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PRODUCTS OF PROTOPOLOGICAL GROUPS

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Abstract. Montgomery and Zippin saied that a group is approximated by Lie groups if every neighborhood of the identity contains an invariant subgroup H such that G/H is topologically isomorphic to a Lie group. Bagley, Wu, and Yang gave a similar definition, which they called a pro-Lie group. Covington extended this concept to a protopological group. Covington showed that protopological groups possess many of the characteristics of topological groups. In particular, Covington showed that in a special case, the product of protopological groups is a protopological group. In this note, we give a characterization theorem for protopological groups and use it to generalize her result about products to the category of all protopological groups.

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1. Introduction. Montgomery and Zippin [5] saied that a group is *approximated by Lie groups* if every neighborhood of the identity contains an invariant subgroup H such that G/H is topologically isomorphic to a Lie group. Using a similar idea, Bagley, Wu, and Yang [1] defined a pro-Lie group. Covington [3] extended this concept to topological groups. She defined a protopological group as a group G with a topology τ and a collection $\mathcal{N}$ of normal subgroups such that (1) for every neighborhood U of the identity, there exists $N \in \mathcal{N}$ such that $N \subseteq U$ and (2) G/N with the quotient topology is a topological group for every $N \in \mathcal{N}$. The collection $\mathcal{N}$ is called a normal system. We denote the quotient topology on G/N by $q_N(\tau)$, and we call the collection $\mathcal{Q} = \{q_N(\tau)\}_{N\in\mathcal{N}}$ a quotient system for (G,τ). In [2], Covington defines a t-protopological group as a protopological group (G,τ) with the additional requirement that the natural map $\eta_N : G \to G/N$ is an open map for all $N \in \mathcal{N}$. She also shows that the product of t-protopological groups is a t-protopological group. Although her proof uses ideas different than those used in the proof that a product of topological groups is a topological group, it uses the fact that $\eta_N : G \to G/N$ is an open map for all $N \in \mathcal{N}$. Since there are protopological groups, which are not t-protopological, it is of interest to determine if the category of protopological groups is closed under products. However, it is apparent that a different proof technique is needed, since we do not have the hypothesis that $\eta_N : G \to G/N$ is an open map for all $N \in \mathcal{N}$. In this note, we will give a characterization theorem for protopological groups, and then use it to show that the product of protopological groups is protopological.

Let (G,τ) be a protopological group with normal system $\mathcal{N}$ and quotient system $\mathcal{Q} = \{q_N(\tau)\}_{N\in\mathcal{N}}$. We note that for each $N \in \mathcal{N}$, the pullback topology from $(G/N, q_N(\tau))$ determines a group topology on G. We will denote this topology on G by $P_N(\tau)$. Since the join of group topologies is a group topology, $\tau_p = \vee_{N\in\mathcal{N}} P_N(\tau)$ is also a group topology on G. We call τ_p the complete pullback topology on G generated by τ. This topology may also be called the weak topology on G. Covington [2] showed that τ_p is the Graev topology when (G,τ) is a protopological group. It is a well-known result, due to Hewitt and Ross [4], that $\tau_p = \vee_{N\in\mathcal{N}} P_N(\tau)$ is the coarsest topology that makes G a topological group. Hence, for a protopological group (G,τ) with normal system $\mathcal{N}$, the complete pullback topology τ_p is the only group topology contained in τ. When using the pullback topologies, we will be interested in saturated sets. In particular, we will say that a set $U \subseteq G$ *is saturated with respect to* $N \in \mathcal{N}$ if for all $x \in U$, $\eta_N^{-1}(\eta_N(x)) \subseteq U$.

2. Characterization and product theorems

Theorem 2.1 (characterization theorem for protopological groups)**.** *Let (G,τ) be a protopological group with normal system $\mathcal{N}$ and quotient system $\mathfrak{Q} = \{q_N(\tau)\}_{N\in\mathcal{N}}$. Let τ^* be a topology on G. Then (G,τ^*) is a protopological group with normal system $\mathcal{N}$ and quotient system $\mathfrak{Q}$ if and only if τ^* satisfies the following properties:*

(a) $\tau_p \subseteq \tau^*$;

(b) *if $U \in \tau^*$, then $U \in \tau_p$ or U is not saturated with respect to N for all $N \in \mathcal{N}$; and*

(c) *if $U \in \tau^*$ is a neighborhood of e, then there exists $N \in \mathcal{N}$ such that $N \subseteq U$.*

Proof. Let (G,τ^*) be a protopological group. Since $\tau_p = \vee_{N\in\mathcal{N}} P_N(\tau) = \vee_{N\in\mathcal{N}} P_N(\tau^*)$ is the coarsest topology that makes G a topological group [4], it follows that $\tau_p \subseteq \tau^*$. If $U \in \tau^*$ is saturated with respect to some $N \in \mathcal{N}$, then $U = \eta_N^{-1}(\eta_N(U))$. But then $U \in P_N(\tau) \subseteq \tau_p$. Now, if $U \in \tau^*$ is a neighborhood of e, there exists $N \in \mathcal{N}$ such that $N \subseteq U$. Conversely, assume that (a), (b), and (c) are satisfied. For $N \in \mathcal{N}$, consider the group G/N with the topology $q_N(\tau^*)$. Let $U \in q_N(\tau) = q_N(\tau_p)$. Since $\eta_N^{-1}(U) \in \tau_p \subseteq \tau^*$, it follows that $U \in q_N(\tau^*)$. Hence, $q_N(\tau) = q_N(\tau_p) \subseteq q_N(\tau^*)$. Now, let $U \in q_N(\tau^*)$. Then $q_N^{-1}(U) \in \tau_p$ and $q_N^{-1}(U)$ is saturated with respect to N. Therefore, $q_N^{-1}(U) \in \tau_p$ which implies that $U \in q_N(\tau_p) = q_N(\tau)$. Thus, $q_N(\tau^*) \subseteq q_N(\tau_p) = q_N(\tau)$. Therefore, $q_N(\tau^*) = q_N(\tau)$ for all $N \in \mathcal{N}$. □

By imposing the additional condition that $\eta_N(\tau^*) \subseteq q_N(\tau)$, we obtain a characterization theorem for t-protopological groups.

