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# FIXED POINT AND COINCIDENCE POINT THEOREMS FOR A PAIR OF SINGLE-VALUED AND MULTI-VALUED MAPS ON A METRIC SPACE

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We obtain fixed point and coincidence point theorems for a pair of single-valued and multi-valued maps on a metric space satisfying a generalized nonexpansive type condition.

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Bogin [1] proved a fixed point theorem for a nonexpansive type self-map on a complete metric space. While Rhoades obtained a generalization of it (see [8, Theorem 1]) by replacing the constant coefficients in the governing inequality of the map with nonnegative real-valued functions of the independent variables, Ćirić [3] obtained a generalization by further weakening the governing inequality without allowing the coefficients to vary. Chandra et al. obtained a coincidence point theorem (see [2, Theorem 2.1]) for a pair of self-maps on a metric space unifying the results of Rhoades and Ćirić. They also obtained a corresponding version for multimaps (see [2, Theorem 2.2]). In this paper, we obtain proper generalizations of Theorems 2.1 and 2.2 of Chandra et al. [2].

Throughout, unless otherwise stated, $(X,d)$ is a metric space, $K(X)$ is the collection of all nonempty compact subsets of $X$, $\mathrm{CL}(X)$ is the collection of all nonempty closed subsets of $X$, $H$ is the extended Hausdorff metric on $\mathrm{CL}(X)$, $F$ is a mapping from $X$ into $\mathrm{CL}(X)$, $f$, $S$ are self-maps on $X$, $I$ is the identity map on $X$, for any self-map $h$ on $X$, $\Re(h)=\{hx : x\in X\}$, $\mathbb{R}^+$ is the set of all nonnegative real numbers, $\mathbb{N}$ is the set of all positive integers, $\Omega : (\mathbb{R}^+)^5\to\mathbb{R}^+$ is monotonically increasing in each coordinate variable, for any $t_1,t_2,t_3,t_4,t_5\in\mathbb{R}^+$, $\Omega(t_1^+,t_2^+,t_3^+,t_4^+,t_5^+) = \inf\{\Omega(s_1,s_2,s_3,s_4,s_5) : s_j\in(t_j,+\infty)$ for all $j=1,2,3,4,5\}$, $\Omega(t_1,t_2^+,t_3^+,t_4^+,t_5^+) = \inf\{\Omega(t_1,s_2,s_3,s_4,s_5) : s_j\in(t_j,+\infty)$ for all $j=2,3,4,5\}$, $\sigma_j:\mathbb{R}^+\to\mathbb{R}^+$ $(j=1,2)$ and $\zeta:\mathbb{R}^+\to\mathbb{R}^+$ are defined as

$$\sigma_1(t)=\Omega(0^+,0^+,t^+,t^+,t^+),\qquad \sigma_2(t)=\Omega(t,0^+,0^+,t^+,0^+),$$
$$\zeta(t)=\max\{\sigma_1(t),\sigma_2(t)\},\tag{1}$$

for all $t\in\mathbb{R}^+$, $\alpha$, $D$ are functions from $X\times X$ to $\mathbb{R}^+$ defined as

$$\alpha(x,y)=\Omega\big(d(Sx,fx),d(Sy,fy),d(Sx,Sy),d(fx,Sy),d(Sx,fy)\big),$$
$$D(x,y)=\Omega\big(d(Sx,Fx),d(Sy,Fy),d(Sx,Sy),d(Sy,Fx),d(Sx,Fy)\big),\tag{2}$$

for all $x,y$ in $X$.

**Definition 1.** We say that $(f,S)$ has property $A$ if there is a sequence $\{x_n\}_{n=0}^{\infty}$ in $X$ such that $Sx_{n+1} = fx_n$ $(= y_n,$ say$)$ for all $n = 0,1,2\ldots.$

**Lemma 2.** *Suppose that $(f,S)$ has property $A$, $\{d(y_n,y_{n+1})\}_{n=0}^{\infty}$ converges to zero, $\sigma_1(t) < t$ for all $t \in (0,\infty)$, and that*

$$d(fx,fy) \le \alpha(x,y) \tag{3}$$

*for all $x,y$ in $X$. Then $\{y_n\}$ is Cauchy.*

**Proof.** If possible, suppose that $\{y_n\}$ is not Cauchy. Then there exists a positive real number $\varepsilon$ with the following property: given $N \in \mathbb{N}$ there exists $m,n \in \mathbb{N} \ni m > n \ge N$ and $d(y_n,y_m) \ge \varepsilon$. Hence there exist strictly increasing sequences $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ in $\mathbb{N}$ such that $k < n_k < m_k$, $d(y_{n_k},y_{m_k}) \ge \varepsilon$, and $d(y_{n_k},y_{m_k-1}) < \varepsilon$ for all $k \in \mathbb{N}$. Since $\{d(y_n,y_{n+1})\}$ converges to zero, it follows that $\{d(y_{n_k},y_{m_k})\}_{k=1}^{\infty}$ converges to $\varepsilon$ and that for any fixed $r,s$ in $\{-1,0,1\}$, the sequence $\{d(y_{n_k+r},y_{m_k+s})\}_{k=1}^{\infty}$ also converges to $\varepsilon$. We have $\alpha(x_{n_k},x_{m_k+1}) = \Omega(d(y_{n_k-1},y_{n_k}),d(y_{m_k},y_{m_k+1}),d(y_{n_k-1},y_{m_k}),$ $d(y_{n_k},y_{m_k}),d(y_{n_k-1},y_{m_k+1}))$ for all $k \in \mathbb{N}$. We note that the limit superior of $\{\alpha(x_{n_k},x_{m_k+1})\}_{k=1}^{\infty}$ is less than or equal to $\Omega(0^+,0^+,\varepsilon^+,\varepsilon^+,\varepsilon^+)$ $(= \sigma_1(\varepsilon))$. We also note that $\{d(fx_{n_k},fx_{m_k+1})\}_{k=1}^{\infty}$ converges to $\varepsilon$. From (3) we have

$$d(fx_{n_k},fx_{m_k+1}) \le \alpha(x_{n_k},x_{m_k+1}) \tag{4}$$

for all $k \in \mathbb{N}$. By taking limit superiors on both sides of (4) as $k \to +\infty$ we obtain $\varepsilon \le \sigma_1(\varepsilon)$. This is a contradiction since $\sigma_1(t) < t$ for all $t \in (0,\infty)$ and $\varepsilon > 0$. Hence $\{y_n\}$ is Cauchy. □

**Definition 3.** We say that $\Omega$ has property $A$ if $\Omega(t,s,t,0,t+s) < s$ for all $s,t \in \mathbb{R}^+$ with $t < s$.

