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ON THE DIOPHANTINE EQUATION $x^3 = dy^2 \pm q^6$

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ABSTRACT. Let $q > 3$ denote an odd prime and d a positive integer without any prime factor $p \equiv 1 \pmod 3$. In this paper, we have proved that if $(x,q) = 1$, then $x^3 = dy^2 \pm q^6$ has exactly two solutions provided $q \not\equiv \pm 1 \pmod{24}$.

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Cohn [1] and recently Zhang [2, 3] have solved the Diophantine equation

$$x^3 = dy^2 \pm q^6 \tag{1}$$

when $q = 1,3,4$, under some conditions on d. In this paper, we consider the general case of (1) where $q \neq 3$ is any odd prime by using arguments similar to those used by Cohn [1].

Let $(x,y) \in \mathbb{Z} \times \mathbb{Z}$ be a solution of (1) with $x, y > 0$, then the solution is trivial if $x = 0$, $\pm q^2$ or $y = \pm 1$. We need the following lemma.

LEMMA 1. *The equation $p^2 = a^4 - 3b^2$, where p denotes an odd prime and $(p,a) = l$, may have a solution in positive integers a and b only if $p \equiv \pm 1 \pmod{24}$.*

PROOF. Suppose $3b^2 = a^4 - p^2$. Then clearly a is odd and b is even. Since $a^4 \equiv 3b^2 \pmod p$, and $(p,a) = 1$ therefore the Legendre symbol $(3/p) = 1$ and so $p \equiv \pm 1 \pmod{12}$. Now $(a^2 + p, a^2 - p) = 2$ implies that

$$a^2 \pm p = 3.2c^2, \tag{2}$$

$$a^2 \mp p = 2d^2, \tag{3}$$

where $2cd = b$ and $(c,d) = 1$. Whence

$$a^2 = 3c^2 + d^2. \tag{4}$$

Here d is odd, otherwise we get a contradiction modulo 4. Then considering (3) modulo 8, we get $p \equiv \pm 1 \pmod 8$. This completes the proof. □

Now we consider the upper sign in (1), our main result is laid down in the following.

THEOREM 2. *Let d be a positive integer without prime factor $p \equiv 1 \pmod 3$ and let $q \neq 3$ be an odd prime. If $q \not\equiv \pm 1 \pmod{24}$ and $(x,q) = 1$, then the Diophantine equation*

$$x^3 = dy^2 + q^6 \tag{5}$$

has exactly two solutions given by

$$x_1 = \frac{3q^4-2q^2-1}{4}, \quad y = ab, \quad \text{where } a = \frac{3q^4+1}{4},\ db^2 = \frac{3q^4-6q^2-1}{4},$$
$$x_2 = \frac{q^4-2q^2-3}{4}, \quad y = 9ab, \quad \text{where } a = \frac{q^4+3}{4},\ db^2 = \frac{q^4-6q^2-3}{4}. \tag{6}$$

Proof. If d has a square factor, then it can be absorbed into y^2, so there is no loss of generality in supposing d a square free integer. Now

$$dy^2 = x^3 - q^6 = (x-q^2)(x^2+q^2x+q^4). \tag{7}$$

If any prime r divides both d and $(x^2+q^2x+q^4)$, then by hypothesis $r \equiv 2 \pmod 3$ or $r = 3$. But $r \mid (x^2+q^2x+q^4)$ implies that $(2x+q^2)^2+3q^4 \equiv 0 \pmod r$ so the Legendre symbol $(-3/r) = 1$, which is a contradiction, whence $r = 1$ or 3. Also since $(x,q) = 1$, therefore $(x-q^2, x^2+q^2x+q^4) = 1$ or 3. So for (7) we have only two possibilities: either

$$x^2+q^2x+q^4 = a^2, \qquad x-q^2 = db^2, \tag{8}$$

or

$$x^2+q^2x+q^4 = 3a^2, \qquad x-q^2 = 3db^2, \tag{9}$$

where $(q,a) = 1$ and $(q,b) = 1$. Consider the first possibility when $(2x+q^2)^2+3q^4 = (2a)^2$ and $y = ab$. This equation is known to have a finite number of solutions. It can be written as

$$3q^4 = (2a+2x+q^2)(2a-(2x+q^2)). \tag{10}$$

Then for the nontrivial solution of this equation we have only two cases:

Case 1.

$$3q^4 = 2a \pm (2x+q^2), \qquad 1 = 2a \mp (2x+q^2), \tag{11}$$

by subtracting and adding these two equations we get

$$x = \frac{3q^4-2q^2-1}{4}, \qquad a = \frac{3q^4+1}{4}. \tag{12}$$

Here $a > 1$, so $y > 1$, and $x-q^2 = db^2$ implies that

$$db^2 = \frac{3q^4-6q^2-1}{4}. \tag{13}$$

Case 2.

$$3 = 2a \pm (2x+q^2), \qquad q^4 = 2a \mp (2x+q^2). \tag{14}$$

As in Case 1 we get the nontrivial solution

$$x = \frac{3q^4-2q^2-1}{4}, \qquad a = \frac{3q^4+1}{4}, \qquad db^2 = \frac{3q^4-6q^2-1}{4}. \tag{15}$$

Now suppose the second possibility. Obviously a is odd and $x^2 \equiv 3a^2 \pmod q$, and since $(q,a) = 1$, so the Legendre symbol $(3/q) = 1$, hence $q \equiv \pm 1 \pmod{12}$. Eliminating x and dividing by 3, we get

$$a^2 = q^4 + 3db^2(q^2 + db^2). \tag{16}$$

Considering (16) modulo 8 we get either $db^2 \equiv -1 \pmod 8$ or $db^2 \equiv 0 \pmod 8$.

