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ON THE CONSTRUCTIONS OF TITS AND FAULKNER: AN ISOMORPHISM THEOREM

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ABSTRACT. Classification theory guarantees the existence of an isomorphism between any two E_8's, at least over an algebraically closed field of characteristic 0. The purpose of this paper is to construct for any Jordan algebra J of degree 3 over a field Φ of characteristic $\neq 2,3$ an explicit isomorphism between the algebra obtained from J by Faulkner's construction and the algebra obtained from the split octonions and J by Tits construction.

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Let $J = J(N,1)$ be a quadratic Jordan algebra with 1 over a field Φ constructed as in [3] from an admissible nondegenerate cubic form N with base point 1. Let

$$E = \left\{ \begin{bmatrix} \alpha & a \\ b & \beta \end{bmatrix} \mid \alpha,\beta \in \Phi,\ a,b \in J \right\}. \tag{1}$$

Then it is shown in [1] that E is a ternary algebra. Moreover, it is shown in [1] that $S = S(E,R(E)) = E \oplus \tilde{E} \oplus \Phi u \oplus \Phi \tilde{u} \oplus \Phi U \oplus D$ is a Lie algebra of dimension 248, where $\tilde{E}$ denotes a second copy of E. (Recall from [1] that D is the algebra of derivations of E and $R(E) = \Phi U \oplus D$, the span of $\{R(x,y) \mid x,y \in E\} \cup \{U\}$. For $R = E \oplus \Phi u$, $U(E) = \{A \in \mathrm{Hom}_\Phi(R,R) \mid uA \in \Phi u \text{ and } EA \subseteq E\}$ is a Lie algebra with $[AB] = AB - BA$. Here $U \in U(E)$ such that $uU = 2u$ and $xU = x$, for all $x \in E$. So, U is in the center of $U(E)$. Also $R(x,y) \in U(E)$, for $x,y \in E$, is defined by $uR(x,y) = \langle x,y\rangle u$, $zR(x,y) = \langle z,x,y\rangle$, for all $z \in E$, where $\langle , \rangle$ is the bilinear form and $\langle , , \rangle$ is the ternary form defining a ternary Φ-module, as in [1].)

On the other hand, let C be the algebra of split octonions and J an arbitrary Jordan algebra of degree 3. Then according to Tits in [4], $L = \mathrm{Der}(C) \oplus C_0 \otimes J_0 \oplus \mathrm{Der}(J)$ is a Lie algebra, where $\mathrm{Der}(C)$ is the algebra of derivations of C and C_0 is the subspace of C whose elements have trace 0.

Theorem 1. *There is an explicit isomorphism between the Lie algebras constructed by Tits and the Faulkner processes.*

The proof of this theorem is the goal of this paper and the proof completes this paper.

Our attempt is to construct an isomorphism $\Phi : L \to S$ by breaking Φ into three parts Φ_1 restricted to $\mathrm{Der}(C)$, Φ_2 restricted to $C_0 \otimes J_0$, and Φ_3 restricted to $\mathrm{Der}(J)$.

First we construct a candidate S_1 for the image of Φ_1 in S which in turn plays a significant role throughout the paper. This S_1 is obtained by replacing J by Φ_1 in the construction of S. Therefore,

$$S_1 = E_1 \oplus \tilde{E}_1 \oplus \Phi u \oplus \Phi \tilde{u} \oplus \Phi U \oplus D_1. \tag{2}$$

Lemma 2. *The dimension of* $D_1 = 3$.

The study of the derivation algebra D gives detailed information about S. So, we need to look at the action of $R(x_1,y_1)$, $x_1,y_1 \in E$, on an arbitrary element

$$z = \begin{bmatrix} \alpha & a \\ b & \beta \end{bmatrix} \in E. \tag{3}$$

Let

$$x_i = \begin{bmatrix} \alpha_i & a_i \\ b_i & \beta_i \end{bmatrix} \in E \quad \text{for } i = 1,2. \tag{4}$$

Recall from [1] that $R(x,y)$ is the close derivation of E if and only if $\rho(x,y) = \langle x,y\rangle = 0$. Then it is straightforward to check that $R(x_{12},1_{22})$, $R(x_{21},1_{11})$, and $R(x_{12},1_{21})$ are the only distinct derivations of E, where x_{12} is the 2×2 matrix with $(1,2)$th element equal to $x \in J_0$ and 0 elsewhere. In case $J = \Phi 1$, similar calculations and tables show that $R(1_{11},1_{21})$, $R(1_{22},1_{12})$, and $R(1,1)$, where 1 is the 2×2 unit matrix in the last term, form a basis of D_1. Thus, Lemma 2 is proved.