**Definition 4.** We say that $\Omega$ has property $B$ if there exist (i) a monotonically increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(t^+) < t$ for all $t \in (0,\infty)$, and (ii) for each $t \in \mathbb{R}^+$ a nonempty index set $I_t$ and nonnegative real numbers $\beta_i$, $\gamma_i$ $(i \in I_t)$ such that $\sup\{\gamma_i : i \in I_t\} < 1$, $\Omega(t,t,2t,t,t+s) \le \sup\{(1+\beta_i)t + \gamma_i s : i \in I_t\}$ for all $s \in [t,2t]$, $\Omega(t,t,t,0,\lambda_t t) \le \varphi(t)$, where $\lambda_t = \sup\{(1+\beta_i)/(1-\gamma_i) : i \in I_t\}$.

**Lemma 5.** *Suppose that $(f,S)$ has property $A$, $\Omega$ has properties $A$ and $B$, and that inequality (3) is true for all $x,y$ in $X$. Then $\{d(y_n,y_{n+1})\}_{n=0}^{\infty}$ converges to zero.*

**Proof.** We have $\alpha(x_n,x_{n+1}) = \Omega(d(y_{n-1},y_n),d(y_n,y_{n+1}),d(y_{n-1},y_n),0,d(y_{n-1},$ $y_{n+1}))$ for all $n \in \mathbb{N}$. From inequality (3) we have

$$d(y_n,y_{n+1}) \le \alpha(x_n,x_{n+1}) \tag{5}$$

for all $n \in \mathbb{N}$. If $d(y_{m-1},y_m) < d(y_m,y_{m+1})$ for some $m \in \mathbb{N}$, then since $d(y_{n-1},y_{n+1})$ $\le d(y_{n-1},y_n) + d(y_n,y_{n+1})$, $\Omega$ is increasing in each coordinate variable and $\Omega$ has property $A$, it follows from (5) that $d(y_m,y_{m+1}) < d(y_m,y_{m+1})$ which is a contradiction. Hence $d(y_n,y_{n+1}) \le d(y_{n-1},y_n)$ for all $n \in \mathbb{N}$.

From (3) we have

$$d(y_n,y_{n+2}) \le \alpha(x_n,x_{n+2}) \tag{6}$$

for all $n \in \mathbb{N}$. But

$$\begin{aligned}\alpha(x_n,x_{n+2}) &= \Omega(d(y_{n-1},y_n),d(y_{n+1},y_{n+2}),d(y_{n-1},y_{n+1}),\\&\qquad d(y_n,y_{n+1}),d(y_{n-1},y_{n+2}))\\&\le \Omega(d(y_{n-1},y_n),d(y_{n+1},y_{n+2}),d(y_{n-1},y_n)+d(y_n,y_{n+1}),\\&\qquad d(y_n,y_{n+1}),d(y_{n-1},y_n)+d(y_n,y_{n+2}))\\&\le \Omega(d(y_{n-1},y_n),d(y_{n-1},y_n),2d(y_{n-1},y_n),d(y_{n-1},y_n),\\&\qquad d(y_{n-1},y_n)+d(y_n,y_{n+2}))\end{aligned} \tag{7}$$

since the sequence $\{d(y_{k-1},y_k)\}$ is monotonically decreasing and $\Omega$ is increasing in each coordinate variable. Since $\Omega$ has property $B$, there exist (i) a monotonically increasing function $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ with $\varphi(t^+)<t$ for all $t\in(0,\infty)$ and (ii) for each $t\in\mathbb{R}^+$ a nonempty index set $I_t$ and nonnegative real numbers $\beta_i$, $\gamma_i$ $(i\in I_t)$ such that $\sup\{\gamma_i : i\in I_t\}<1$, $\Omega(t,t,2t,t,t+s)\le\sup\{(1+\beta_i)t+\gamma_i s : i\in I_t\}$ for all $s\in[t,2t]$, and $\Omega(t,t,t,0,\lambda_t t)\le\varphi(t)$, where $\lambda_t=\sup\{(1+\beta_i)/(1-\gamma_i) : i\in I_t\}$. Since $d(y_n,y_{n+2})\le d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})\le 2d(y_{n-1},y_n)$, we have

$$\begin{aligned}&\Omega(d(y_{n-1},y_n),d(y_{n-1},y_n),2d(y_{n-1},y_n),d(y_{n-1},y_n),d(y_{n-1},y_n)+d(y_n,y_{n+2}))\\&\qquad\le \sup\{(1+\beta_i)d(y_{n-1},y_n)+\gamma_i d(y_n,y_{n+2}) : i\in I_r\}\end{aligned} \tag{8}$$

provided $d(y_{n-1},y_n)\le d(y_n,y_{n+2})$, where $r=d(y_{n-1},y_n)$. We assume that $d(y_{n-1},y_n)\le d(y_n,y_{n+2})$. Then, from (6), (7), and (8), we have

$$d(y_n,y_{n+2})\le\sup\{(1+\beta_i)d(y_{n-1},y_n)+\gamma_i d(y_n,y_{n+2}) : i\in I_r\}. \tag{9}$$

Hence given $\varepsilon>0$ there exists $j\in I_r$ such that

$$d(y_n,y_{n+2})\le(1+\beta_j)d(y_{n-1},y_n)+\gamma_j d(y_n,y_{n+2})+\varepsilon. \tag{10}$$

Hence in view of the hypothesis that $1>\sup\{\gamma_i : i\in I_r\}$ $(=\mu,\ \text{say})$, we have

$$d(y_n,y_{n+2})\le\left(\frac{1+\beta_j}{1-\gamma_j}\right)d(y_{n-1},y_n)+\left(\frac{\varepsilon}{1-\gamma_j}\right)\le\lambda_r d(y_{n-1},y_n)+\left(\frac{\varepsilon}{1-\mu}\right). \tag{11}$$

Since $\varepsilon>0$ is arbitrary, from (11) it follows that

$$d(y_n,y_{n+2})\le\lambda_r d(y_{n-1},y_n). \tag{12}$$

Since $\lambda_r\ge 1$, (12) is evidently true if $d(y_{n-1},y_n)>d(y_n,y_{n+2})$. Hence (12) is true for all $n\in\mathbb{N}$. Hence we have

$$\begin{aligned}\alpha(x_{n+1},x_{n+2}) &= \Omega(d(y_n,y_{n+1}),d(y_{n+1},y_{n+2}),d(y_n,y_{n+1}),0,d(y_n,y_{n+2}))\\&\le\Omega(d(y_{n-1},y_n),d(y_{n-1},y_n),d(y_{n-1},y_n),0,d(y_n,y_{n+2}))\\&\le\Omega(d(y_{n-1},y_n),d(y_{n-1},y_n),d(y_{n-1},y_n),0,\lambda_r d(y_{n-1},y_n))\\&\le\varphi(d(y_{n-1},y_n))\end{aligned} \tag{13}$$

for all $n \in \mathbb{N}$. Hence from (3) we have

$$d(y_{n+1}, y_{n+2}) \le \varphi(d(y_{n-1}, y_n)) \tag{14}$$

for all $n \in \mathbb{N}$. Since $\varphi$ is monotonically increasing on $\mathbb{R}^+$, by repeatedly using inequality (14) we obtain

$$d(y_{2n}, y_{2n+1}) \le \varphi^n(d(y_0, y_1)), \qquad d(y_{2n+1}, y_{2n+2}) \le \varphi^n(d(y_1, y_2)) \tag{15}$$

for all $n \in \mathbb{N}$. Since $\varphi$ is monotonically increasing on $\mathbb{R}^+$ and $\varphi(t^+) < t$ for all $t \in (0,\infty)$, $\{\varphi^n(t)\}$ converges to zero for all $t$ in $\mathbb{R}^+$. Hence from (15) it follows that $\{d(y_n, y_{n+1})\}$ converges to zero. □

**Definition 6** (see [5]). A pair $(f_1, f_2)$ of self-maps on $(X,d)$ is said to be compatible (co.) if $\{d(f_1f_2x_n, f_2f_1x_n)\}$ converges to zero whenever $\{x_n\}$ is a sequence in $X$ such that $\{f_1x_n\}$ and $\{f_2x_n\}$ are convergent in $X$ and have the same limit.

