We notice that if we put $p(z) = z(f'(z)/f(z))$, then $p(z)$ is analytic in U with $p(0) = 1$ and G_b gets transformed to

$$G_b = \left\{ f \in A,\ p(z) = z\frac{f'(z)}{f(z)} \;\middle|\; \left| z\frac{p'(z)}{p^2(z)} \right| < b,\ z \in U \right\}. \tag{1.3}$$

Definition 1.3. An analytic function $f(z)$ is said to be subordinate to another analytic function $g(z)$, denoted symbolically as $f(z) \prec g(z)$, if $f(0) = g(0)$ and there exists an analytic function $\omega(z) \in A$, $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in U$ such that $f(z) = g(\omega(z))$.

Theorem 1.4. *Let* $-1 \le \alpha \le 1$, $0 \le a < 1$, $\lambda > 0$ *and let* $p(z)$ *be an analytic function in* U, $p(0) = 1$, $p(z) \ne 0$, $z \in U$ *satisfy the subordination*

$$z\frac{p'(z)}{p^2(z)} \prec \frac{\lambda z}{(1+az)^{1+\alpha}}. \tag{1.4}$$

Then

$$\begin{gathered} \frac{1}{p(z)} \prec 1 - \frac{\lambda}{a\alpha}\left(1-(1+az)^{-\alpha}\right), \quad \alpha \ne 0, \\ \operatorname{Re}\frac{1}{p(z)} > 0 \quad \text{if } 0 < \lambda \le \frac{a\alpha}{1-(1+a)^{-\alpha}}, \ \alpha \ne 0. \end{gathered} \tag{1.5}$$

For $\alpha = 0$ *and* $0 < \lambda \le a/\log(1+a)$

$$p(z) \prec \frac{1}{1-(\lambda/a)\log(1+az)}, \qquad \operatorname{Re} p(z) > \left(1 - \frac{\lambda}{a}\log(1-a)\right)^{-1}. \tag{1.6}$$

The special case of (1.4) for $\alpha = 1$, $\lambda = b-a$, $-1 \le b < a \le 1$ had been considered in [1]. Silverman's case corresponds to $\alpha = -1$.

In the notation of subordination the class G_b defined by (1.3) can equivalently be written as

$$G_b = \left\{ f \in A,\ p(z) = z\frac{f'(z)}{f(z)} \;\middle|\; z\frac{p'(z)}{p^2(z)} \prec bz,\ z \in U \right\}. \tag{1.7}$$

We need the following result from [3].

Theorem 1.5. *If* h *is starlike in* U, $h(0) = 0$ *and* p *is analytic in* U, $p(0) = 1$ *satisfies*

$$zp'(z) \prec h(z), \tag{1.8}$$

then

$$p(z) \prec q(z) = 1 + \int_0^z \frac{h(t)}{t}\,dt, \tag{1.9}$$

where q *is a convex function.*

2. Proof of Theorem 1.2. From (1.7), $f \in G_b$ is equivalent to

$$z\frac{p'(z)}{p^2(z)} = b\omega(z), \quad \omega(0) = 0, \ |\omega(z)| < 1. \tag{2.1}$$

By integration from 0 to z and using $p(0) = 1$, we get

$$\frac{1}{p(z)} = 1 - b\int_0^1 \frac{\omega(tz)}{t}\,dt. \tag{2.2}$$

From (2.2) using Schwartz lemma for $\omega(z)$, we get

$$\left|1-\frac{1}{p(z)}\right| \le b|z|, \tag{2.3}$$

or equivalently, $|z| = r$ and

$$|p^2(z)| - 2\operatorname{Re} p(z) + 1 \le b^2 r^2 |p^2(z)|. \tag{2.4}$$

Therefore, if $b \le 1$,

$$(1-b^2r^2)|p^2(z)| - 2\operatorname{Re} p(z) + 1 \le 0. \tag{2.5}$$

This is equivalent to

$$\left|p(z) - \frac{1}{1-b^2r^2}\right| \le \frac{br}{1-b^2r^2}, \quad \text{if } 0 \le b \le 1, \tag{2.6}$$

$$\left|p(z) + \frac{1}{b^2-1}\right| \ge \frac{b}{b^2-1}, \quad \text{if } b > 1. \tag{2.7}$$

