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ON θ-PRECONTINUOUS FUNCTIONS

TAKASHI NOIRI

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Abstract. We introduce a new class of functions called θ-precontinuous functions which is contained in the class of weakly precontinuous (or almost weakly continuous) functions and contains the class of almost precontinuous functions. It is shown that the θ-precontinuous image of a p-closed space is quasi H-closed.

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1. Introduction. A subset A of a topological space X is said to be preopen [14] or nearly open [26] if $A \subset \text{Int}(\text{Cl}(A))$. A function $f : X \to Y$ is called precontinuous [14] if the preimage $f^{-1}(V)$ of each open set V of Y is preopen in X. Precontinuity was called near continuity by Pták [26] and also called almost continuity by Frolík [9] and Husain [10]. In 1985, Janković [12] introduced almost weak continuity as a weak form of precontinuity. Popa and Noiri [23] introduced weak precontinuity and showed that almost weak continuity is equivalent to weak precontinuity. Paul and Bhattacharyya [21] called weakly precontinuous functions quasi precontinuous and obtained the further properties of quasi precontinuity. Recently, Nasef and Noiri [16] have introduced and investigated the notion of almost precontinuity. Quite recently, Jafari and Noiri [11] investigated the further properties of almost precontinuous functions.

In this paper, we introduce a new class of functions called θ-precontinuous functions which is contained in the class of weakly precontinuous functions and contains the class of almost precontinuous functions. We obtain basic properties of θ-precontinuous functions. It is shown in the last section that the θ-precontinuous images of p-closed (resp., β-connected) spaces are quasi H-closed (resp., semi-connected).

2. Preliminaries. Throughout, by (X,τ) and (Y,σ) (or simply X and Y) we denote topological spaces. Let S be a subset of X. We denote the interior and the closure of S by $\text{Int}(S)$ and $\text{Cl}(S)$, respectively. A subset S is said to be *preopen* [14] (resp., *semi-open* [13], *α-open* [17]) if $S \subset \text{Int}(\text{Cl}(S))$ (resp., $S \subset \text{Cl}(\text{Int}(S))$, $S \subset \text{Int}(\text{Cl}(\text{Int}(S)))$). The complement of a preopen set is called *preclosed*. The intersection of all preclosed sets containing S is called the *preclosure* [8] of S and is denoted by $\text{pCl}(S)$. The *preinterior* of S is defined by the union of all preopen sets contained in S and is denoted by $\text{pInt}(S)$. The family of all preopen sets of X is denoted by $\text{PO}(X)$. We set $\text{PO}(X,x) = \{U : x \in U \text{ and } U \in \text{PO}(X)\}$. A point x of X is called a *θ-cluster* point of S if $\text{Cl}(U) \cap S \neq \emptyset$ for every open set U of X containing x. The set of all θ-cluster points of S is called the *θ-closure* of S and is denoted by $\text{Cl}_\theta(S)$. A subset S is said to be *θ-closed* [27] if $S = \text{Cl}_\theta(S)$. The complement of a θ-closed set is said to be *θ-open*. A point x of X

is called a *pre θ-cluster* point of S if $\mathrm{pCl}(U)\cap S\neq\varnothing$ for every preopen set U of X containing x. The set of all pre θ-cluster points of S is called the *pre θ-closure* of S and is denoted by $\mathrm{pCl}_\theta(S)$. A subset S is said to be *pre θ-closed* [20] if $S=\mathrm{pCl}_\theta(S)$. The complement of a pre θ-closed set is said to be *pre θ-open*.

Definition 2.1. A function $f:X\to Y$ is said to be *precontinuous* [14] (resp., *almost precontinuous* [16], *weakly precontinuous* [23] or *quasi precontinuous* [21]) if for each $x\in X$ and each open set V of Y containing $f(x)$, there exists $U\in\mathrm{PO}(X,x)$ such that $f(U)\subset V$ (resp., $f(U)\subset\mathrm{Int}(\mathrm{Cl}(V))$, $f(U)\subset\mathrm{Cl}(V)$).

Definition 2.2. A function $f:X\to Y$ is said to be *almost weakly continuous* [12] if $f^{-1}(V)\subset\mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Cl}(V))))$ for every open set V of Y.

