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DERIVATIONS IN BANACH ALGEBRAS

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We present some conditions which imply that a derivation on a Banach algebra maps the algebra into its Jacobson radical.

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1. Introduction. Throughout this paper, A represents an associative algebra over the complex field $\mathbb{C}$, and the *Jacobson radical* of A and the *center* of A are denoted by $\operatorname{rad}(A)$ and $Z(A)$, respectively. Let I be any closed (2-sided) ideal of the Banach algebra A. Then let Q_I denote the canonical quotient map from A onto A/I. Recall that an algebra A is *prime* if $aAb = \{0\}$ implies that either $a = 0$ or $b = 0$. A mapping $f : A \to A$ is called *commuting* (resp., *centralizing*) if $[f(x), x] = 0$ (resp., $[f(x), x] \in Z(A)$) for all $x \in A$. More generally, for a positive integer n, we define a mapping f to be n-*commuting* (resp., n-*centralizing*) if $[f(x), x^n] = 0$ (resp., $[f(x), x^n] \in Z(A)$) for all $x \in A$. A linear mapping $d : A \to A$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in A$.

The Singer-Wermer theorem, which is a classical theorem of Banach algebra theory, states that every continuous derivation on a commutative Banach algebra maps into its Jacobson radical [9], and Thomas [10] proved that the Singer-Wermer theorem remains true without assuming the continuity of the derivation. (This generalization is called the Singer-Wermer conjecture.) On the other hand, Posner [6] obtained two fundamental results in 1957: (i) the first result (the so-called Posner's first theorem) asserts that if d and g are derivations on a 2-torsion free prime ring such that the product dg is also a derivation, then either $d = 0$ or $g = 0$. (ii) The second result (the so-called Posner's second theorem) states that if d is a centralizing derivation on a noncommutative prime ring, then $d = 0$. As an analytic analogue of Posner's second theorem, Mathieu and Runde [5, Theorem 1] generalized the Singer-Wermer conjecture by proving that every centralizing derivation on a Banach algebra maps into its Jacobson radical. The main objective of this paper is to obtain a generalization (Theorem 2.3) of the above Singer-Wermer conjecture which is inspired by Posner's first theorem.

2. Results. To prove our main result we need the following two lemmas.

Lemma 2.1. *Let d and g be derivations on a noncommutative prime algebra A. If there exist a positive integer n and $\alpha \in \mathbb{C}$ such that $\alpha d^2 + g$ is n-commuting on A, then both $d = 0$ and $g = 0$ on A.*

Proof. For the convenience, we write f instead of αd^2+g. Then the assumption of the lemma can be written in the form

$$[f(x),x^n]=0 \tag{2.1}$$

for all $x\in A$. For $\alpha=0$, the result is obtained from [3, Corollary, page 3713]. Let $\alpha\neq 0$. Substituting $x+\lambda y$ $(\lambda\in\mathbb{C})$ for x in (2.1), we obtain

$$\lambda Q_1(x,y)+\lambda^2Q_2(x,y)+\cdots+\lambda^nQ_n(x,y)=0,\quad x,y\in A, \tag{2.2}$$

where $Q_i(x,y)$ denotes the sum of terms involving i factors of y in the expansion of $[f(x+\lambda y),(x+\lambda y)^n]=0$. Since λ is arbitrary, we have

$$\begin{aligned}Q_1(x,y)&=[f(y),x^n]+[f(x),x^{n-1}y]\\&\quad+[f(x),x^{n-2}yx]+\cdots+[f(x),yx^{n-1}]=0,\quad x,y\in A.\end{aligned} \tag{2.3}$$

Substituting xy for y in (2.3), we get

$$\begin{aligned}0&=x[f(x),x^{n-1}y]+[f(x),x]x^{n-1}y\\&\quad+x[f(x),x^{n-2}yx]+[f(x),x]x^{n-2}yx\\&\quad+\cdots+x[f(x),yx^{n-1}]+[f(x),x]yx^{n-1}\\&\quad+f(x)[y,x^n]+2\alpha[d(x)d(y),x^n]+x[f(y),x^n],\quad x,y\in A;\end{aligned} \tag{2.4}$$

and left multiplying (2.3) by x and subtracting the result from (2.4), we have

$$\begin{aligned}0&=[f(x),x]x^{n-1}y+[f(x),x]x^{n-2}yx+\cdots+[f(x),x]yx^{n-1}\\&\quad+f(x)[y,x^n]+2\alpha[d(x)d(y),x^n],\quad x,y\in A.\end{aligned} \tag{2.5}$$