**Definition 7** (see [4]). A pair $(f_1, f_2)$ of self-maps on an arbitrary set $E$ is said to be weakly compatible (w.co.) if $f_1f_2x = f_2f_1x$ whenever $x \in E$ is such that $f_1x = f_2x$.

**Remark 8.** If $(f,S)$ is co. then it is w.co.

**Definition 9** (see [7]). A pair $(f_1, f_2)$ of self-maps on $(X,d)$ is said to be reciprocally continuous on $X$ if $\{f_1f_2x_n\}$ converges to $f_1u$ and $\{f_2f_1x_n\}$ converges to $f_2u$ whenever $\{x_n\}$ is a sequence in $X$ such that $\{f_1x_n\}$ and $\{f_2x_n\}$ converge to $u$ for some $u \in X$.

**Definition 10.** A pair $(f_1, f_2)$ of self-maps on $(X,d)$ is said to be reciprocally continuous at $u \in X$ if $\{f_1f_2x_n\}$ converges to $f_1u$ and $\{f_2f_1x_n\}$ converges to $f_2u$ whenever $\{x_n\}$ is a sequence in $X$ such that $\{f_1x_n\}$ and $\{f_2x_n\}$ converge to $u$.

**Lemma 11.** *Suppose that $(f,S)$ has property A, $\{y_n\}$ converges to an element $z$ of $X$ and that (3) is true for all $x, y$ in $X$. Then the following statements are true:*

(i) *If $\sigma_1(t) < t$ for all $t \in (0,\infty)$ and $fp = Sp$ for some $p \in X$, then $fp = z$. In particular, $f$ and $S$ cannot have a common fixed point or coincidence value other than $z$ if $\sigma_1(t) < t$ for all $t \in (0,\infty)$.*

(ii) *If $\sigma_2(t) < t$ for all $t \in (0,\infty)$ and $z \in \Re(S)$, then there exists $w \in X$ such that $fw = Sw = z$.*

(iii) *If $\zeta(t) < t$ for all $t \in (0,\infty)$, $z \in \Re(S)$ and $(f,S)$ is w.co., then $fz = Sz = z$.*

(iv) *If $\sigma_1(t) < t$ for all $t \in (0,\infty)$, $S$ is continuous at $z$ and $(f,S)$ is co., then $Sz = z$.*

(v) *If $\sigma_1(t) < t$ for all $t \in (0,\infty)$, $f$ is continuous at $z$ and $(f,S)$ is co., then $fz = z$.*

(vi) *If $\sigma_1(t) < t$ for all $t \in (0,\infty)$, $(f,S)$ is co. and reciprocally continuous at $z$, then $fz = Sz = z$.*

**Proof.** (i) Suppose that $\sigma_1(t) < t$ for all $t \in (0,\infty)$ and $fp = Sp$ for some $p \in X$. We have

$$\alpha(p, x_{n+1}) = \Omega(0, d(y_n, y_{n+1}), d(fp, y_n), d(fp, y_n), d(fp, y_{n+1})) \tag{16}$$

for all $n \in \mathbb{N}$. We note that the limit superior of the sequence $\{\alpha(p, x_{n+1})\}$ is less than or equal to $\Omega(0,0^+,d(fp,z)^+,d(fp,z)^+,d(fp,z)^+)$ which in turn is less than or equal to $\sigma_1(d(fp,z))$. From (3) we have

$$d(fp,y_{n+1}) \le \alpha(p,x_{n+1}) \tag{17}$$

for all $n \in \mathbb{N}$. By taking limit superiors on both sides of (17) as $n \to +\infty$ we obtain $d(fp,z) \le \sigma_1(d(fp,z))$. Since $\sigma_1(t) < t$ for all $t \in (0,\infty)$, we have $d(fp,z) = 0$. Hence $fp = z$. Hence $f$ and $S$ cannot have a common fixed point other than $z$. If $p,q \in X$ are such that $fp = Sp$ and $fq = Sq$, then we have $fp = z = fq$. Hence $f$ and $S$ cannot have a coincidence value other than $z$.

(ii) Suppose that $\sigma_2(t) < t$ for all $t \in (0,\infty)$ and $z \in \Re(S)$. Then there exists $w \in X \ni Sw = z$. We have

$$\alpha(w,x_{n+1}) = \Omega(d(z,fw),d(y_n,y_{n+1}),d(z,y_n),d(fw,y_n),d(z,y_{n+1})) \tag{18}$$

for all $n \in \mathbb{N}$. We note that the limit superior of the sequence $\{\alpha(w,x_{n+1})\}$ is less than or equal to $\Omega(d(fw,z),0^+,0^+,d(fw,z)^+,0^+)$ which in turn is less than or equal to $\sigma_2(d(fw,z))$. From (3) we have

$$d(fw,y_{n+1}) \le \alpha(w,x_{n+1}) \tag{19}$$

for all $n \in \mathbb{N}$. By taking limit superiors on both sides of (19) as $n \to +\infty$ we obtain $d(fw,z) \le \sigma_2(d(fw,z))$. Hence $d(fw,z) = 0$. Hence $fw = z$.

(iii) Suppose that $\zeta(t) < t$ for all $t \in (0,\infty)$, $z \in \Re(S)$ and $(f,S)$ is w.co. From statement (ii) it follows that there exists $w \in X \ni fw = Sw = z$. Hence from the weak compatibility of $(f,S)$ we have $fz = fSw = Sfw = Sz$. We have

$$\alpha(z,x_{n+1}) = \Omega(d(Sz,fz),d(y_n,y_{n+1}),d(Sz,y_n),d(fz,y_n),d(Sz,y_{n+1})) \tag{20}$$

for all $n \in \mathbb{N}$. We note that the limit superior of the sequence $\{\alpha(z,x_{n+1})\}$ is less than or equal to $\Omega(0,0^+,d(fz,z)^+,d(fz,z)^+,d(fz,z)^+)$ which in turn is less than or equal to $\sigma_1(d(fz,z))$. From (3) we have

$$d(fz,y_{n+1}) \le \alpha(z,x_{n+1}) \tag{21}$$

for all $n \in \mathbb{N}$. By taking limit superiors on both sides of (21) as $n \to +\infty$ we obtain $d(fz,z) \le \sigma_1(d(fz,z))$. Since $\sigma_1(t) < t$ for all $t \in (0,\infty)$, we have $d(fz,z) = 0$. Hence $fz = z$. Hence $Sz = z$.