Now it is clear that $\mathrm{Dim}(S_1) = 14$ which is the same as $\mathrm{Dim}(\mathrm{Der}(C))$ as in [4]. So, Φ_1 could be defined conveniently. Moreover, it is necessary to test that the role of S_1 in S is the same as the role of $\mathrm{Der}(C)$ in L. As a matter of fact, this test will provide us with an idea for the definition of Φ_2.

Now let $\{1_{11},1_{12},1_{21},1_{22},\tilde{1}_{11},\tilde{1}_{12},\tilde{1}_{21},\tilde{1}_{22},u,\tilde{u},U,R(1_{11},1_{21}),R(1_{22},1_{12}),R(1,1)\}$ be a basis of S_1. There are 7 copies of J_0 in S. Two copies in E, two in $\tilde{E}$ and three in D. Let these copies of J_0 be represented by x_{12}, x_{21}, $\tilde{x}_{12}$, $\tilde{x}_{21}$, $R(x_{12},1_{22})$, $R(x_{21},1_{11})$, and $R(x_{12},1_{21})$.

Lemma 3. *Let $x \in J_0$, then the Φ-vector space spanned by the seven copies of x is an S_1-module.*

Proof. It is enough to show that the Lie product of the seven copies of J_0 with each of the generators of S_1 are elements of J_0. The computations are lengthy but straightforward. Only two parts are shown below and the rest are similar. The Lie product rule in [1, equation (2.8)] is used.

(1) $[x_{12},1_{11}] = [x_{12},1_{12}] = [x_{12},1_{21}] = [x_{12},1_{22}] = 0$, $[x_{12},\tilde{1}_{11}] = 0$, $[x_{12},\tilde{1}_{12}] = R(x_{21},1_{11})$, $[x_{12},\tilde{1}_{21}] = R(x_{12},1_{21})$, $[x_{12},\tilde{1}_{22}] = [x_{12},1_{22}]$, $[x_{12},u] = 0$, $[x_{12},\tilde{u}] = \tilde{x}_{12}$, $[x_{12},U] = x_{12}$, $[x_{12},R(1_{11},1_{21})] = 0$, $[x_{12},R(1_{22},1_{12})] = -\tilde{x}_{21}$, $[x_{12},R(1,1)] = x_{12}$.

(2) $[R(x_{12},1_{21}),1_{11}] = [R(x_{12},1_{21}),1_{22}] = 0$, $[R(x_{12},1_{21}),1_{12}] = 2x_{12}$, $[R(x_{12},1_{21}),1_{21}] = 2x_{21}$, $[R(x_{12},1_{21}),\tilde{1}_{11}] = [R(x_{12},1_{21}),\tilde{1}_{22}] = 0$, $[R(x_{12},1_{21}),\tilde{1}_{12}] = 2\tilde{x}_{12}$, $[R(x_{12},1_{21}),\tilde{1}_{21}] = 2\tilde{x}_{21}$, $[R(x_{12},1_{21}),u] = [R(x_{12},1_{21}),\tilde{u}] = [R(x_{12},1_{21}),U] = 0$, $[R(x_{12},1_{21}),R(1_{11},1_{21})] = -2R(x_{21},1_{11})$, $[R(x_{12},1_{21}),R(1_{22},1_{12})] = 2R(x_{12},1_{22})$, $[R(x_{12},1_{21}),R(1,1)] = 0$. □