Definition 2.3. A function $f:X\to Y$ is said to be *strongly θ-precontinuous* [19] if for each $x\in X$ and each open set V of Y containing $f(x)$, there exists $U\in\mathrm{PO}(X,x)$ such that $f(\mathrm{pCl}(U))\subset V$.

Definition 2.4. A function $f:X\to Y$ is said to be *θ-precontinuous* if for each $x\in X$ and each open set V of Y containing $f(x)$, there exists $U\in\mathrm{PO}(X,x)$ such that $f(\mathrm{pCl}(U))\subset\mathrm{Cl}(V)$.

Remark 2.5. By the above definitions and Theorem 3.3 below, we have the following implications and none of these implications is reversible by [19, Example 2.2], [11, Example 2.9], and Examples 2.6 and 5.11 below.

$$\begin{aligned}&\text{strongly }\theta\text{-precontinuous}\Longrightarrow\text{precontinuous}\Longrightarrow\text{almost precontinuous}\\&\qquad\Longrightarrow\theta\text{-precontinuous}\Longrightarrow\text{weakly precontinuous}.\end{aligned}\tag{2.1}$$

Example 2.6. This example is due to Arya and Deb [4]. Let X be the set of all real numbers. The topology τ on X is the cocountable topology. Let $Y=\{a,b,c\}$, $\sigma=\{\varnothing,Y,\{a\},\{c\},\{a,c\}\}$. We define a function $f:(X,\tau)\to(Y,\sigma)$ by $f(x)=a$ if x is rational; $f(x)=b$ if x is irrational. Then f is a θ-precontinuous function which is not almost precontinuous

3. Characterizations

Theorem 3.1. *For a function $f:X\to Y$ the following properties are equivalent:*
(1) *f is θ-precontinuous;*
(2) $\mathrm{pCl}_\theta(f^{-1}(B))\subset f^{-1}(\mathrm{Cl}_\theta(B))$ *for every subset B of Y;*
(3) $f(\mathrm{pCl}_\theta(A))\subset\mathrm{Cl}_\theta(f(A))$ *for every subset A of X.*

Proof. (1)⇒(2). Let B be any subset of Y. Suppose that $x\notin f^{-1}(\mathrm{Cl}_\theta(B))$. Then $f(x)\notin\mathrm{Cl}_\theta(B)$ and there exists an open set V containing $f(x)$ such that $\mathrm{Cl}(V)\cap B=\varnothing$. Since f is $\theta.p.c.$, there exists $U\in\mathrm{PO}(X,x)$ such that $f(\mathrm{pCl}(U))\subset\mathrm{Cl}(V)$. Therefore, we have $f(\mathrm{pCl}(U))\cap B=\varnothing$ and $\mathrm{pCl}(U)\cap f^{-1}(B)=\varnothing$. This shows that $x\notin\mathrm{pCl}_\theta(f^{-1}(B))$. Thus, we obtain $\mathrm{pCl}_\theta(f^{-1}(B))\subset f^{-1}(\mathrm{Cl}_\theta(B))$.

(2)⇒(3). Let A be any subset of X. Then we have $\mathrm{pCl}_\theta(A) \subset \mathrm{pCl}_\theta(f^{-1}(f(A))) \subset f^{-1}(\mathrm{Cl}_\theta(f(A)))$ and hence $f(\mathrm{pCl}_\theta(A)) \subset \mathrm{Cl}_\theta(f(A))$.

(3)⇒(2). Let B be a subset of Y. We have $f(\mathrm{pCl}_\theta(f^{-1}(B))) \subset \mathrm{Cl}_\theta(f(f^{-1}(B))) \subset \mathrm{Cl}_\theta(B)$ and hence $\mathrm{pCl}_\theta(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}_\theta(B))$.