(iv) Suppose that $\sigma_1(t) < t$ for all $t \in (0,\infty)$, $S$ is continuous at $z$ and $(f,S)$ is co. Since $\{y_n\}$ converges to $z$ and $S$ is continuous at $z$, $\{Sy_n\}$ converges to $Sz$. Hence the sequences $\{SSx_n\}$ and $\{Sfx_n\}$ converge to $Sz$. Since $(f,S)$ is co., and $\{fx_n\}$ and $\{Sx_n\}$ are convergent sequences having the same limit $z$, it follows that $\{d(Sfx_n,fSx_n)\}$ converges to zero. Since $\{Sfx_n\}$ converges to $Sz$, it follows that $\{fSx_n\}$ also converges to $Sz$. We have

$$\begin{aligned}\alpha(Sx_n,x_{n+1}) = \Omega(d(SSx_n,fSx_n),d(y_n,y_{n+1}),d(SSx_n,y_n),\\ d(fSx_n,y_n),d(SSx_n,y_{n+1}))\end{aligned} \tag{22}$$

for all $n \in \mathbb{N}$. We note that the limit superior of the sequence $\{\alpha(Sx_n, x_{n+1})\}$ is less than or equal to $\Omega(0^+, 0^+, d(Sz,z)^+, d(Sz,z)^+, d(Sz,z)^+) = \sigma_1(d(Sz,z))$. From (3) we have

$$d(fSx_n, y_{n+1}) \leq \alpha(Sx_n, x_{n+1}) \tag{23}$$

for all $n \in \mathbb{N}$. By taking limit superiors on both sides of (23) as $n \to +\infty$ we obtain $d(Sz,z) \leq \sigma_1(d(Sz,z))$. Hence $d(Sz,z) = 0$. Hence $Sz = z$.

(v) Suppose that $\sigma_1(t) < t$ for all $t \in (0,\infty)$, $f$ is continuous at $z$ and $(f,S)$ is co. Since $\{y_n\}$ converges to $z$ and $f$ is continuous at $z$, $\{fy_n\}$ converges to $fz$. Hence the sequences $\{ffx_n\}$ and $\{fSx_n\}$ converge to $fz$. Since $(f,S)$ is co., and $\{fx_n\}$ and $\{Sx_n\}$ are convergent sequences having the same limit $z$, it follows that $\{d(Sfx_n, fSx_n)\}$ converges to zero. Since $\{fSx_n\}$ converges to $fz$, it follows that $\{Sfx_n\}$ also converges to $fz$. We have

$$\begin{aligned}\alpha(fx_n, x_{n+1}) = \Omega(&d(Sfx_n, ffx_n), d(y_n, y_{n+1}), d(Sfx_n, y_n), \\ &d(ffx_n, y_n), d(Sfx_n, y_{n+1}))\end{aligned} \tag{24}$$

for all $n \in \mathbb{N}$. We note that the limit superior of the sequence $\{\alpha(fx_n, x_{n+1})\}$ is less than or equal to $\Omega(0^+, 0^+, d(fz,z)^+, d(fz,z)^+, d(fz,z)^+) = \sigma_1(d(fz,z))$. From (3) we have

$$d(ffx_n, y_{n+1}) \leq \alpha(fx_n, x_{n+1}) \tag{25}$$

for all $n \in \mathbb{N}$. By taking limit superiors on both sides of (25) as $n \to +\infty$ we obtain $d(fz,z) \leq \sigma_1(d(fz,z))$. Hence $d(fz,z) = 0$. Hence $fz = z$.

(vi) Suppose that $\sigma_1(t) < t$ for all $t \in (0,\infty)$, and $(f,S)$ is co. and reciprocally continuous at $z$. Since $\{y_n\}$ converges to $z$, the sequences $\{fx_n\}$ and $\{Sx_n\}$ are convergent and have the same limit $z$. Hence from the reciprocal continuity of $(f,S)$ at $z$ it follows that $\{Sfx_n\}$ converges to $Sz$ and $\{fSx_n\}$ converges to $fz$ and from the compatibility of $(f,S)$ it follows that $\{d(Sfx_n, fSx_n)\}$ converges to zero. Hence $fz = Sz$. We have

$$\alpha(z, x_{n+1}) = \Omega(d(Sz, fz), d(y_n, y_{n+1}), d(Sz, y_n), d(fz, y_n), d(Sz, y_{n+1})) \tag{26}$$

for all $n \in \mathbb{N}$. We note that the limit superior of the sequence $\{\alpha(z, x_{n+1})\}$ is less than or equal to $\Omega(0, 0^+, d(fz,z)^+, d(fz,z)^+, d(fz,z)^+)$ which in turn is less than or equal to $\sigma_1(d(fz,z))$. From (3) we have

$$d(fz, y_{n+1}) \leq \alpha(z, x_{n+1}) \tag{27}$$

for all $n \in \mathbb{N}$. By taking limit superiors on both sides of the above inequality as $n \to +\infty$ we obtain $d(fz,z) \leq \sigma_1(d(fz,z))$. Hence $d(fz,z) = 0$. Hence $fz = z$. Hence $Sz = z$. □

**Theorem 12.** *Suppose that $(f,S)$ has property A, $\Omega$ has properties A and B, $\sigma_1(t) < t$ for all $t \in (0,\infty)$ and that inequality (3) is true for all $x, y$ in $X$. Then $\{y_n\}$ is Cauchy. Suppose that it converges to an element $z$ of $X$. Then the following statements are true:*

(i) *$f$ and $S$ cannot have a common fixed point or coincidence value other than $z$.*
(ii) *If $S$ is continuous at $z$ and $(f,S)$ is co., then $Sz = z$.*
(iii) *If $f$ is continuous at $z$ and $(f,S)$ is co., then $fz = z$.*
(iv) *If $(f,S)$ is co. and reciprocally continuous at $z$, then $fz = Sz = z$.*

(v) *If* $\sigma_2(t) < t$ *for all* $t \in (0,\infty)$ *and* $z \in \Re(S)$*, then there exists* $w \in X$ *such that* $fw = Sw = z$.

(vi) *If* $\sigma_2(t) < t$ *for all* $t \in (0,\infty)$*,* $z \in \Re(S)$ *and* $(f,S)$ *is w.co., then* $fz = Sz = z$.