Using the Lie product rule in the Faulkner's construction in [1], we have for $x,y \in J_0$, $[x_{12},y_{12}] = \langle x_{12},y_{12}\rangle u = 0$. Similarly, $[x_{21},y_{21}] = [\tilde{x}_{12},\tilde{y}_{12}] = [\tilde{x}_{21},\tilde{y}_{21}] = 0$. Let

$$z = \begin{bmatrix} \alpha & a \\ b & \beta \end{bmatrix} \in E, \qquad \bar{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \bar{A} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}. \tag{5}$$

It is easy to check that $z[R(x_{12},1_{22}),R(y_{12},1_{22})] = z[R(x_{21},1_{11}),R(y_{21},1_{11})] = \bar{O}$ and $z[R(x_{12},1_{21}),R(y_{12},1_{21})] = 4\bar{A}[R_x,R_y]$. The use of the ternary product $\langle ,,\rangle$ as in [1], equations (1.6) and (2.3), may be needed. For future reference, we note that using similar computations implies $[R(x_{12},1_{21}),R(y_{12},1_{21})] = R(x_{12},y_{21}) - R(xy_{12},1_{21})$.

As in [2, page 105], let $\{e1,e2,f1,f2,f3,g1,g2,g3\}$ be a basis of C. Then C_0 is spanned by $\{e2-e1,f1,f2,f3,g1,g2,g3\}$. Moreover, $\mathrm{Der}(C)$ is 14-dimensional, and is spanned by $\{\mathrm{D}_{f1,f2},\mathrm{D}_{f1,f3},\mathrm{D}_{f2,f3},\mathrm{D}_{g1,g2},\mathrm{D}_{g1,g3},\mathrm{D}_{g2,g3},\mathrm{D}_{e1,gi},\mathrm{D}_{e2,fi}$, $(i = 1,2,3)$, $\mathrm{D}_{f1,g1},(\mathrm{D}_{f2,g2}-\mathrm{D}_{f3,g3})\}$ from [4, page 77]. [Recall that $\mathrm{D}_{x,z} = [R_x,R_y]+[L_x,R_z]+[L_x,L_z]$, where R_x, L_x are the right and left multiplications]. But C_0 is 7-dimensional tells us that L contains 7 copies of J_0. We name them, for $x \in J_0$, as $(e2-e1)\otimes x$, $f1\otimes x$, $f2\otimes x$, $f3\otimes x$, $g1\otimes x$, $g2\otimes x$, $g3\otimes x$. Note that $C_0\otimes J_0$ is a $\mathrm{Der}(C)$-module, because, for $a \in C_0$, $x \in J_0$, $[a\otimes x,\mathrm{D}] = a\mathrm{D}\otimes x \in C_0\otimes J_0$, where $\mathrm{D}\in\mathrm{Der}(C)$. Moreover, for $x,y \in J_0$, $[(e2-e1)\otimes x,(e2-e1)\otimes y] = [R_x,R_y] \in \mathrm{Der}(J)$, and $[fi\otimes x,fi\otimes y] = [gi\otimes x,gi\otimes y] = 0$, $i = 1,2,3$.