(2)⇒(1). Let $x \in X$ and V be an open set of Y containing $f(x)$. Then we have $\mathrm{Cl}(V) \cap (Y - \mathrm{Cl}(V)) = \varnothing$ and $f(x) \notin \mathrm{Cl}_\theta(Y - \mathrm{Cl}(V))$. Hence, $x \notin f^{-1}(\mathrm{Cl}_\theta(Y - \mathrm{Cl}(V)))$ and $x \notin \mathrm{pCl}_\theta(f^{-1}(Y - \mathrm{Cl}(V)))$. There exists $U \in \mathrm{PO}(X,x)$ such that $\mathrm{pCl}(U) \cap f^{-1}(Y - \mathrm{Cl}(V)) = \varnothing$; hence $f(\mathrm{pCl}(U)) \subset \mathrm{Cl}(V)$. Therefore, f is $\theta.p.c.$ □

Theorem 3.2. *For a function $f : X \to Y$ the following properties are equivalent:*

(1) *f is θ-precontinuous;*

(2) *$f^{-1}(V) \subset \mathrm{pInt}_\theta(f^{-1}(\mathrm{Cl}(V)))$ for every open set V of Y;*

(3) *$\mathrm{pCl}_\theta(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ for every open set V of Y.*

Proof. (1)⇒(2). Suppose that V is any open set of Y and $x \in f^{-1}(V)$. Then $f(x) \in V$ and there exists $U \in \mathrm{PO}(X,x)$ such that $f(\mathrm{pCl}(U)) \subset \mathrm{Cl}(V)$. Therefore, $x \in U \subset \mathrm{pCl}(U) \subset f^{-1}(\mathrm{Cl}(V))$. This shows that $x \in \mathrm{pInt}_\theta(f^{-1}(\mathrm{Cl}(V)))$. Therefore, we obtain $f^{-1}(V) \subset \mathrm{pInt}_\theta(f^{-1}(\mathrm{Cl}(V)))$.

(2)⇒(3). Suppose that V is any open set of Y and $x \notin f^{-1}(\mathrm{Cl}(V))$. Then $f(x) \notin \mathrm{Cl}(V)$ and there exists an open set W containing $f(x)$ such that $W \cap V = \varnothing$; hence $\mathrm{Cl}(W) \cap V = \varnothing$. Therefore, we have $f^{-1}(\mathrm{Cl}(W)) \cap f^{-1}(V) = \varnothing$. Since $x \in f^{-1}(W)$, by (2) $x \in \mathrm{pInt}_\theta(f^{-1}(\mathrm{Cl}(W)))$. There exists $U \in \mathrm{PO}(X,x)$ such that $\mathrm{pCl}(U) \subset f^{-1}(\mathrm{Cl}(W))$. Thus we have $\mathrm{pCl}(U) \cap f^{-1}(V) = \varnothing$ and hence $x \notin \mathrm{pCl}_\theta(f^{-1}(V))$. This shows that $\mathrm{pCl}_\theta(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$.

(3)⇒(1). Suppose that $x \in X$ and V is any open set of Y containing $f(x)$. Then $V \cap (Y - \mathrm{Cl}(V)) = \varnothing$ and $f(x) \notin \mathrm{Cl}(Y - \mathrm{Cl}(V))$. Therefore, $x \notin f^{-1}(\mathrm{Cl}(Y - \mathrm{Cl}(V)))$ and by (3) $x \notin \mathrm{pCl}_\theta(f^{-1}(Y - \mathrm{Cl}(V)))$. There exists $U \in \mathrm{PO}(X,x)$ such that $\mathrm{pCl}(U) \cap f^{-1}(Y - \mathrm{Cl}(V)) = \varnothing$. Therefore, we obtain $f(\mathrm{pCl}(U)) \subset \mathrm{Cl}(V)$. This shows that f is $\theta.p.c.$ □

Theorem 3.3. *For a function $f : X \to Y$ the following properties hold:*

(1) *if f is almost precontinuous, then it is θ-precontinuous;*

(2) *if f is θ-precontinuous, then it is weakly precontinuous.*

Proof. Statement (2) is obvious. We will show statement (1). Suppose that $x \in X$ and V is any open set of Y containing $f(x)$. Since f is almost precontinuous, $f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)))$ is preopen and $f^{-1}(\mathrm{Cl}(V))$ is preclosed in X by [16, Theorem 3.1]. Now, set $U = f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)))$. Then we have $U \in \mathrm{PO}(X,x)$ and $\mathrm{pCl}(U) \subset f^{-1}(\mathrm{Cl}(V))$. Therefore, we obtain $f(\mathrm{pCl}(U)) \subset \mathrm{Cl}(V)$. This shows that f is $\theta.p.c.$ □