**Proof.** The proof follows from Lemmas 2, 5, and 11. □

**Corollary 13.** *Suppose that* $(f,S)$ *has property* $A$ *and that there is a monotonically decreasing function* $\delta : \mathbb{R}^+ \to (0,1/3]$ *such that*

$$d(fx,fy) \le \sup\{ad(Sx,Sy) + b\max\{d(Sx,fx),d(Sy,fy)\} + c[d(fx,Sy)+d(Sx,fy)] : a \ge 0,\ b \ge \delta(\theta(x,y)), c \ge \delta(\theta(x,y)),\ a+b+2c \le 1\} \tag{28}$$

*for all* $x,y$ *in* $X$*, where*

$$\theta(x,y) = \max\left\{d(Sx,Sy),d(Sx,fx),d(Sy,fy),\frac{1}{2}[d(fx,Sy)+d(Sx,fy)]\right\}. \tag{29}$$

*Then* $\{y_n\}$ *is Cauchy. Suppose that it converges to an element* $z$ *of* $X$*. Then the following statements are true:*

(i) $f$ *and* $S$ *cannot have a common fixed point or coincidence value other than* $z$.

(ii) *If* $z \in \Re(S)$*, then there exists* $w \in X$ *such that* $fw = Sw = z$.

(iii) *If* $z \in \Re(S)$ *and* $(f,S)$ *is w.co., then* $fz = Sz = z$.

(iv) *If* $S$ *is continuous at* $z$ *and* $(f,S)$ *is co., then* $fz = Sz = z$.

(v) *If* $f$ *is continuous at* $z$ *and* $(f,S)$ *is co., then* $fz = z$.

(vi) *If* $(f,S)$ *is co. and reciprocally continuous at* $z$*, then* $fz = Sz = z$.

**Proof.** Define $\Lambda : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ as $\Lambda(t_1,t_2,t_3,t_4,t_5) = \max\{t_1,t_2,t_3,(1/2)(t_4+t_5)\}$. Define $I : \mathbb{R}^+ \to \mathbb{R}^+$ as $I(t) = \{(a,b,c) \in (\mathbb{R}^+)^3 : a \ge 0,\ b \ge \delta(t),\ c \ge \delta(t)$, and $a+b+2c \le 1\}$. Define $\Omega : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ as

$$\Omega(t_1,t_2,t_3,t_4,t_5) = \sup\{at_3 + b\max\{t_1,t_2\} + c(t_4+t_5) : (a,b,c) \in I(\Lambda(t_1,t_2,t_3,t_4,t_5))\} \tag{30}$$

for all $t_1,t_2,t_3,t_4,t_5 \in \mathbb{R}^+$. It is clear that (3) is true for all $x,y$ in $X$. Since $\delta$ is monotonically decreasing on $\mathbb{R}^+$, it is evident that $\Omega$ is increasing in each coordinate variable. It can be verified that $\sigma_1(t) \le [1-\delta(t^+)]t$ and $\sigma_2(t) \le [1-\delta(t)]t$ for all $t \in (0,\infty)$ so that $\zeta(t) < t$ for all $t \in (0,\infty)$. It can be seen that $\Omega(t,s,t,0,t+s) \le t + [1-\delta(s)](s-t) < s$ if $0 \le t < s < +\infty$. Hence $\Omega$ has property $A$. It can be shown that $\Omega(t,t,2t,t,t+s) \le \sup\{(1+a)t + cs : (a,b,c) \in I(2t)\}$ for all $t(\mathbb{R})^+$ and $s \in [t,2t]$, $(1+a)/(1-c) \le 2-b \le 2-\delta(2t)$ for all $(a,b,c) \in I(2t)$ and that $\Omega(t,t,t,0,\lambda_t t) \le [1-(2-\lambda_t)\delta(t)]t \le [1-\delta(2t)\delta(t)]t < t$ for all $t \in (0,+\infty)$, where $\lambda_t = \sup\{(1+a)/(1-c) : (a,b,c) \in I(2t)\} \le 2-\delta(2t)$. Clearly, $\sup\{c : (a,b,c) \in I(2t)\} \le (1/2)[1-\delta(2t)] \le 1/2$. Define $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\varphi(t) = [1-\delta(2t)\delta(t)]t$ for all $t \in \mathbb{R}^+$. Since $\delta$ is monotonically decreasing on $\mathbb{R}^+$, $\varphi$ is monotonically increasing on $\mathbb{R}^+$. Clearly, $\varphi(t^+) < t$ for all $t \in (0,+\infty)$ and $\Omega(t,t,t,0,\lambda_t t) \le \varphi(t)$ for all $t \in \mathbb{R}^+$. Hence $\Omega$ satisfies property $B$ with $I_t = I(2t)$, $\beta_i = a$, and $\gamma_i = c$, where $i = (a,b,c) \in I(2t)$. Now the corollary is evident from Theorem 12. □

**Corollary 14** (see [2, Theorem 2.1]). *Suppose that $f(X) \subseteq S(X)$ and that there are nonnegative real-valued functions $a, b, c$ on $X \times X$ such that*

$$d(fx,fy) \le a(x,y)d(Sx,Sy) + b(x,y)\max\{d(Sx,fx),d(Sy,fy)\} + c(x,y)[d(fx,Sy)+d(Sx,fy)] \tag{31}$$

*for all $x, y$ in $X$, $\inf\{b(u,v) : u,v \in X\} > 0$, $\inf\{c(u,v) : u,v \in X\} > 0$, and $\sup\{a(u,v) + b(u,v) + 2c(u,v) : u,v \in X\} = 1$. Suppose also that either* (a) *$X$ is complete and $S$ is surjective; or* (b) *$X$ is complete, $S$ is continuous and $(f,S)$ is co.; or* (c) *$S(X)$ is complete: or* (d) *$f(X)$ is complete. Then $f$ and $S$ have a coincidence point in X. Further, the coincidence value is unique.*

**Proof.** The proof follows from Corollary 13 by taking $\delta$ as a constant function on $\mathbb{R}^+$ with its value in $(0, \min\{1/3, \delta_1, \delta_2\}]$, where $\delta_1 = \inf\{b(u,v) : u,v \in X\}$ and $\delta_2 = \inf\{c(u,v) : u,v \in X\}$. □

**Corollary 15.** *Suppose that $(f,S)$ has property A and that there are a positive integer $N$, nonnegative constants $a_1,\dots,a_N$, and positive constants $b_1,\dots,b_N,c_1,\dots,c_N$ such that $a_i + b_i + 2c_i \le 1$ for all $i = 1,2,\dots,N$ and*

$$d(fx,fy) \le \max\{a_i d(Sx,Sy) + b_i \max\{d(Sx,fx),d(Sy,fy)\} + c_i[d(fx,Sy)+d(Sx,fy)] : i \in \{1,2,\dots,N\}\} \tag{32}$$

*for all $x, y$ in $X$. Then $\{y_n\}$ is Cauchy. Suppose that it converges to an element $z$ of $X$. Then statements (i) to (vi) of Corollary 13 are true here also.*

**Proof.** The proof follows from Corollary 13 by taking $\delta$ as a constant function on $\mathbb{R}^+$ with its value in $(0, \min\{1/3, \delta_1, \delta_2\}]$, where $\delta_1 = \min\{b_k : k = 1,2,\dots,N\}$ and $\delta_2 = \min\{c_k : k = 1,2,\dots,N\}$. □

