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A NEW INEQUALITY FOR A POLYNOMIAL

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ABSTRACT. Let $p(z) = a_0 + \sum_{j=t}^{n} a_j z^j$ be a polynomial of degree n, having no zeros in $|z| < k$, $k \ge 1$, then it has been shown that for $R > 1$ and $|z| = 1$, $|p(Rz) - p(z)| \le (R^n - 1)(1 + A_t B_t k^{t+1})/(1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t}))\max_{|z|=1}|p(z)| - \{1 - (1 + A_t B_t k^{t+1})/(1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t}))\}((R^n - 1)m/k^n)$, where $m = \min_{|z|=k}|p(z)|$, $1 \le t < n$, $A_t = (R^t - 1)/(R^n - 1)$, and $B_t = |a_t/a_0|$. Our result generalizes and improves some well-known results.

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1. Introduction and statements of results. Let $p(z)$ be a polynomial of degree n, then

$$\max_{|z|=1} |p'(z)| \le n \max_{|z|=1} |p(z)|, \tag{1.1}$$

$$\max_{|z|=R>1} |p(z)| \le R^n \max_{|z|=1} |p(z)|. \tag{1.2}$$

Inequality (1.1) is a famous result known as Bernstein's inequality (see [9]) where as inequality (1.2) is a simple consequence of maximum modulus principle [7]. Here in both inequalities (1.1) and (1.2) the equality holds if and only if $p(z)$ has all its zeros at the origin.

If $p(z)$ does not vanish in $|z| < 1$, then (1.1) and (1.2) can be respectively replaced by

$$\max_{|z|=1} |p'(z)| \le \frac{n}{2} \max_{|z|=1} |p(z)|, \tag{1.3}$$

$$\max_{|z|=R>1} |p(z)| \le \frac{R^n + 1}{2} \max_{|z|=1} |p(z)|. \tag{1.4}$$

Inequality (1.3) was conjectured by Erdös and later proved by Lax [5], whereas inequality (1.4) is due to Ankeny and Rivlin [1]. Here in both inequalities (1.3) and (1.4), the equality holds for $p(z) = \alpha + \beta z^n$, $|\alpha| = |\beta|$. Inequalities (1.3) and (1.4) are, respectively, much better than inequalities (1.1) and (1.2). As a generalization of (1.3), it was shown by Malik [6] that if $p(z)$ does not vanish in $|z| < k$, $k \ge 1$, then

$$\max_{|z|=1} |p'(z)| \le \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.5}$$

The result is sharp and the extremal polynomial is $p(z) = (z + k)^n$.

Chan and Malik [3] considered the class of polynomials $p(z) = a_0 + \sum_{j=t}^{n} a_j z^j$, $1 \le t \le n$, and proved the following extension of inequality (1.5).

Theorem 1.1. *If $p(z) = a_0 + \sum_{j=t}^{n} a_j z^j$ is a polynomial of degree n, having no zeros in the disk $|z| < k$ where $k \geq 1$, then for $1 \leq t \leq n$,*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^t} \max_{|z|=1} |p(z)|. \tag{1.6}$$

The result is best possible and equality holds for the polynomial $p(z) = (z^t + k^t)^{n/t}$, where n is a multiple of t.

Inequality (1.6) was also independently proved by Qazi [8, Lemma 1] who, in fact, has also proved the following result.

Theorem 1.2. *If $p(z) = a_0 + \sum_{j=t}^{n} a_j z^j$ is a polynomial of degree n, having no zeros in the disk $|z| < k$ where $k \geq 1$, then for $1 \leq t \leq n$,*

$$\max_{|z|=1} |p'(z)| \leq n \frac{1 + (t/n)|a_t/a_0| k^{t+1}}{1 + k^{t+1} + (t/n)|a_t/a_0|(k^{t+1} + k^{2t})} \max_{|z|=1} |p(z)|. \tag{1.7}$$

In this paper, we improve inequality (1.7) for the class of polynomials $p(z) = a_0 + \sum_{j=t}^{n} a_j z^j$, $1 \leq t < n$, not vanishing in the disk $|z| < k$, $k \geq 1$. More precisely, we prove the following result.