Corollary 3.4. *Let Y be a regular space. Then, for a function $f : X \to Y$ the following properties are equivalent:*

(1) *f is strongly θ-precontinuous;*

(2) *f is precontinuous;*

(3) *f is almost precontinuous;*

(4) *f is θ-precontinuous;*

(5) *f is weakly precontinuous.*

Proof. This is an immediate consequence of [19, Theorem 3.2]. □

Definition 3.5. A topological space X is said to be *pre-regular* [20] if for each preclosed set F and each point $x \in X - F$, there exist disjoint preopen sets U and V such that $x \in U$ and $F \subset V$.

Lemma 3.6 (see [20]). *A topological space X is pre-regular if and only if for each $U \in \mathrm{PO}(X)$ and each point $x \in U$, there exists $V \in \mathrm{PO}(X,x)$ such that $x \in V \subset \mathrm{pCl}(V) \subset U$.*

Theorem 3.7. *Let X be a pre-regular space. Then $f : X \to Y$ is $\theta.p.c.$ if and only if it is weakly precontinuous.*

Proof. Suppose that f is weakly precontinuous. Let $x \in X$ and V is any open set of Y containing $f(x)$. Then, there exists $U \in \mathrm{PO}(X,x)$ such that $f(U) \subset \mathrm{Cl}(V)$. Since X is pre-regular, there exists $U_* \in \mathrm{PO}(X,x)$ such that $x \in U_* \subset \mathrm{pCl}(U_*) \subset U$. Therefore, we obtain $f(\mathrm{pCl}(U_*)) \subset \mathrm{Cl}(V)$. This shows that f is $\theta.p.c.$ □

Theorem 3.8. *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function of f defined by $g(x) = (x, f(x))$ for each $x \in X$. Then g is $\theta.p.c.$ if and only if f is $\theta.p.c.$*

Proof

Necessity. Suppose that g is $\theta.p.c.$ Let $x \in X$ and V be an open set of Y containing $f(x)$. Then $X \times V$ is an open set of $X \times Y$ containing $g(x)$. Since g is $\theta.p.c.$, there exists $U \in \mathrm{PO}(X,x)$ such that $g(\mathrm{pCl}(U)) \subset \mathrm{Cl}(X \times V)$. It follows that $\mathrm{Cl}(X \times V) = X \times \mathrm{Cl}(V)$. Therefore, we obtain $f(\mathrm{pCl}(U)) \subset \mathrm{Cl}(V)$. This shows that f is $\theta.p.c.$

Sufficiency. Let $x \in X$ and W be any open set of $X \times Y$ containing $g(x)$. There exist open sets $U_1 \subset X$ and $V \subset Y$ such that $g(x) = (x, f(x)) \in U_1 \times V \subset W$. Since f is $\theta.p.c.$, there exists $U_2 \in \mathrm{PO}(X,x)$ such that $f(\mathrm{pCl}(U_2)) \subset \mathrm{Cl}(V)$. Let $U = U_1 \cap U_2$, then $U \in \mathrm{PO}(X,x)$. Therefore, we obtain $g(\mathrm{pCl}(U)) \subset \mathrm{Cl}(U_1) \times f(\mathrm{pCl}(U_2)) \subset \mathrm{Cl}(U_1) \times \mathrm{Cl}(V) \subset \mathrm{Cl}(W)$. This shows that g is $\theta.p.c.$ □

4. Some properties

Lemma 4.1 (see [15]). *Let A and X_0 be subsets of a space X.*

(1) *If $A \in \mathrm{PO}(X)$ and X_0 is semi-open in X, then $A \cap X_0 \in \mathrm{PO}(X_0)$.*

(2) *If $A \in \mathrm{PO}(X_0)$ and $X_0 \in \mathrm{PO}(X)$, then $A \in \mathrm{PO}(X)$.*

Lemma 4.2 (see [7]). *Let A and X_0 be subsets of a space X such that $A \subset X_0 \subset X$. Let $\mathrm{pCl}_{X_0}(A)$ denote the preclosure of A in the subspace X_0.*

(1) *If X_0 is semi-open in X, then $\mathrm{pCl}_{X_0}(A) \subset \mathrm{pCl}(A)$.*