**Corollary 16** (see uniqueness part of [2, Theorem 2.1]). *Suppose that $(X,d)$ is complete and*

$$d(fx,fy) \le a(\max\{d(x,y),d(x,fx),d(y,fy),[d(fx,y)+d(x,fy)]/2\}) + b(\max\{d(x,fx),d(y,fy)\}) + c[d(fx,y)+d(x,fy)] \tag{33}$$

*for all $x, y$ in $X$ and for some constants $a, b, c$ with $a \ge 0$, $b > 0$, $c > 0$ and $a + b + 2c = 1$. Then $f$ has a unique fixed point in $X$.*

**Proof.** The proof follows from Corollary 15 by taking $S = I$, $N = 3$, $a_1 = a$, $b_1 = b_3 = b$, $c_1 = c_2 = c$, $a_2 = a_3 = 0$, $b_2 = a + b$, $c_3 = (1/2)a + c$. □

**Corollary 17.** *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonically increasing map such that $\varphi(t^+) < t$ for all $t \in (0,\infty)$. Suppose that $(f,S)$ has property A and*

$$d(fx,fy) \le \varphi\left(\max\left\{d(Sx,Sy),d(Sx,fx),d(Sy,fy),\frac{1}{2}[d(fx,Sy)+d(Sx,fy)]\right\}\right) \tag{34}$$

*for all $x,y$ in $X$. Then $\{y_n\}$ is Cauchy. Suppose that it converges to an element $z$ of $X$. Then statements (i) to (vi) of Corollary 13 are true here also.*

**Proof.** Define $\Omega : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ as

$$\Omega(t_1,t_2,t_3,t_4,t_5) = \varphi\left(\max\left\{t_1,t_2,t_3,\frac{1}{2}(t_4+t_5)\right\}\right) \tag{35}$$

for all $t_1,t_2,t_3,t_4,t_5 \in \mathbb{R}^+$. It is clear that $\Omega$ is increasing in each coordinate variable, $\zeta(t) < t$ for all $t \in (0,\infty)$, $\Omega$ has property $A$ and that (3) is true for all $x,y$ in $X$. Clearly, $\Omega(t,t,2t,t,t+s) = \varphi(2t) \le 2t$ if $t \in \mathbb{R}^+$ and $t \le s \le 2t$. Also $\Omega(t,t,t,0,2t) = \varphi(t)$ for all $t \in \mathbb{R}^+$. Hence $\Omega$ satisfies property $B$ with $I_t$ being a singleton, $\beta_i = 1$, $\gamma_i = 0$ $(i \in I_t)$ and $\lambda_t = 2$ for all $t \in \mathbb{R}^+$. Now the corollary is evident from Theorem 12. □

**Remark 18.** Example 19 shows that Corollary 17 cannot be deduced from Corollary 13 and therefore from Corollary 14 also. In particular, it follows that Theorem 12 is a proper generalization of [2, Theorem 2.1].

**Example 19.** Define $f : [0,1] \to [0,1]$ as $f(x) = x/(1+x)$ for all $x \in [0,1]$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\varphi(t) = t/(1+t)$ for all $t \in \mathbb{R}^+$. Then $\varphi$ is a strictly increasing continuous function on $\mathbb{R}^+$, $\varphi(t) < t$ for all $t \in (0,\infty)$ and $|fx - fy| \le \varphi(|x-y|)$ for all $x,y$ in $X$. For $x,y \in [0,1]$ let $\theta(x,y) = \max\{|x-y|, \max\{|x-fx|,|y-fy|\}, (1/2)[|fx-y| + |x-fy|]\}$. Let $\delta : \mathbb{R}^+ \to (0,1/3]$ be a monotonically decreasing function. For $t \in \mathbb{R}^+$ let $I(t) = \{(a,b,c) \in (\mathbb{R}^+)^3 : a \ge 0,\ b \ge \delta(t),\ c \ge \delta(t) \text{ and } a+b+2c \le 1\}$. If possible, suppose that

$$\begin{aligned}|fx - fy| \le \sup\{&a|x-y| + b\max\{|x-fx|,|y-fy|\} \\ &+ c[|fx-y| + |x-fy|] : (a,b,c) \in I(\theta(x,y))\}\end{aligned} \tag{36}$$

for all $x,y$ in [0, 1]. Then, since $I(1) \supseteq I(\theta(x,y))$ for all $x,y$ in [0, 1], we have

$$\begin{aligned}|fx - fy| \le \sup\{&a|x-y| + b\max\{|x-fx|,|y-fy|\} \\ &+ c[|fx-y| + |x-fy|] : (a,b,c) \in I(1)\}\end{aligned} \tag{37}$$

for all $x,y$ in [0, 1]. Hence for $x \in (0,(1/2)\delta(1)]$ and $y = 0$ we have

$$\frac{x}{1+x} \le \sup\left\{ax + b\frac{x^2}{1+x} + c\left[\frac{x}{1+x} + x\right] : (a,b,c) \in I(1)\right\} \tag{38}$$

so that we have

$$\begin{aligned}1 &\le \sup\{a(1+x) + bx + c(2+x) : (a,b,c) \in I(1)\} \\ &= \sup\{a + 2c + x(a+b+c) : (a,b,c) \in I(1)\} \\ &\le \sup\{a + 2c + x : (a,b,c) \in I(1)\} \\ &\le \sup\left\{a + 2c + \frac{1}{2}\delta(1) : (a,b,c) \in I(1)\right\} \\ &\le \sup\{a + b + 2c : (a,b,c) \in I(1)\} - \frac{1}{2}\delta(1) \\ &= 1 - \frac{1}{2}\delta(1) < 1\end{aligned} \tag{39}$$

which is a contradiction.

**Definition 20.** We say that $\Omega$ has property $C$ if there exist (i) a monotonically increasing function $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ with $\sum_{n=1}^{\infty}\varphi^n(t)<+\infty$ for all $t\in\mathbb{R}^+$, and (ii) for each $t\in\mathbb{R}^+$ a nonempty index set $I_t$ and nonnegative real numbers $\beta_i$, $\gamma_i$ $(i\in I_t)$ such that $\sup\{\gamma_i : i\in I_t\}<1$, $\Omega(t,t,2t,t,t+s)\le\sup\{(1+\beta_i)t+\gamma_i s : i\in I_t\}$ for all $s\in[t,2t]$, $\Omega(t,t,t,0,\lambda_t t)\le\varphi(t)$, where $\lambda_t=\sup\{(1+\beta_i)/(1-\gamma_i) : i\in I_t\}$.

**Definition 21.** We say that $(F,S)$ has property $A$ if there is a sequence $\{u_n\}_{n=0}^{\infty}$ in $X$ such that $Su_{n+1}\in Fu_n$ and $d(Su_n,Fu_n)=d(Su_n,Su_{n+1})$ for all $n=0,1,2\ldots$. (Let $v_n$ stand for $Su_{n+1}$.)