In (2.5), replace y by yx to obtain

$$\begin{aligned}0&=[f(x),x]x^{n-1}yx+[f(x),x]x^{n-2}yx^2\\&\quad+\cdots+[f(x),x]yx^n+f(x)[y,x^n]x\\&\quad+2\alpha[d(x)d(y),x^n]x+2\alpha[d(x)yd(x),x^n],\quad x,y\in A;\end{aligned} \tag{2.6}$$

and multiply by x on the right in (2.5) to obtain

$$\begin{aligned}0&=[f(x),x]x^{n-1}yx+[f(x),x]x^{n-2}yx^2+\cdots+[f(x),x]yx^n\\&\quad+f(x)[y,x^n]x+2\alpha[d(x)d(y),x^n]x,\quad x,y\in A.\end{aligned} \tag{2.7}$$

We now subtract (2.7) from (2.6) to get

$$d(x)yd(x)x^n-x^nd(x)yd(x)=0,\quad x,y\in A. \tag{2.8}$$

Replacing y by $yd(x)z$ in (2.8), we obtain

$$d(x)yd(x)zd(x)x^n-x^nd(x)yd(x)zd(x)=0,\quad x,y,z\in A. \tag{2.9}$$

According to (2.8), we can write, in relation (2.9), $x^n d(x)zd(x)$ for $d(x)zd(x)x^n$ and $d(x)yd(x)x^n$ instead of $x^n d(x)yd(x)$, which gives

$$d(x)y[d(x),x^n]zd(x) = 0, \quad x,y,z \in A. \tag{2.10}$$

From (2.10) and primeness of A, it follows that, for any $x \in A$ we have either $[d(x),x^n] = 0$ or $d(x) = 0$. In any case $[d(x),x^n] = 0$ for all $x \in A$, which yields $d = 0$ on A by [3, Corollary, page 3713]. Now the initial hypothesis yields that $[g(x),x^n] = 0$, $x \in A$, so $g = 0$ on A, which completes the proof of the lemma. □

Lemma 2.2. *Let d be a derivation on a Banach algebra A and J a primitive ideal of A. If there exists a real constant $K > 0$ such that $\|Q_J d^n\| \le K^n$ for all $n \in \mathbb{N}$, then $d(J) \subseteq J$.*

Proof. See [11, Lemma 1.2]. □

Now we prove our main result.

Theorem 2.3. *Let d and g be derivations on a Banach algebra A. If there exist a positive integer n and $\alpha \in \mathbb{C}$ such that $\alpha d^2 + g$ is n-commuting on A, then both d and g map A into* $\text{rad}(A)$.

Proof. Let J be any primitive ideal of A. Using Zorn's lemma, we find a minimal prime ideal P contained in J, and hence $d(P) \subseteq P$ and $g(P) \subseteq P$ (see [5, Lemma]). Suppose first that P is closed. Then the derivations d and g on A induce the derivations $\bar{d}$ and $\bar{g}$ on the Banach algebra A/P, defined by $\bar{d}(x+P) = d(x)+P$ and $\bar{g}(x+P) = g(x)+P$ $(x \in A)$. In case A/P is commutative, both $\bar{d}(A/P)$ and $\bar{g}(A/P)$ are contained in the Jacobson radical of A/P by [10]. We consider the case when A/P is noncommutative. The assumption that $\alpha d^2 + g$ is n-commuting on A gives that the mapping $\alpha\bar{d}^2 + \bar{g}$ is n-commuting on A/P. Since A/P is a prime algebra, it follows from Lemma 2.1 that both $\bar{d} = 0$ and $\bar{g} = 0$ on A/P. Consequently, we see that both $d(A) \subseteq J$ and $g(A) \subseteq J$. If P is not closed, then we see that $\mathcal{S}(d) \subseteq P$ by [2, Lemma 2.3], where $\mathcal{S}(T)$ is the separating space of a linear operator T. Then we have, by [8, Lemma 1.3], $\mathcal{S}(Q_{\bar{P}}d) = \overline{Q_{\bar{P}}(\mathcal{S}(d))} = \{0\}$ whence $Q_{\bar{P}}d$ is continuous on A. This means that $Q_{\bar{P}}d(\bar{P}) = \{0\}$, that is, $d(\bar{P}) \subseteq \bar{P}$. Hence, we see that d induces a derivation $\tilde{d}$ on the Banach algebra $A/\bar{P}$, defined by $\tilde{d}(x+\bar{P}) = d(x)+\bar{P}$ $(x \in A)$. This shows that we can define a map