(2) *If $A \in \mathrm{PO}(X_0)$ and $X_0 \in \mathrm{PO}(X)$, then $\mathrm{pCl}(A) \subset \mathrm{pCl}_{X_0}(A)$.*

Theorem 4.3. *If $f : X \to Y$ is $\theta.p.c.$ and X_0 is a semi-open subset of X, then the restriction $f/X_0 : X_0 \to Y$ is $\theta.p.c.$*

Proof. For any $x \in X_0$ and any open neighborhood V of $f(x)$, there exists $U \in \mathrm{PO}(X,x)$ such that $f(\mathrm{pCl}(U)) \subset \mathrm{Cl}(V)$ since f is $\theta.p.c.$ Put $U_0 = U \cap X_0$, then by Lemmas 4.1 and 4.2 $U_0 \in \mathrm{PO}(X_0,x)$ and $\mathrm{pCl}_{X_0}(U_0) \subset \mathrm{pCl}(U_0)$. Therefore, we obtain

$$(f/X_0)(\mathrm{pCl}_{X_0}(U_0)) = f(\mathrm{pCl}_{X_0}(U_0)) \subset f(\mathrm{pCl}(U_0)) \subset f(\mathrm{pCl}(U)) \subset \mathrm{Cl}(V). \quad (4.1)$$

This shows that f/X_0 is $\theta.p.c.$ □

Theorem 4.4. *A function $f : X \to Y$ is $\theta.p.c.$ if for each $x \in X$ there exists $X_0 \in \mathrm{PO}(X,x)$ such that the restriction $f/X_0 : X_0 \to Y$ is $\theta.p.c.$*

Proof. Let $x \in X$ and V be any open neighborhood of $f(x)$. There exists $X_0 \in \mathrm{PO}(X,x)$ such that $f/X_0 : X_0 \to Y$ is $\theta.p.c.$ Thus, there exists $U \in \mathrm{PO}(X_0,x)$ such that $(f/X_0)(\mathrm{pCl}_{X_0}(U)) \subset \mathrm{Cl}(V)$. By Lemmas 4.1 and 4.2, $U \in \mathrm{PO}(X,x)$ and $\mathrm{pCl}(U) \subset \mathrm{pCl}_{X_0}(U)$. Hence, we have $f(\mathrm{pCl}(U)) = (f/X_0)(\mathrm{pCl}(U)) \subset (f/X_0)(\mathrm{pCl}_{X_0}(U)) \subset \mathrm{Cl}(V)$. This shows that f is $\theta.p.c.$ □

Corollary 4.5. *Let $\{U_\lambda : \lambda \in \Lambda\}$ be an α-open cover of a topological space X. A function $f : X \to Y$ is $\theta.p.c.$ if and only if the restriction $f/U_\lambda : U_\lambda \to Y$ is $\theta.p.c.$ for each $\lambda \in \Lambda$.*

Proof. This is an immediate consequence of Theorems 4.3 and 4.4. □

Let $\{X_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces, A_α a nonempty subset of X_α for each $\alpha \in \mathcal{A}$ and $X = \Pi\{X_\alpha : \alpha \in \mathcal{A}\}$ denote the product space, where $\mathcal{A}$ is nonempty.

Lemma 4.6 (see [8]). *Let n be a positive integer and $A = \Pi_{j=1}^{n} A_{\alpha_j} \times \Pi_{\alpha \neq \alpha_j} X_\alpha$.*
(1) *$A \in \mathrm{PO}(X)$ if and only if $A_{\alpha_j} \in \mathrm{PO}(X_{\alpha_j})$ for each $j = 1,2,\dots,n$.*
(2) *$\mathrm{pCl}(\Pi_{\alpha\in\mathcal{A}} A_\alpha) \subset \Pi_{\alpha\in\mathcal{A}} \mathrm{pCl}(A_\alpha)$.*

Theorem 4.7. *If a function $f_\alpha : X_\alpha \to Y_\alpha$ is $\theta.p.c.$ for each $\alpha \in \mathcal{A}$. Then the product function $f : \Pi X_\alpha \to \Pi Y_\alpha$, defined by $f(\{x_\alpha\}) = \{f_\alpha(x_\alpha)\}$ for each $x = \{x_\alpha\}$, is $\theta.p.c.$*