**Lemma 22.** *Suppose that $(F,S)$ has property $A$, $\Omega$ has properties $A$ and $C$, and*

$$H(Fx,Fy)\le\Omega\big(d(Sx,Fx),d(Sy,Fy),d(Sx,Sy),d(Sy,Fx),d(Sx,Fy)\big) \tag{40}$$

*for all $x,y$ in $X$. Then $\{v_n\}_{n=1}^{\infty}$ is Cauchy.*

**Proof.** We have $D(u_n,u_{n+1})\le\Omega(d(v_{n-1},v_n),d(v_n,v_{n+1}),d(v_{n-1},v_n),0,d(v_{n-1},v_{n+1}))$ for all $n\in\mathbb{N}$ since $\Omega$ is increasing in each coordinate variable. From (40) we have

$$d(v_n,v_{n+1})=d(Su_{n+1},Fu_{n+1})\le H(Fu_n,Fu_{n+1})\le D(u_n,u_{n+1}) \tag{41}$$

for all $n\in\mathbb{N}$. Now proceeding as in the proof of Lemma 5 it can be seen that $d(v_n,v_{n+1})\le d(v_{n-1},v_n)$ for all $n\in\mathbb{N}$. From (40) we have

$$d(v_n,Fu_{n+2})=d(Su_{n+1},Fu_{n+2})\le H(Fu_n,Fu_{n+2})\le D(u_n,u_{n+2}) \tag{42}$$

for all $n\in\mathbb{N}$. But

$$\begin{aligned}
D(u_n,u_{n+2}) &\le \Omega\big(d(v_{n-1},v_n),d(v_{n+1},v_{n+2}),d(v_{n-1},v_{n+1}),\\
&\qquad d(v_n,v_{n+1}),d(v_{n-1},Fu_{n+2})\big)\\
&\le \Omega\big(d(v_{n-1},v_n),d(v_{n+1},v_{n+2}),d(v_{n-1},v_n)+d(v_n,v_{n+1}),\\
&\qquad d(v_n,v_{n+1}),d(v_{n-1},v_n)+d(v_n,Fu_{n+2})\big)\\
&\le \Omega\big(d(v_{n-1},v_n),d(v_{n-1},v_n),2d(v_{n-1},v_n),d(v_{n-1},v_n),\\
&\qquad d(v_{n-1},v_n)+d(v_n,Fu_{n+2})\big)
\end{aligned} \tag{43}$$

since the sequence $\{d(v_{k-1},v_k)\}$ is monotonically decreasing and $\Omega$ is increasing in each coordinate variable. Since $\Omega$ has property $C$, there exist (i) a monotonically increasing function $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ with $\sum_{n=1}^{\infty}\varphi^n(t)<+\infty$ for all $t\in\mathbb{R}^+$ and (ii) for each $t\in\mathbb{R}^+$ a nonempty index set $I_t$ and nonnegative real numbers $\beta_i$, $\gamma_i$ $(i\in I_t)$ such that $\sup\{\gamma_i : i\in I_t\}<1$, $\Omega(t,t,2t,t,t+s)\le\sup\{(1+\beta_i)t+\gamma_i s : i\in I_t\}$ for all $s\in[t,2t]$, and $\Omega(t,t,t,0,\lambda_t t)\le\varphi(t)$, where $\lambda_t=\sup\{(1+\beta_i)/(1-\gamma_i) : i\in I_t\}$. Since $d(v_n,Fu_{n+2})\le d(v_n,v_{n+2})\le d(v_n,v_{n+1})+d(v_{n+1},v_{n+2})\le 2d(v_{n-1},v_n)$, we have

$$\begin{aligned}
&\Omega\big(d(v_{n-1},v_n),d(v_{n-1},v_n),2d(v_{n-1},v_n),d(v_{n-1},v_n),d(v_{n-1},v_n)+d(v_n,Fu_{n+2})\big)\\
&\quad\le \sup\big\{(1+\beta_i)d(v_{n-1},v_n)+\gamma_i d(v_n,Fu_{n+2}) : i\in I_r\big\}
\end{aligned} \tag{44}$$

provided $d(v_{n-1},v_n) \le d(v_n,Fu_{n+2})$, where $r = d(v_{n-1},v_n)$. Hence from (42), (43), and (44) we have

$$d(v_n,Fu_{n+2}) \le \sup\{(1+\beta_i)d(v_{n-1},v_n)+\gamma_i d(v_n,Fu_{n+2}) : i \in I_r\} \tag{45}$$

provided $d(v_{n-1},v_n) \le d(v_n,Fu_{n+2})$. Now proceeding as in the proof of Lemma 5, it can be shown that

$$d(v_n,Fu_{n+2}) \le \lambda_r d(v_{n-1},v_n) \tag{46}$$

for all $n \in \mathbb{N}$. Hence we have

$$\begin{aligned} D(u_{n+1},u_{n+2}) &= \Omega(d(v_n,v_{n+1}),d(v_{n+1},v_{n+2}),d(v_n,v_{n+1}),0,d(v_n,Fu_{n+2})) \\ &\le \Omega(d(v_{n-1},v_n),d(v_{n-1},v_n),d(v_{n-1},v_n),0,d(v_n,Fu_{n+2})) \\ &\le \Omega(d(v_{n-1},v_n),d(v_{n-1},v_n),d(v_{n-1},v_n),0,\lambda_r d(v_{n-1},v_n)) \\ &\le \varphi(d(v_{n-1},v_n)) \end{aligned} \tag{47}$$

for all $n \in \mathbb{N}$. Hence from (40) we have

$$\begin{aligned} d(v_{n+1},v_{n+2}) &= d(Su_{n+2},Fu_{n+2}) \\ &\le H(Fu_{n+1},Fu_{n+2}) \le D(u_{n+1},u_{n+2}) \le \varphi(d(v_{n-1},v_n)), \end{aligned} \tag{48}$$

that is,

$$d(v_{n+1},v_{n+2}) \le \varphi(d(v_{n-1},v_n)) \tag{49}$$

for all $n \in \mathbb{N}$. Since $\varphi$ is monotonically increasing on $\mathbb{R}^+$, by repeatedly using (49) we obtain

$$d(v_{2n},v_{2n+1}) \le \varphi^n(d(v_0,v_1)), \qquad d(v_{2n+1},v_{2n+2}) \le \varphi^n(d(v_1,v_2)), \tag{50}$$

for all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty}\varphi^n(t) < +\infty$ for all $t \in \mathbb{R}^+$, from (50) it follows that $\sum_{n=1}^{\infty} d(v_n, v_{n+1})$ is convergent. Hence $\{v_n\}_{n=1}^{\infty}$ is Cauchy. □

**Theorem 23.** *Suppose that $(F,S)$ has property A, $\Omega$ has properties A and C and that (40) is true for all $x,y$ in $X$. Then $\{v_n\}$ is Cauchy. Suppose that it converges to an element $z$ of $S(X)$ and $\sigma_2(t) < t$ for all $t \in (0,\infty)$. Then $Sw \in Fw$ for any $w \in X \ni Sw = z$.*