(1) $db^2 \equiv -1 \pmod 8$. Then from (16) we get

$$3d^2b^4 = (2a + 2q^2 + 3db^2)(2a - 2q^2 - 3db^2). \tag{17}$$

Let S be a common prime divisor of the two factors in the right-hand side of (17), then S is odd, $S \mid 4a$ and $S \mid 2(2q^2 + 3db^2)$. But S^2 divides the left-hand side implies that $S \mid 3db^2$, so $S \mid q^2$. Here $S = 1$, otherwise $x - q^2 = 3db^2$ implies that $q \mid x$ which is not true. Thus from (17) we get

$$2a \pm (2q^2 + 3db^2) = d_1^2 b_1^4, \qquad 2a \mp (2q^2 + 3db^2) = 3d_2^2 b_2^4, \tag{18}$$

where $d = d_1 d_2$ and $b = b_1 b_2$. Whence

$$\pm 2(2q^2 + 3db^2) = d_1^2 b_1^4 - 3d_2^2 b_2^4. \tag{19}$$

Considering this equation modulo 3, we get

$$4q^2 = d_1^2 b_1^4 - 3d_2^2 b_2^4 - 6db^2. \tag{20}$$

Now we prove that $d_1 = 1$. Since d is odd, therefore d_1 must be odd. Let t be any odd prime dividing d_1 then by hypothesis $t \equiv 2 \pmod 3$ but then from (20) we get

$$4q^2 \equiv -3d_2^2 b_2^4 \pmod t, \tag{21}$$

so $(-3/t) = 1$, which is not true. Thus $d_1 = 1$ and (20) becomes

$$q^2 = b_1^4 - 3\left(\frac{b_1^2 + db_2^4}{2}\right)^2, \tag{22}$$

since $(q, b_1) = 1$, therefore by Lemma 1, $q \equiv \pm 1 \pmod{24}$.

(2) $db^2 \equiv 0 \pmod 8$. Now we prove that if (16) has a solution, then $q \equiv \pm 1 \pmod{24}$. Since d is a square free, b should be even. Suppose $b = 2m$, then (16) can be written as

$$12d^2m^4 = (a + q^2 + 6dm^2)(a - q^2 - 6dm^2). \tag{23}$$

As before we can prove that the common divisor of the two factors in the right-hand side of (23) is 2, so

$$a \pm (q^2 + 6dm^2) = 2d_1^2 m_1^4, \qquad a \mp (q^2 + 6dm^2) = 6d_2^2 m_2^4, \tag{24}$$

where $d = d_1 d_2$ and $m = m_1 m_2$. It is clear that $(a,q) = 1$ implies that $(m_1, q) = 1$.

Subtracting the two equations in (24) we get

$$\pm(q^2+6dm^2)=d_1^2m_1^4-3d_2^2m_2^4, \tag{25}$$

again considering this equation modulo 3, we get $q^2=d_1^2m_1^4-3d_2^2m_2^4-6dm^2$. As before d_1 cannot have any odd prime divisor, so $d_1=1$ or 2.

If $d_1=1$, then

$$q^2=4m_1^4-3(m_1^2+dm_2^2). \tag{26}$$

Here m_1 is odd, otherwise we get a contradiction modulo 8. Since $(m_1,q)=1$, then from (26) we get

$$2m_1^2\pm q=3s^2,\qquad 2m_1^2\mp q=n^2, \tag{27}$$

where $sn=m_1^2+dm_2^2$, so s and n are both odd. Hence $q\equiv\pm1 \pmod 8$, combining this result with $q\equiv\pm1 \pmod{12}$, we get $q\equiv\pm1 \pmod{24}$.

If $d_1=2$, then

$$q^2=16b_1^4-3(b_1^2+db_2^2)^2 \tag{28}$$

which is impossible modulo 8. □

Using the same argument as in Theorem 2 we can prove the following theorem.

Theorem 3. *Let d be a positive integer without prime factor $p\equiv1 \pmod 3$ and $q\neq3$ an odd prime. If $q\not\equiv\pm1 \pmod{24}$ and $(x,q)=1$, then the Diophantine equation $x^3=dy^2-q^6$ has exactly two solutions given by*

$$\begin{aligned} x_1&=\frac{3q^4+2q^2-1}{4}, &\quad y&=ab, &\quad \text{where } a&=\frac{3q^4+1}{4},\ db^2=\frac{3q^4+6q^2-1}{4},\\ x_2&=\frac{q^4+2q^2-3}{4}, &\quad y&=9ab, &\quad \text{where } a&=\frac{q^4+3}{4},\ db^2=\frac{q^4+6q^2-3}{4}. \end{aligned} \tag{29}$$

Sometimes, combining our results with Cohn's result [1] we can solve the title equation completely when d has no prime factor $\equiv1 \pmod 3$, as we show in the following example.

Example 4. Consider the Diophantine equation $x^3=dy^2\pm5^6$ where d has no prime factor $\equiv1 \pmod 3$ and $(5,d)=1$.

Here $q=5$, when $(x,5)=1$, using Theorem 2 for the positive sign this equation has only two solutions given by $x_1=456$, $db^2=431$, and $x_2=143$, $db^2=118$. So $d=431,118$. Now let $5\mid x$, then because $(5,d)=1$, the equation reduces to the form $x^3=5dy^2+1$, which by [1, Theorem 1] has no solution in positive integers.

So the equation $x^3=dy^2+5^6$ has a solution only if $d=431,118$.

For the negative sign this equation has two solutions when $(x,5)=1$ given by

$$x_1=481,\qquad db^2=506,\qquad x_2=168,\qquad db^2=193, \tag{30}$$

that is, when $d=506,193$. If $5\mid x$, then the equation reduces to the form $x^3=5dy^2-1$, which by [1, Theorem 2] has no solution in positive integers.

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