$$\Psi\tilde{d}^n Q_{\bar{P}} : A \longrightarrow A/\bar{P} \longrightarrow A/\bar{P} \longrightarrow A/J \tag{2.11}$$

by $\Psi\tilde{d}^n Q_{\bar{P}}(x) = Q_J d^n(x)$ $(x \in A, n \in \mathbb{N})$, where Ψ is the canonical induced map from $A/\bar{P}$ onto A/J (the relation $\bar{P} \subseteq J$ guarantees its existence). The continuity of $\tilde{d}$ is clear from [8, Lemma 1.4], and hence yields that $\|Q_J d^n\| \le \|\tilde{d}\|^n$ for all $n \in \mathbb{N}$. Now, according to Lemma 2.2, we obtain that $d(J) \subseteq J$. Following the same argument with g, we see that $g(J) \subseteq J$. Then the derivations d and g on A induce the derivations $\hat{d}$ and $\hat{g}$ on the Banach algebra A/J, defined by $\hat{d}(x+J) = d(x)+J$ and $\hat{g}(x+J) = g(x)+J$ $(x \in A)$. The rest follows as when P is closed since the primitive algebra A/J is prime. So we also obtain that $d(A) \subseteq J$ and $g(A) \subseteq J$. Since J was arbitrary, we arrive at the conclusion that $d(A) \subseteq \text{rad}(A)$ and $g(A) \subseteq \text{rad}(A)$. □

A mapping $f : A \to A$ is said to be *skew-centralizing* if $\langle f(x), x\rangle \in Z(A)$ for all $x \in A$, where $\langle a, b\rangle$ denotes the Jordan product $ab + ba$.

Corollary 2.4. *Let d and g be derivations on a Banach algebra A. If there exists $\alpha \in \mathbb{C}$ such that $\alpha d^2 + g$ is skew-centralizing on A, then both d and g map A into* $\operatorname{rad}(A)$.

Proof. Since $\langle \alpha d^2(x) + g(x), x\rangle \in Z(A)$ for all $x \in A$, we obtain that $[\langle \alpha d^2(x) + g(x), x\rangle, x] = 0$ for all $x \in A$. From the relation

$$\begin{aligned} 0 &= [\langle \alpha d^2(x) + g(x), x\rangle, x] \\ &= \langle [\alpha d^2(x) + g(x), x], x\rangle \\ &= [\alpha d^2(x) + g(x), x^2], \end{aligned} \tag{2.12}$$

we see that $\alpha d^2 + g$ is 2-commuting, and hence Theorem 2.3 guarantees the conclusion. □

As a noncommutative version of the Singer-Wermer theorem, we also obtain the next result by using Lemma 2.1.

Theorem 2.5. *Let d and g be continuous derivations on a Banach algebra A. If there exist a positive integer n and $\alpha \in \mathbb{C}$ such that the mapping $\alpha d^2 + g$ is n-centralizing on A, then both d and g map A into* $\operatorname{rad}(A)$.

Proof. Given any primitive ideal J of A, we have $d(J) \subseteq J$ and $g(J) \subseteq J$ by [7, Theorem 2.2]. Thus we can suppose that A is primitive. From $[\alpha d^2(x) + g(x), x^n] \in Z(A)$ for all $x \in A$, we obtain $[[\alpha d^2(x) + g(x), x^n], x^n] = 0$, and hence $[\alpha d^2(x) + g(x), x^n]$ is quasinilpotent by the Kleinecke-Shirokov theorem [1, Proposition 18.13]. Since $Z(A)$ is trivial, it follows that $[\alpha d^2(x) + g(x), x^n]$ is a scalar multiple of 1, and so $[\alpha d^2(x) + g(x), x^n] = 0$ for all $x \in A$. Note that a commutative primitive Banach algebra is isomorphic to the complex field $\mathbb{C}$. Hence we also can assume that A is noncommutative. Now, the primeness of A and Lemma 2.1 allows that both $d = 0$ and $g = 0$ on A, which gives the result. □

We do not know whether Theorem 2.5 can be proved without the continuity assumption. However, in the special case when the Banach algebra is semisimple, we obtain the following result.

Corollary 2.6. *Let d and g be derivations on a semisimple Banach algebra A. If there exist a positive integer n and $\alpha \in \mathbb{C}$ such that $\alpha d^2 + g$ is n-centralizing on A, then both $d = 0$ and $g = 0$ on A.*

Proof. The fact that every derivation on a semisimple Banach algebra is continuous [4, Remark 4.3] guarantees the conclusion. □

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