Proof. Let $x = \{x_\alpha\} \in \Pi X_\alpha$ and W be any open set of ΠY_α containing $f(x)$. Then, there exists an open set V_{α_j} of Y_{α_j} such that

$$f(x) = \{f_\alpha(x_\alpha)\} \in \Pi_{j=1}^{n} V_{\alpha_j} \times \Pi_{\alpha\neq\alpha_j} Y_\alpha \subset W. \tag{4.2}$$

Since f_α is $\theta.p.c.$ for each α, there exists $U_{\alpha_j} \in \mathrm{PO}(X_{\alpha_j}, x_{\alpha_j})$ such that $f_{\alpha_j}(\mathrm{pCl}(U_{\alpha_j})) \subset \mathrm{Cl}(V_{\alpha_j})$ for $j = 1,2,\dots,n$. Now, put $U = \Pi_{j=1}^{n} U_{\alpha_j} \times \Pi_{\alpha\neq\alpha_j} X_\alpha$. Then, it follows from Lemma 4.6 that $U \in \mathrm{PO}(\Pi X_\alpha, x)$. Moreover, we have

$$\begin{aligned} f(\mathrm{pCl}(U)) &\subset f(\Pi_{j=1}^{n} \mathrm{pCl}(U_{\alpha_j}) \times \Pi_{\alpha\neq\alpha_j} X_\alpha) \\ &\subset \Pi_{j=1}^{n} f_{\alpha_j}(\mathrm{pCl}(U_{\alpha_j})) \times \Pi_{\alpha\neq\alpha_j} Y_\alpha \\ &\subset \Pi_{j=1}^{n} \mathrm{Cl}(V_{\alpha_j}) \times \Pi_{\alpha\neq\alpha_j} Y_\alpha \subset \mathrm{Cl}(W). \end{aligned} \tag{4.3}$$

This shows that f is $\theta.p.c.$ □

5. Preservation property

Definition 5.1. A topological space X is said to be
1. *p-closed* [7] (resp., *p-Lindelöf*) if every cover of X by preopen sets has a finite (resp., countable) subfamily whose preclosures cover X,
2. *countably p-closed* if every countable cover of X by preopen sets has a finite subfamily whose preclosures cover X;
3. *quasi H-closed* [25] (resp., *almost Lindelöf* [6]) if every cover of X by open sets has a finite (resp., countable) subfamily whose closures cover X,
4. *lightly compact* [5] if every countable cover of X by open sets has a finite subfamily whose closures cover X.

Definition 5.2. A subset K of a space X is said to be

(1) *p-closed relative to* X [7] if for every cover $\{V_\alpha : \alpha \in \mathcal{A}\}$ of K by preopen sets of X, there exists a finite subset $\mathcal{A}_*$ of $\mathcal{A}$ such that $K \subset \cup\{\text{pCl}(V_\alpha) : \alpha \in \mathcal{A}_*\}$,

(2) *quasi H-closed relative* to X [25] if for every cover $\{V_\alpha : \alpha \in \mathcal{A}\}$ of K by open sets of X, there exists a finite subset $\mathcal{A}_*$ of $\mathcal{A}$ such that $K \subset \cup\{\text{Cl}(V_\alpha) : \alpha \in \mathcal{A}_*\}$.

Theorem 5.3. *If $f : X \to Y$ is a $\theta.p.c.$ function and K is p-closed relative to X, then $f(K)$ is quasi H-closed relative to Y.*

Proof. Suppose that $f : X \to Y$ is $\theta.p.c.$ and K is p-closed relative to X. Let $\{V_\alpha : \alpha \in \mathcal{A}\}$ be a cover of $f(K)$ by open sets of Y. For each point $x \in K$, there exists $\alpha(x) \in \mathcal{A}$ such that $f(x) \in V_{\alpha(x)}$. Since f is $\theta.p.c.$, there exists $U_x \in \text{PO}(X,x)$ such that $f(\text{pCl}(U_x)) \subset \text{Cl}(V_{\alpha(x)})$. The family $\{U_x : x \in K\}$ is a cover of K by preopen sets of X and hence there exists a finite subset K_* of K such that $K \subset \cup_{x\in K_*}\text{pCl}(U_x)$. Therefore, we obtain $f(K) \subset \cup_{x\in K_*}\text{Cl}(V_{\alpha(x)})$. This shows that $f(K)$ is quasi H-closed relative to Y. □