Thus, comparing the above two paragraphs, we conclude that $(e2-e1)\otimes x$ and $R(x_{12},1_{21})$ have similar roles in L and S, respectively. Moreover, the annihilator of $(e2-e1)\otimes x$ is spanned by $\{\mathrm{D}_{f1,g1},(\mathrm{D}_{f2,g2}-\mathrm{D}_{f3,g3}),\mathrm{D}_{f1,g2},\mathrm{D}_{f2,g3},\mathrm{D}_{f3,g1},\mathrm{D}_{f1,g3},\mathrm{D}_{f2,g1},\mathrm{D}_{f3,g2}\}$. The annihilator of $R(x_{12},1_{21})$ is spanned by $\{1_{11},1_{22},\tilde{1}_{11},\tilde{1}_{22},u,\tilde{u},U,R(1,1)\}$ which can be seen in the proof of Lemma 3. Studying the actions of the annihilators of $R(x_{12},1_{21})$ and $(e2-e1)\otimes x$ on the rest of the six copies of J_0, respectively, we define ϕ_1 and ϕ_2 in the following way: $\mathrm{D}_{f1,g2}\phi_1 = -31_{11}$, $\mathrm{D}_{f3,g1}\phi_1 = 31_{22}$, $\mathrm{D}_{f2,g3}\phi_1 = 3\tilde{u}$, $\mathrm{D}_{f1,g3}\phi_1 = 3\tilde{1}_{11}$, $\mathrm{D}_{f2,g1}\phi_1 = 3\tilde{1}_{22}$, $\mathrm{D}_{f3,g2}\phi_1 = -3u$, $\mathrm{D}_{f1,g1}\phi_1 = R(1,1)$, $(\mathrm{D}_{f2,g2}-\mathrm{D}_{f3,g3})\phi_1 = -3U$, $\mathrm{D}_{e2,g1}\phi_1 = R(1_{22},1_{12})$, $\mathrm{D}_{e2,g2}\phi_1 = -1_{12}$, $\mathrm{D}_{e2,g3}\phi_1 = \tilde{1}_{12}$, $\mathrm{D}_{e1,f1}\phi_1 = R(1_{11},1_{21})$, $\mathrm{D}_{e1,f2}\phi_1 = \tilde{1}_{21}$, $\mathrm{D}_{e1,f3}\phi_1 = 1_{21}$, and $(f_1\otimes x)\phi_2 = R((1/2)x_{21},1_{11})$, $(f_2\otimes x)\phi_2 = (1/2)\tilde{x}_{21}$, $(f_3\otimes x)\phi_2 = (1/2)x_{21}$, $(g_1\otimes x)\phi_2 = R((1/2)x_{12},1_{22})$, $(g_2\otimes x)\phi_2 = -(1/2)x_{12}$, $(g_3\otimes x)\phi_2 = (1/2)\tilde{x}_{12}$. Recall from [4] that $[a\otimes x,\mathrm{D}] = a\mathrm{D}\otimes x$, for all $\mathrm{D}\in\mathrm{Der}(C)$, $a\in C_0$, $x\in J_0$.

Lemma 4. *The mapping ϕ_1 is an isomorphism and ϕ_2 is a bijection.*

Proof. The map ϕ_1 is injective: let $\mathrm{D}_1\phi_1 = \mathrm{D}_2\phi_1$ for $\mathrm{D}_1,\mathrm{D}_2\in\mathrm{Der}(C)$ and $\mathrm{D}1\neq\mathrm{D}2$. We can write $\mathrm{D}_1\phi_1 = (c_1\mathrm{D}_{f1,g2}+c_2\mathrm{D}_{f3,g1}+\cdots+c_{14}\mathrm{D}_{e1,f3})\phi_1 = c_1(-31_{11})+c_2(31_{22})+\cdots+c_{14}1_{21} = S_1$. Similarly, $\mathrm{D}_2\phi_1 = k_1(-31_{11})+k_2(31_{22})+\cdots+k_{14}1_{21} = S_2$, $c_i,k_i\in\Phi$ and ϕ_1 is clearly linear. But $S_1 = S_2$ implies $c_i = k_i$ and hence $\mathrm{D}_1 = \mathrm{D}_2$.

The map ϕ_1 is surjective: let $S\in S_1$. Then $S = c_1(-31_{11})+c_2(31_{22})+\cdots+c_{14}1_{21} = (c_1\mathrm{D}_{f1,g2}+c_2\mathrm{D}_{f3,g1}+\cdots+c_{14}\mathrm{D}_{e1,f3})\phi_1 = \mathrm{D}_{\phi_1}$ (say). But $\mathrm{D}\in\mathrm{Der}(C)$.