Theorem 1.3. *If $p(z) = a_0 + \sum_{j=t}^{n} a_j z^j$ is a polynomial of degree n which does not vanish in $|z| < k$ where $k \geq 1$, then for every $R > 1$ and $|z| = 1$,*

$$\begin{aligned} |p(Rz) - p(z)| \leq{}& (R^n - 1) \frac{1 + A_t B_t k^{t+1}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \max_{|z|=1} |p(z)| \\ & - \left\{ 1 - \frac{1 + A_t B_t k^{t+1}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \right\} \frac{(R^n - 1)m}{k^n}, \end{aligned} \tag{1.8}$$

where $m = \min_{|z|=k} |p(z)|$, $1 \leq t < n$, $A_t = (R^t - 1)/(R^n - 1)$ and $B_t = |a_t/a_0|$.

Remark 1.4. If we divide the two sides of (1.8) by $(R-1)$ and let $R \to 1$, we get

$$\begin{aligned} \max_{|z|=1} |p'(z)| \leq{}& n \frac{1 + (t/n)|a_t/a_0| k^{t+1}}{1 + k^{t+1} + (t/n)|a_t/a_0|(k^{t+1} + k^{2t})} \max_{|z|=1} |p(z)| \\ & - \left\{ 1 - \frac{1 + (t/n)|a_t/a_0| k^{t+1}}{1 + k^{t+1} + (t/n)|a_t/a_0|(k^{t+1} + k^{2t})} \right\} \frac{mn}{k^n} \end{aligned} \tag{1.9}$$

which is an improvement of (1.7) due to Qazi [8] for $1 \leq t < n$.

If we use the fact that

$$|p(Rz) - p(z)| \geq |p(Rz)| - |p(z)| \tag{1.10}$$

or

$$|p(Rz)| \leq |p(Rz) - p(z)| + |p(z)|, \tag{1.11}$$

the following corollary is an immediate consequence of the above theorem.

Corollary 1.5. *If $p(z) = a_0 + \sum_{j=t}^{n} a_j z^j$ is a polynomial of degree n which does not vanish in $|z| < k$ where $k \geq 1$, then for $R > 1$*

$$\begin{aligned}\max_{|z|=R} |p(z)| \leq{}& \frac{R^n\{1 + A_t B_t k^{t+1}\} + k^{t+1} + A_t B_t k^{2t}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \max_{|z|=1} |p(z)| \\ &- \left\{1 - \frac{1 + A_t B_t k^{t+1}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})}\right\} \frac{(R^n - 1)m}{k^n},\end{aligned} \tag{1.12}$$

where $m = \min_{|z|=k} |p(z)|$, $1 \leq t < n$, $A_t = (R^t - 1)/(R^n - 1)$, and $B_t = |a_t/a_0|$.

The inequality

$$\frac{R^t - 1}{R^n - 1} \leq \frac{t}{n} \tag{1.13}$$

holds for all $R > 1$ and $1 \leq t \leq n$. To prove this inequality, we observe that for every $R > 1$, it easily follows when $t = n$. Hence to establish (1.13), it suffices to consider the case $1 \leq t \leq n-1$ and $R > 1$. So, we assume that $R > 1$ and $1 \leq t \leq n-1$, and then we have

$$\begin{aligned} tR^n - nR^t + (n-t) &= tR^t(R^{n-t} - 1) - (n-t)(R^t - 1) \\ &= (R-1)\{tR^t(R^{n-t-1} + R^{n-t-2} + \cdots + 1) \\ &\qquad - (n-t)(R^{t-1} + \cdots + R + 1)\} \\ &= (R-1)\{t(n-t)R^t - (n-t)tR^{t-1}\} \\ &= t(n-t)(R-1)^2 R^{t-1} > 0. \end{aligned} \tag{1.14}$$

This implies that $t(R^n - 1) \geq n(R^t - 1)$, for all values of $R > 1$ and $1 \leq t \leq n-1$ which is equivalent to (1.13).

With the help of (1.13) a simple direct calculation yields

$$\begin{aligned} &\frac{R^n\{1 + A_t B_t k^{t+1}\} + k^{t+1} + A_t B_t k^{2t}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \max_{|z|=1} |p(z)| \\ &\quad - \left\{1 - \frac{1 + A_t B_t k^{t+1}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})}\right\} \frac{(R^n - 1)m}{k^n} \\ &\leq \frac{R^n\{1 + (t/n) B_t k^{t+1}\} + k^{t+1} + (t/n) B_t k^{2t}}{1 + k^{t+1} + (t/n) B_t (k^{t+1} + k^{2t})} \max_{|z|=1} |p(z)| \\ &\quad - \left\{1 - \frac{1 + (t/n) B_t k^{t+1}}{1 + k^{t+1} + (t/n) B_t (k^{t+1} + k^{2t})}\right\} \frac{(R^n - 1)m}{k^n}. \end{aligned} \tag{1.15}$$

Hence from Theorem 1.3, we easily deduce the following corollary.