**Proof.** The proof that $\{v_n\}$ is Cauchy follows from Lemma 22. Suppose that it converges to an element $z$ of $S(X)$. Let $w \in X$ be such that $z = Sw$. We have

$$D(w,u_{n+1}) = \Omega(d(z,Fw),d(v_n,v_{n+1}),d(z,v_n),d(v_n,Fw),d(z,Fu_{n+1})) \tag{51}$$

for all $n \in \mathbb{N}$. We note that the limit superior of the sequence $\{D(w,u_{n+1})\}$ is less than or equal to $\Omega(d(z,Fw),0^+,0^+,d(z,Fw)^+,0^+)$ which in turn is less than or equal to $\sigma_2(d(z,Fw))$. From (40) we have

$$d(v_{n+1},Fw) \le H(Fw,Fu_{n+1}) \le D(w,u_{n+1}) \tag{52}$$

for all $n \in \mathbb{N}$. By taking limit superiors on both sides of (52) as $n \to +\infty$ we obtain $d(z,Fw) \le \sigma_2(d(z,Fw))$. Since $\sigma_2(t) < t$ for all $t \in (0,\infty)$, we have $d(z,Fw) = 0$. Since $Fw$ is closed, $z \in Fw$. □

**Corollary 24** (see [2, Theorem 2.2]). *Suppose that $F(x) \in K(X)$ for all $x \in X$, $F(x) \subseteq S(X)$ for all $x \in X$ and that there are nonnegative real-valued functions $a,b,c$ on $X \times X$ such that*

$$H(Fx,Fy) \le a(x,y)d(Sx,Sy) + b(x,y)\max\{d(Sx,Fx),d(Sy,Fy)\} + c(x,y)[d(Sy,Fx)+d(Sx,Fy)] \tag{53}$$

*for all $x,y$ in $X$, $\inf\{b(u,v) : u,v \in X\} > 0$, $\inf\{c(u,v) : u,v \in X\} > 0$, and $\sup\{a(u,v)+b(u,v)+2c(u,v) : u,v \in X\} = 1$. Suppose also that either* (a) *$X$ is $(F,S)$ orbitally complete and $S$ is surjective; or* (b) *$S(X)$ is $(F,S)$ orbitally complete: or* (c) *$F(X)$ is $(F,S)$ orbitally complete. Then $F$ and $S$ have a coincidence point in $X$.*

**Proof.** Let $\delta = \min\{1/3,\delta_1,\delta_2\}$, where $\delta_1 = \inf\{b(u,v) : u,\ v \in X\}$ and $\delta_2 = \inf\{c(u,v) : u,\ v \in X\}$. Define $\Omega : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ as

$$\Omega(t_1,t_2,t_3,t_4,t_5) = \sup\{at_3 + b\max\{t_1,t_2\} + c(t_4+t_5) : a \ge 0,\ b \ge \delta,\ c \ge \delta, \text{ and } a+b+2c \le 1\}. \tag{54}$$

As in the proof of Corollary 13 it can be seen that $\Omega$ has properties $A$ and $C$ and that $\sigma_2(t) < t$ for all $t \in (0,\infty)$. Evidently (40) is true for all $x,y$ in $X$. Since the values of $F$ are compact and $Fx \subseteq S(X)$ for all $x$ in $X$, $(F,S)$ has property $A$. Now the corollary is evident from Theorem 23. □

**Corollary 25.** *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonically increasing map such that $\sum_{n=1}^{\infty}\varphi^n(t) < +\infty$ for all $t \in (0,\infty)$. Suppose that $(F,S)$ has property $A$ and*

$$H(Fx,Fy) \le \varphi\left(\max\left\{d(Sx,Sy),d(Sx,Fx),d(Sy,Fy),\frac{1}{2}[d(Sy,Fx)+d(Sx,Fy)]\right\}\right) \tag{55}$$

*for all $x,y$ in $X$. Then $\{y_n\}$ is Cauchy. Suppose that it converges to an element $z$ of $S(X)$. Then $Sw \in Fw$ for any $w \in X \ni Sw = z$.*

**Proof.** The hypothesis on $\varphi$ ensures that $\varphi(t) < t$ for all $t \in (0,\infty)$. The corollary follows from Theorem 23 by defining $\Omega$ as in the proof of Corollary 17 and noting that $\sigma_2(t) = \varphi(t)$ for all $t \in (0,\infty)$. □

**Remark 26.** Example 27 shows that Corollary 25 cannot be deduced from Corollary 24 and hence Theorem 23 is a proper generalization of [2, Theorem 2.2].

**Example 27.** Let $X = [0,1]$. Define $F : X \to K(X)$ as $F(x) = [0,x/(1+\sqrt{x})^2]$ for all $x \in X$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\varphi(t) = t/(1+\sqrt{t})^2$ for all $t \in \mathbb{R}^+$. Then $\varphi$ is a strictly increasing continuous function on $\mathbb{R}^+$, $\varphi(t) < t$ for all $t \in (0,\infty)$, $\sum_{n=1}^{\infty}\varphi^n(t) < +\infty$ for all $t \in \mathbb{R}^+$ and $H(Fx,Fy) \le \varphi(|x-y|)$ for all $x,y$ in $X$. Let $\delta \in (0,1/3]$ and

$I = \{(a,b,c) \in (\mathbb{R}^+)^3 : a \geq 0,\ b \geq \delta,\ c \geq \delta,$ and $a+b+2c \leq 1\}$. If possible, suppose that

$$H(Fx,Fy) \leq \sup\{a|x-y| + b\max\{d(x,Fx),d(y,Fy)\} + c[d(x,Fy)+d(y,Fx)] : (a,b,c) \in I\} \tag{56}$$

for all $x,y$ in $[0,1]$. Then for $x \in (0,\delta^2/9)$ and $y = 0$ we have

$$\frac{x}{(1+\sqrt{x})^2} \leq \sup\left\{ax + b\left[x - \frac{x}{(1+\sqrt{x})^2}\right] + cx : (a,b,c) \in I\right\} \tag{57}$$

so that we have

$$\begin{aligned} 1 &\leq \sup\{a(1+x+2\sqrt{x}) + b(x+2\sqrt{x}) + c(1+x+2\sqrt{x}) : (a,b,c) \in I\} \\ &= \sup\{a+c+(a+b+c)(x+2\sqrt{x}) : (a,b,c) \in I\} \\ &\leq \sup\{a+c+(a+b+c)(3\sqrt{x}) : (a,b,c) \in I\} \\ &\leq \sup\{a+c+(a+b+c)\delta : (a,b,c) \in I\} \\ &\leq (1-2\delta)+(1-\delta)\delta \\ &= 1-\delta-\delta^2 < 1 \end{aligned} \tag{58}$$

which is a contradiction.

**Remark 28.** Following the proof of [6, Theorem 1] it can be shown that Corollary 25 remains valid if the condition $'(F,S)$ has property $A'$ is replaced with the condition $'Fx \subseteq S(X)$ for all $x$ in $X'$ provided $\varphi$ is subjected to the additional condition $'\varphi(t^+) < t$ for all $t \in (0,\infty)'$. With this modification Corollary 25 is a generalization of [2, Theorem 2.3]. Example 27 shows that the generalization is proper.

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