The map ϕ_1 preserves Lie product: let $\mathrm{D}_1,\mathrm{D}_2\in\mathrm{Der}(C)$. To show that $[\mathrm{D}_1\phi_1,\mathrm{D}_2\phi_1] = [\mathrm{D}_1,\mathrm{D}_2]\phi_1$. First we notice that for $a\otimes x\in C_0\otimes J_0$ and $\mathrm{D}\in\mathrm{Der}(C)$, $[(a\otimes x)\phi_2,\mathrm{D}\phi_1] = [a\otimes x,\mathrm{D}]\phi_2 = (a\mathrm{D}\otimes x)\phi_2$ from the study of correspondences between $\mathrm{Der}(C)$ and S_1, and $\mathrm{Der}(C)$-module in L and S_1-module in S we discussed before. Using the Jacobi identity, we have, $[[a\otimes x,\mathrm{D}_1],\mathrm{D}_2]+[[\mathrm{D}_1,\mathrm{D}_2],a\otimes x]+[[\mathrm{D}_2,a\otimes x],\mathrm{D}_1] = 0$. Thus we obtain $a[\mathrm{D}_1,\mathrm{D}_2]\otimes x = a\mathrm{D}_1\mathrm{D}_2\otimes x - a\mathrm{D}_2\mathrm{D}_1\otimes x$. Therefore, $[(a\otimes x)\phi_2,[\mathrm{D}_1\phi_1,\mathrm{D}_2\phi_1]] = [[(a\otimes x)\phi_2,\mathrm{D}_1\phi_1],\mathrm{D}_2\phi_1] - [[(a\otimes x)\phi_2,\mathrm{D}_2\phi_1],\mathrm{D}_1\phi_1] = (a\mathrm{D}_1\mathrm{D}_2\otimes x - a\mathrm{D}_2\mathrm{D}_1\otimes x)\phi_2 = (a[\mathrm{D}_1,\mathrm{D}_2]\otimes x)\phi_2 = [a\otimes x,[\mathrm{D}_1,\mathrm{D}_2]]\phi_2$. Thus, $[(a\otimes x)\phi_2,[\mathrm{D}_1\phi_1,\mathrm{D}_2\phi_1]] = [(a\otimes x)\phi_2,[\mathrm{D}_1,\mathrm{D}_2]\phi_1]$. But, ϕ_2 is bijective in the same way as ϕ_1 is and by dimension count. Hence, ϕ_1 preserves Lie product and ϕ_1 is an isomorphism. □

Lemma 5. *The centralizer of S_1 is $\{R\in \mathrm{Der}(E) \mid E_1R = 0\}$.*

Proof. Let the centralizer of S_1 be denoted by $\mathrm{cent}(S_1)$. Then $\mathrm{cent}(S_1) = \{\mathrm{S}\in S \mid [\mathrm{S},S_1] = 0\}$. Let $\mathrm{S} = y+\tilde{e}+\alpha u+\beta\tilde{u}+R$ and $\mathrm{S}_1 = y_1+\tilde{e}_1+\alpha_1u+\beta_1\tilde{u}+R_1$. Then, $[\mathrm{S},y_1] = 0$ implies $\langle y,y_1\rangle u - R(y_1,e)+\beta y_1 - y_1R = 0$. Hence, $\langle y,y_1\rangle = 0$, $y_1R = -R(y_1,e)$, $\beta = 0$. Similarly, $[\mathrm{S},\tilde{e}_1] = 0$ implies $\alpha = 0$, $[\mathrm{S},u] = 0$ implies $e = 0$, $\rho(R) = 0$, $[\mathrm{S},\tilde{u}] = 0$ implies $\tilde{y} = 0$ and finally, $[\mathrm{S},R_1] = 0$ implies $[R,R_1] = 0$. Moreover, $\tilde{y} = 0$ implies $y = 0$, $e = 0$ implies $\tilde{e} = 0$, $y_1R = 0$ implies $E_1R = 0$. Hence $\mathrm{S} = R$ and R is a derivation E. Thus, $\mathrm{cent}(S_1) = \{R\in \mathrm{Der}(E) \mid E_1R = 0 \text{ and } [R,R_1] = 0\}$. If $R_1 = R_1(x_1,y_1)$ for $x_1,y_1\in E_1$ then $[R,R_1(x_1,y_1)] = -[R_1(x_1,y_1),R] = R_1(x_1R,y_1)+R_1(x_1,y_1R) = 0$ for $E_1R = 0$. Hence, $\mathrm{cent}(S_1) = \{R\in\mathrm{Der}(E)\mid E_1R = 0\}$. □