Corollary 5.4. *Let $f : X \to Y$ be a $\theta.p.c.$ surjection. Then, the following properties hold:*

(1) *If X is p-closed, then Y is quasi H-closed.*

(2) *If X is p-Lindelöf, then Y is almost Lindelöf.*

(3) *If X is countably p-closed, then Y is lightly compact.*

A subset S of a topological space X is said to be *β-open* [1] or *semipreopen* [3] if $S \subset \text{Cl}(\text{Int}(\text{Cl}(S)))$. It is well known that α-openness implies both preopenness and semi-openness which imply β-openness. The complement of a semipreopen set is said to be *semipreclosed* [3]. The intersection of all semipreclosed sets of X containing a subset S is the *semipreclosure* of S and is denoted by $\text{spCl}(S)$ [3].

Definition 5.5. A topological space X is said to be

(1) *β-connected* [24] or *semipreconnected* [2] if X cannot be expressed as the union of two nonempty disjoint β-open sets,

(2) *semi-connected* [22] if X cannot be expressed as the union of two nonempty disjoint semi-open sets.

Remark 5.6. We have the following implications:

$$\beta\text{-connected} \Longrightarrow \text{semi-connected} \Longrightarrow \text{connected}. \tag{5.1}$$

But, the converses need not be true as the following simple examples show.

Example 5.7. (1) Let $X = \{a,b,c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\}$. Then (X,τ) is connected but not semi-connected.

(2) Let $X = \{a,b,c\}$ and $\tau = \{X, \emptyset, \{b,c\}\}$. Then (X,τ) is semi-connected but not β-connected.

Lemma 5.8. *For a topological space X, the following properties are equivalent:*

(1) *X is β-connected or semipreconnected.*

(2) *The intersection of two nonempty semipreopen subsets of X is always nonempty.*

(3) *The intersection of two nonempty preopen subsets of X is always nonempty.*

(4) $\text{pCl}(V) = X$ *for every nonempty preopen subset* V *of* X.
(5) $\text{spCl}(V) = X$ *for every nonempty semipreopen subset* V *of* X.

Proof. The proofs of equivalences of (1), (2), and (3) are given in [2, Theorem 6.4]. The other properties (4) and (5), which are stated in [18], are easily equivalent to (3) and (2), respectively. □

Theorem 5.9. *If* $f : X \to Y$ *is a* $\theta.p.c.$ *surjection and* X *is* β*-connected, then* Y *is semi-connected.*

Proof. Let V be any nonempty open set of Y. Let $y \in V$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. Since f is $\theta.p.c.$, there exists $U \in \text{PO}(X,x)$ such that $f(\text{pCl}(U)) \subset \text{Cl}(V)$. Since X is β-connected, by Lemma 5.8 $\text{pCl}(U) = X$ and hence $\text{Cl}(V) = Y$ since f is surjective. Therefore, it follows from [22, Theorem 4.3] that Y is semi-connected. □

Remark 5.10. The following example shows that the image of β-connectedness under weakly precontinuous surjections is not necessarily semi-connected.

Example 5.11. Let X be the set of real numbers, $\tau = \{\emptyset\} \cup \{V \subset X : 0 \in V\}$, $Y = \{a,b,c\}$, and $\sigma = \{Y, \emptyset, \{a\}, \{b\}, \{a,b\}\}$. Define a function $f : (X,\tau) \to (Y,\sigma)$ as follows: $f(x) = a$ if $x < 0$; $f(x) = c$ if $x = 0$; $f(x) = b$ if $x > 0$. Then f is a weakly precontinuous surjection which is not $\theta.p.c.$ The topological space (X,τ) is β-connected by Lemma 5.8. By Example 5.7(1), (Y,σ) is connected but not semi-connected.

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Takashi Noiri: Department of Mathematics, Yatsushiro College of Technology, Yatsushiro, Kumamoto, 866-8501, Japan

E-mail address: noiri@as.yatsushiro-nct.ac.jp