Corollary 1.6. *If $p(z) = a_0 + \sum_{j=t}^{n} a_j z^j$ is a polynomial of degree n which does not vanish in $|z| < k$ where $k \geq 1$, then for every $R > 1$,*

$$\begin{aligned}\max_{|z|=R} |p(z)| \leq{}& \frac{R^n\{1 + (t/n) B_t k^{t+1}\} + k^{t+1} + (t/n) B_t k^{2t}}{1 + k^{t+1} + (t/n) B_t (k^{t+1} + k^{2t})} \max_{|z|=1} |p(z)| \\ &- \left\{1 - \frac{1 + (t/n) B_t k^{t+1}}{1 + k^{t+1} + (t/n) B_t (k^{t+1} + k^{2t})}\right\} \frac{(R^n - 1)m}{k^n},\end{aligned} \tag{1.16}$$

where $m = \min_{|z|=k} |p(z)|$, $1 \leq t < n$, and $B_t = |a_t/a_0|$.

Next, if we take $t = 1$ in Theorem 1.3, we get the following corollary.

Corollary 1.7. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n which does not vanish in the disk $|z| < k$, $k \geq 1$, then for every $R > 1$*

$$\begin{aligned} |p(Rz) - p(z)| \leq (R^n - 1)\frac{1 + A_1 B_1 k^2}{1 + k^2 + A_1 B_1 (2k^2)} \max_{|z|=1} |p(z)| \\ - \left\{1 - \frac{1 + A_1 B_1 k^2}{1 + k^2 + A_1 B_1 (2k^2)}\right\} \frac{(R^n - 1)m}{k^n}, \end{aligned} \tag{1.17}$$

where $m = \min_{|z|=k} |p(z)|$, $A_1 = (R-1)/(R^n - 1)$, and $B_1 = |a_1/a_0|$.

Remark 1.8. If we divide the two sides of (1.17) by $(R-1)$ and let $R \to 1$, it easily follows that, if $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that $p(z) \neq 0$ in $|z| < k$, $k \geq 1$, then for $|z| \leq 1$,

$$\begin{aligned} |p'(z)| \leq n \frac{n|a_0| + k^2|a_1|}{n(1 + k^2)|a_0| + 2k^2|a_1|} \max_{|z|=1} |p(z)| \\ - \left\{1 - \frac{n|a_0| + k^2|a_1|}{n(1 + k^2)|a_0| + 2k^2|a_1|}\right\} \frac{mn}{k^n} \end{aligned} \tag{1.18}$$

which is an improvement of a result due to Govil et al. [4].

It is known that

$$\frac{t}{n}\left|\frac{a_t}{a_0}\right| k^t \leq 1. \tag{1.19}$$

Using this fact and inequality (1.13), it is easy to verify that

$$\frac{1 + A_t B_t k^{t+1}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \leq \frac{1}{1 + k^t}. \tag{1.20}$$

By using these observations, the following result is an immediate consequence of Theorem 1.3.

Corollary 1.9. *If $p(z) = a_0 + \sum_{j=t}^{n} a_j z^j$ is a polynomial of degree n which does not vanish in the disk $|z| < k$ where $k \geq 1$, then for every $R > 1$ and $|z| = 1$,*

$$\begin{aligned} |p(Rz) - p(z)| &\leq \frac{R^n - 1}{1 + k^t} \max_{|z|=1} |p(z)| - \left(1 - \frac{1}{1 + k^t}\right) \frac{(R^n - 1)m}{k^n} \\ &= \frac{R^n - 1}{1 + k^t} \left\{\max_{|z|=1} |p(z)| - \frac{m}{k^{n-t}}\right\} \end{aligned} \tag{1.21}$$

and in the fortiori

$$\max_{|z|=R} |p(z)| \leq \frac{R^n + k^t}{1 + k^t} \max_{|z|=1} |p(z)| - \left(\frac{R^n - 1}{1 + k^t}\right) \frac{m}{k^{n-t}}. \tag{1.22}$$

Remark 1.10. For $k = t = 1$, (1.22) reduces to

$$M(p,R) \le \frac{R^n+1}{2}M(p,1) - \left(\frac{R^n-1}{2}\right)m, \tag{1.23}$$

which is an improvement of (1.4) due to Ankeny and Rivlin [1].

Inequality (1.23) was proved by Aziz and Dawood [2].