Example 6. The bracket $[R(x_{12},1_{21}),R(y_{12},1_{21})]$ is an element of $\mathrm{cent}(S_1)$. This is clear from the computations after the proof of Lemma 3. For z and $\bar{A}$ as defined in these computations, $z[R((1/2)x_{12},1_{21}),R((1/2)y_{12},1_{21})] = \bar{A}[R_x,R_y]$, $x,y\in J_0$. But, $[R_x,R_y]\in\mathrm{Der}(J)$, as we have seen before. It is an easy exercise that for $x,y\in J$ there exist $x0,y0\in J_0$ such that $[R_x,R_y] = [R_{x0},R_{y0}]$. Hence, for $x,y\in J_0,[R_x,R_y]$ span $\mathrm{Der}(J)$. Write $[R_x,R_y] = D_{x,y}$. Then we define ϕ_3 as follows: $\phi_3 : \mathrm{Der}(J)\to \mathrm{cent}(S_1)$ by $(D_{x,y})\phi_3 = [R((1/2)x_{12},1_{21}),R((1/2)y_{12},1_{21})]$. (Recall from [4] that for a central simple Jordan algebra J of degree 3 there is a linear function $T()$ such that $x = (1/3)T(x)+x0$, $x\in J$, $x0\in J_0$. $T(1) = 3\cdot x*y = x\cdot y-(1/3)T(x\cdot y)1$, for $x,y\in J_0$.)

Lemma 7. *The map ϕ_3 is an isomorphism.*

Proof. The map ϕ_3 is injective: let $(\mathrm{D}_{x,y})\phi_3 = 0$. Then clearly, $\mathrm{D}_{x,y} = 0$ from the above discussions.

The map ϕ_3 is surjective: recall that $S = E\oplus\tilde{E}\oplus\Phi u\oplus\Phi\tilde{u}\oplus\Phi U\oplus D$, where $D = \{\sum_i R(x_i,y_i)\mid\sum_i\langle x_i,y_i\rangle = 0\}$ as in [1, Section 4].

We have $S_1 = E_1\oplus\tilde{E}_1\oplus\Phi u\oplus\Phi\tilde{u}\oplus\Phi U\oplus D_1$. Moreover, $J = J_0\oplus\Phi 1$, as in [4]. These show that we write S as a direct sum of S_1, the vector space generated by the copies of J_0 and some derivations in D. In fact we will show that $S = S_1\oplus\mathrm{Im}(\phi_2)\oplus\mathrm{Im}(\phi_3)$. So, it is enough to show that each element of D can be written as a direct sum of elements in these three parts. For each i, let

$$R(x_i,y_i) = R\left(\begin{bmatrix}1 & x\\ x' & 1\end{bmatrix},\begin{bmatrix}1 & y\\ y' & 1\end{bmatrix}\right),\quad x,x',y,y'\in J. \tag{6}$$

Using linearity, $R(x_i,y_i) = R(1_{11},1_{11})+R(1_{11},y_{12})+R(1_{11},y'_{21})+R(1_{11},1_{22})+R(x_{12},1_{11})+R(x_{12},y_{12})+R(x_{12},y'_{21})+R(x_{12},1_{22})+R(x'_{21},1_{11})+R(x'_{21},y_{12})+R(x'_{21},y'_{21})+R(x'_{21},1_{22})+R(1_{22},1_{11})+R(1_{22},y_{12})+R(1_{22},y'_{21})+R(1_{22},1_{22})$. Using