Theorem 2.2 (product theorem)**.** *Let (G_α,τ_α) be a protopological group with normal system $\mathcal{N}_\alpha$ and quotient system $\mathfrak{Q}_\alpha = \{q_N(\tau_\alpha)\}_{N\in\mathcal{N}_\alpha}$, for all $\alpha \in A$. Let $G = \Pi_{\alpha\in A} G_\alpha$, and let $\tau = \Pi_{\alpha\in A}\tau_\alpha$ be the product topology on G. Then (G,τ) is a protopological group with normal system $\mathcal{N} = \{\Pi_{\alpha\in A} N_\alpha \mid N_\alpha \in \mathcal{N}_\alpha$ for all $\alpha \in A$ and $N_\alpha = G_\alpha$ for all but finitely many $\alpha \in A\}$.*

Proof. For each $\alpha \in A$, let τ_{p_α} be the complete pullback topology on G_α. Then $\{(G_\alpha,\tau_{p_\alpha})\}_{\alpha\in A}$ is a collection of topological groups, and (G,τ_p) is a topological group, where $\tau_p = \Pi_{\alpha\in A}\tau_{p_\alpha}$ is the product topology of $\{\tau_{p_\alpha}\}_{\alpha\in A}$. By the characterization theorem, $\tau_{p_\alpha} \subseteq \tau_\alpha$ for each $\alpha \in A$. So, $\tau_p = \Pi_{\alpha\in A}\tau_{p_\alpha} \subseteq \Pi_{\alpha\in A}\tau_\alpha = \tau$. Now, if $U \in \tau$ is a neighborhood of $e = \langle e_\alpha\rangle_{\alpha\in A} \in G$ then there exists $U_{\alpha_i} \in \tau_{\alpha_i}$, for $i = 1,\dots,n$, such that $e \in \Pi_{i=1}^n U_{\alpha_i} \times \Pi_{\alpha\notin\{\alpha_1,\dots,\alpha_n\}} G_\alpha \subseteq U$. Then for each $\alpha_i \in \{\alpha_1,\dots,\alpha_n\}$, there exists $N_{\alpha_i} \in \mathcal{N}_{\alpha_i}$ with $N_{\alpha_i} \subseteq U_{\alpha_i}$. Hence, $N = \Pi_{i=1}^n N_{\alpha_i} \times \Pi_{\alpha\notin\{\alpha_1,\dots,\alpha_n\}} G_\alpha \in \mathcal{N}$ and $e \in N \subseteq \Pi_{i=1}^n U_{\alpha_i} \times \Pi_{\alpha\notin\{\alpha_1,\dots,\alpha_n\}} G_\alpha \subseteq U$. Since (G,τ_p) is a topological group, $(G/N, q_N(\tau_p))$ is a topological group for all $N \in \mathcal{N}$. Hence, (G, τ_p) is a protopological group with normal system $\mathcal{N}$ and quotient system $\mathcal{Q} = \{q_N(\tau_p)\}_{N \in \mathcal{N}}$. For each $N \in \mathcal{N}$, let $P_N(\tau_p)$ be the pullback topology on G from $(G/N, q_N(\tau_p))$. Since the complete pullback topology is the only group topology that makes G a protopological group with normal system $\mathcal{N}$ and quotient system $\mathcal{Q}$, we have that $\tau_p = \vee_{N \in \mathcal{N}} P_N(\tau_p)$. For $N \in \mathcal{N}$, let $\eta_N : G \to G/N$ be defined by $\eta_N(g) = gN$. Now, if $U \in \tau$ is saturated with respect to some $N \in \mathcal{N}$, then, $\eta_N(U) \subseteq q_N(\tau_p)$. But then, $U = \eta_N^{-1}(\eta_N(U)) \in P_N(\tau_p) \subseteq \tau_p$. So, $U \in \tau_p$. By the characterization theorem, (G, τ) is a protopological group. □

References

[1] R. W. Bagley, T. S. Wu, and J. S. Yang, *Pro-Lie groups*, Trans. Amer. Math. Soc. **287** (1985), no. 2, 829–838. MR 86e:22006. Zbl 0575.22006.

[2] J. L. Covington, *T-protopological groups*, Topology Proc. **19** (1994), 87–96. MR 96m:54070. Zbl 0847.54036.

[3] ———, *Protopological groups*, Kyungpook Math. J. **35** (1995), no. 2, 323–328. MR 96m:22001. Zbl 0858.22002.

[4] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, Springer, New York, 1963.

[5] D. Montgomery and L. Zippin, *Topological Transformation Groups*, Interscience Publishers, New York, 1955. MR 17,383b. Zbl 0068.01904.

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