2. A lemma

Lemma 2.1. *Let $p(z) = a_0 + \sum_{j=t}^{n} a_j z^j$ be a polynomial of degree n which does not vanish in $|z| < k$ where $k \ge 1$, then for every $R > 1$ and $|z| = 1$,*

$$|p(Rz) - p(z)| \le (R^n - 1)\frac{1 + A_t B_t k^{t+1}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \max_{|z|=1} |p(z)|, \tag{2.1}$$

where $1 \le t < n$, $A_t = (R^t - 1)/(R^n - 1)$ and $B_t = |a_t/a_0|$.

This lemma is due to Shah [10].

3. Proof of Theorem 1.3. By Rouche's theorem, the polynomial $p(z) + m\beta z^n$, $|\beta| < 1/k^n$, has no zero in $|z| < k$, $k \ge 1$. So on applying Lemma 2.1 to the polynomial $p(z) + m\beta z^n$, $|\beta| < 1/k^n$, we get

$$\begin{aligned} &|(p(Rz) + m\beta R^n z^n) - (p(z) + m\beta z^n)| \\ &\quad \le (R^n - 1)\frac{1 + A_t B_t k^{t+1}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \max_{|z|=1} |p(z) + m\beta z^n| \end{aligned} \tag{3.1}$$

or

$$\begin{aligned} &|p(Rz) - p(z) + m\beta z^n (R^n - 1)| \\ &\quad \le (R^n - 1)\frac{1 + A_t B_t k^{t+1}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \max_{|z|=1} \{|p(z)| + |m\beta z^n|\}. \end{aligned} \tag{3.2}$$

Now choosing the argument of β suitably, the above inequality becomes

$$\begin{aligned} &|p(Rz) - p(z)| + |m\beta z^n (R^n - 1)| \\ &\quad \le (R^n - 1)\frac{1 + A_t B_t k^{t+1}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \max_{|z|=1} \{|p(z)| + m|\beta|\} \end{aligned} \tag{3.3}$$

or

$$\begin{aligned} |p(Rz) - p(z)| \le{}& (R^n - 1)\frac{1 + A_t B_t k^{t+1}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})} \max_{|z|=1} |p(z)| \\ &- \left\{1 - \frac{1 + A_t B_t k^{t+1}}{1 + k^{t+1} + A_t B_t (k^{t+1} + k^{2t})}\right\}(R^n - 1)m|\beta|. \end{aligned} \tag{3.4}$$

Finally letting $|\beta| \to 1/k^n$, we get the desired result. □

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References

[1] N. C. Ankeny and T. J. Rivlin, *On a theorem of S. Bernstein*, Pacific J. Math. **5** (1955), 849–852. MR 17,833e. Zbl 0067.01001.

[2] A. Aziz and Q. M. Dawood, *Inequalities for a polynomial and its derivative*, J. Approx. Theory **54** (1988), no. 3, 306–313. MR 89m:30009. Zbl 0663.41019.

[3] T. N. Chan and M. A. Malik, *On Erdös-Lax theorem*, Proc. Indian Acad. Sci. Math. Sci. **92** (1983), no. 3, 191–193. MR 86c:30009. Zbl 0551.30002.

[4] N. K. Govil, Q. I. Rahman, and G. Schmeisser, *On the derivative of a polynomial*, Illinois J. Math. **23** (1979), no. 2, 319–329. MR 81e:30010. Zbl 0408.30003.

[5] P. D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc. **50** (1944), 509–513. MR 6,61f. Zbl 0061.01802.

[6] M. A. Malik, *On the derivative of a polynomial*, J. London Math. Soc. (2) **1** (1969), 57–60. MR 40#2827. Zbl 0179.37901.

[7] G. Pólya and G. Szegö, *Problems and Theorems in Analysis. Vol. I: Series, Integral Calculus, Theory of Functions*, Aeppli Die Grundlehren der mathematischen Wissenschaften, vol. 193, Springer-Verlag, New York, 1972. MR 49#8782. Zbl 0236.00003.

[8] M. A. Qazi, *On the maximum modulus of polynomials*, Proc. Amer. Math. Soc. **115** (1992), no. 2, 337–343. MR 92i:30002. Zbl 0772.30006.

[9] A. C. Schaeffer, *Inequalities of A. Markoff and S. Bernstein for polynomials and related functions*, Bull. Amer. Math. Soc. **47** (1941), 565–579. MR 3,111a. Zbl 0027.05205.

[10] W. M. Shah, *Extremal properties and bounds of the zeros of polynomials*, Ph.D. thesis, submitted to the University of Kashmir, 1998.

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