$y = (1/3)T(y)1 + y_0$, $y_0 \in J_0$, we get $R(1_{11}, y_{12}) = (1/3)T(y)R(1_{11}, 1_{12}) + R(1_{11}, (y_0)_{12})$. Similarly, $R(1_{12}, y'_{12})$ and $R(x_{12}, y_{12})$ are sum of elements in S_1 and elements in the copies of J_0. Using a remark after Lemma 3, we have $R((x_0)_{12}, (y_0)_{21}) = [R((x_0)_{12}, 1_{21}), R((y_0)_{12}, 1_{21})] + R((x_0y_0)_{12}, 1_{21})$. Thus, $R(x_{12}, y_{21})$ is a sum of elements in S_1, in $\mathrm{Im}(\phi_2)$, and in $\mathrm{Im}(\phi_3)$. Moreover, $R(x_{12}, y_{21}) - R(y_{21}, x_{12}) = \langle x_{12}, y_{21}\rangle U$, by [1, equation (2.5)]. Hence, we have shown that $S = S_1 \oplus \mathrm{Im}(\phi_2) \oplus \mathrm{Im}(\phi_3)$. Since ϕ_3 is injective, we obtain $= S_1 \oplus \mathrm{Im}(\phi_2) \oplus \mathrm{cent}(S_1)$. Thus, $\mathrm{cent}(S_1) = \mathrm{Im}(\phi_3)$ and ϕ_3 is surjective.

The map ϕ_3 preserves Lie product: let $A = [R((1/2)x_{12}, 1_{21}), R((1/2)y_{12}, 1_{21})]$ and $B = [R((1/2)x'_{12}, 1_{21}), R((1/2)y'_{12}, 1_{21})]$ for $x, y, x', y' \in J_0$. Then $z[A,B] = zAB - zBA = \bar{A}[\mathrm{D}_{x,y}, \mathrm{D}_{x',y'}]$. By definition, $[\mathrm{D}_{x,y}, \mathrm{D}_{x',y'}]\phi_3 = [A,B] = [(\mathrm{D}_{x,y})\phi_3, (\mathrm{D}_{x',y'})\phi_3]$. □

Remark 8. The Lie product of an element in $\mathrm{cent}(S_1)$ with an element in a copy of J_0 gives an element in a copy of J_0.

This shows that the behavior of $\mathrm{cent}(S_1)$ in S is the same as that of $\mathrm{Der}(J)$ in L.

To complete the proof of the Theorem the only thing left is to check that ϕ_3 defined on $C_0 \otimes J_0$ preserves Lie product. From [4, page 22], for $x, y \in J_0$ we have $[f1 \otimes x, g2 \otimes y] = (1/12)\langle x, y\rangle \mathrm{D}_{f1,g2} + f1 * g2 \otimes x * y - (f1, g2)[R_x, R_y] = (1/12)T(xy)\mathrm{D}_{f1,g2} + (f1g2 - (1/2)T(f1g2)) \otimes x * y + (1/2)T(f1g2)[R_x, R_y] = (1/12)T(xy)\mathrm{D}_{f1,g2}$. (Note that thus $f1g2 = 0, (f1, g2) = (1/2)T(f1, \tilde{g}2) = (1/2)T(f1, -g2) = -(1/2)T(f1g2)$.) Therefore, $[f1 \otimes x, g2 \otimes y]\phi_3 = (1/12)T(xy)(-31_{11})$. On the other hand, for

$$H = \begin{bmatrix} -\dfrac{1}{4}T(x,y) & 0 \\ 0 & 0 \end{bmatrix}, \tag{7}$$

$[(f_1 \otimes x)\phi_3, (g2 \otimes y)\phi_3] = [R((1/2)x_{12}, 1_{11}), -(1/2)y_{12}] = (1/2)y_{12}R((1/2)x_{12}, 1_{11}) = H = (1/12)T(xy)(-31_{11})$, where $T(x,y) = T(xy)$. Similarly, $[f1 \otimes x, g3 \otimes y]\phi_3 = (1/12)T(xy)(3\tilde{1}_{11}) = [(f1 \otimes x)\phi_3, (g3 \otimes y)\phi_3]$. Also $[f2 \otimes x, g1 \otimes y]\phi_3 = (1/12) \times T(xy)(3\tilde{1}_{22}) = [(f2 \otimes x)\phi_3, (g1 \otimes y)\phi_3]$. And $[f2 \otimes x, g3 \otimes y]\phi_3 = (1/12)T(xy)(3\tilde{u}) = [(f2 \otimes x)\phi_3, (g3 \otimes y)\phi_3]$. Similarly, one can continue to check for $[f3 \otimes x, g1 \otimes y]$ and $[f3 \otimes x, g2 \otimes y]$. And $[f1 \otimes x, f2 \otimes y]\phi_3 = (-(1/12)T(xy)\mathrm{D}_{e2,g3} + g3 \otimes x * y)\phi_3 = (1/4)(x \times y)\tilde{1}_{12} = [(f1 \otimes x)\phi_3, (f2 \otimes y)\phi_3]$, where $x \times y = 2xy - T(xy)$. Similar computations for $[f1 \otimes x, f3 \otimes y]$, $[g1 \otimes x, g2 \otimes y]$, and $[g1 \otimes x, g3 \otimes y]$. We also have, $[f2 \otimes x, f3 \otimes y]\phi_3 = -(1/12)T(xy)R(1_{22}, 1_{12}) + R((1/2)(x * y)_{12}, 1_{22})$. For