Equation (2.6) gives

$$\operatorname{Re} p(z) \ge \frac{1}{1+b} \tag{2.8}$$

and this is sharp because

$$p(z) = \frac{1}{1+bz} \Longrightarrow f(z) = \frac{z}{1+bz} \tag{2.9}$$

satisfies (2.6). The function $p(z)$ given by (2.9) satisfies (1.7) even for $b > 1$. However, (2.9) shows that for $b > 1$ both $p(z)$ and $f(z)$ have a pole at $z = -1/b$ and $\operatorname{Re} p(z)$ can be negative. Thus, the functions $f \in G_b$ for $b > 1$ need not even be regular. □

3. Proof of Theorem 1.4. We notice that the function $z/(1+az)^{1+\alpha}$, $0 \le a < 1$, is starlike for $0 < \alpha \le 1$ and convex for $-1 \le \alpha \le 0$. Since every convex function is starlike, we obtain, from (1.4) and Theorem 1.5,

$$\begin{aligned} \frac{1}{p(z)} &\prec 1 - \frac{\lambda}{a\alpha}\left(1-(1+az)^{-\alpha}\right), \quad \alpha \ne 0, \\ \frac{1}{p(z)} &\prec 1 - \frac{\lambda}{a}\log(1+az), \quad \alpha = 0. \end{aligned} \tag{3.1}$$

As $(1+az)^{-\alpha}$, $|\alpha| \le 1$, $\alpha \ne 0$, is a convex function with real coefficients, we obtain

$$\begin{aligned} \operatorname{Re}\frac{1}{p(z)} > 0 \quad &\text{if } 0 < \lambda \le \frac{a\alpha}{1-(1+a)^{-\alpha}}, \ |\alpha| \le 1, \ \alpha \ne 0, \\ \operatorname{Re}\frac{1}{p(z)} > 0 \quad &\text{if } 0 < \lambda \le \frac{a}{\log(1+a)}, \ \alpha = 0. \end{aligned} \tag{3.2}$$

Hence,

$$\operatorname{Re} p(z) \geq \frac{1}{1-(\lambda/a\alpha)\{1-(1-a)^{-\alpha}\}}, \quad \alpha \neq 0 \tag{3.3}$$

and $f(z)$ satisfying $p(z) = z(f'(z)/f(z))$ is starlike of order $1/(1-(\lambda/a\alpha)\{1-(1-a)^{-\alpha}\})$, $\alpha \neq 0$ and $1/(1-(\lambda/a)\log(1-a))$ for $\alpha = 0$.

In the special case $\alpha = 1$ and $\lambda = a-b$ we obtain $\operatorname{Re} p(z) \geq (1-a)/(1-b)$, $-1 \leq b < a \leq 1$ which corresponds to the case in [1]. If $\alpha = -1$, we obtain $\operatorname{Re} p(z) > 1/(1+\lambda)$ which agrees with Theorem 1.2. □

References

[1] M. Obradović and S. Owa, *A criterion for starlikeness*, Math. Nachr. **140** (1989), 97–102. MR 90i:30020. Zbl 676.30009.

[2] H. Silverman, *Convex and starlike criteria*, Int. J. Math. Math. Sci. **22** (1999), no. 1, 75–79. MR 2000a:30028. Zbl 921.30009.

[3] T. J. Suffridge, *Some remarks on convex maps of the unit disk*, Duke Math. J. **37** (1970), 775–777. MR 42#4722. Zbl 206.36202.

Vikramaditya Singh: 3A/95 Azad Nagar, Kanpur UP 208002, India
E-mail address: rasingh@rocketmail.com