$$z = \begin{bmatrix} \alpha & a \\ b & \beta \end{bmatrix} \in E, \qquad K = \frac{1}{4}\begin{bmatrix} 0 & \alpha x \times y \\ a \times (x \times y) & T(b, x \times y) \end{bmatrix}, \tag{8}$$

$z([f2 \otimes x, f3 \otimes y]\phi_3) = K = z([(f2 \otimes x)\phi_3, (f3 \otimes y)\phi_3])$. Similarly, $[g2 \otimes x, g3 \otimes y]\phi_3 = ((1/12)T(xy)\mathrm{D}_{f1,g1} + (1/2)(e2 - e1) \otimes x * y + (1/2)\mathrm{D}_{y,x})\phi_3 = (1/12)T(xy) R(1,1) - (1/12)T(xy)R(1_{12}, 1_{21}) + (1/4)R(y_{12}, x_{21})$. Then for

$$L = \frac{1}{4}\begin{bmatrix} \alpha T(xy) & aT(xy) - \{axy\} \\ -bT(xy) + \{byx\} & -\beta T(xy) \end{bmatrix}, \tag{9}$$

$z([f1\otimes x, g1\otimes y]\phi_3) = L = z[(f1\otimes x)\phi_3,(g1\otimes y)\phi_3]$. Similar computations are for $[f3\otimes x, g3\otimes y]$. We also have, $[f2\otimes x, g2\otimes y]\phi_3 = (1/4)R(y_{12}, x_{21}) = [(f2\otimes x)\phi_3,(g2\otimes y)\phi_3]$. To check the above two equations, one may need that $(\mathrm{D}_{f2,g2})\phi_3 = [\mathrm{D}_{e1,f2},\mathrm{D}_{e2,g2}]\phi_3 = [\tilde{1}_{21}, -1_{12}] = R(1_{12}, 1_{21})$. Similarly, $(\mathrm{D}_{f3,g3})\phi_3 = R(1_{21}, 1_{12})$. Since $[(e2-e1)\otimes x, f1\otimes y]\phi_3 = (-(1/6)T(xy)\mathrm{D}_{e1,f1} - f1\otimes x * y)\phi_3 = -(1/2)R((xy)_{21}, 1_{11})$. One could use that $\mathrm{D}_{e2-e1,f1} = -2\mathrm{D}_{e1,f1}$. For

$$N = \frac{1}{2}\begin{bmatrix} T(a,xy) & b\times xy \\ \beta xy & 0 \end{bmatrix}, \tag{10}$$

$z([(e2-e1)\otimes x, f1\otimes y]\phi_3) = N = z[((e2-e1)\otimes x)\phi_3,(f1\otimes y)\phi_3]$. Similar computations are for $[(e2-e1)\otimes x, g1\otimes y]$. We also have, $[(e2-e1)\otimes x, f2\otimes y]\phi_3 = -(1/2)xy\tilde{1}_{21} = [((e2-e1)\otimes x)\phi_3,(f2\otimes y)\phi_3]$. Similar computations are for $[(e2-e1)\otimes x, f3\otimes y]$, $[(e2-e1)\otimes x, g2\otimes y]$, and $[(e2-e1)\otimes x, g3\otimes y]$.

Thus, we have shown that ϕ_3 preserves Lie product. Hence ϕ is an isomorphism and the proof of the theorem is completed.

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