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FAMILIES OF $(1,2)$-SYMPLECTIC METRICS ON FULL FLAG MANIFOLDS

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We obtain new families of $(1,2)$-symplectic invariant metrics on the full complex flag manifolds $F(n)$. For $n \geq 5$, we characterize $n-3$ different n-dimensional families of $(1,2)$-symplectic invariant metrics on $F(n)$. Each of these families corresponds to a different class of nonintegrable invariant almost complex structures on $F(n)$.

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1. Introduction. Mo and Negreiros [13], by using moving frames and tournaments, showed explicitly the existence of an n-dimensional family of invariant $(1,2)$-symplectic metrics on $F(n) = U(n)/(U(1)\times\cdots\times U(1))$. This family corresponds to the family of the parabolic almost complex structures on $F(n)$. In this paper, we study the existence of other families of invariant $(1,2)$-symplectic metrics corresponding to classes of nonintegrable invariant almost complex structures on $F(n)$, different to the parabolic one.

Eells and Sampson [10] proved that if $\phi\colon M \to N$ is a holomorphic map between Kähler manifolds then ϕ is harmonic. This result was generalized by Lichnerowicz (see [12] or [20]) as follows: let (M,g,J_1) and (N,h,J_2) be almost Hermitian manifolds with M cosymplectic and N $(1,2)$-symplectic, then any $\pm$-holomorphic map $\phi\colon (M,g,J_1)\to(N,h,J_2)$ is harmonic.

If we want to obtain harmonic maps, $\phi\colon M^2 \to F(n)$, from a closed Riemann surface M^2 to a full flag manifold $F(n)$ by the Lichnerowicz theorem, we must study $(1,2)$-symplectic metrics on $F(n)$ because a Riemann surface is a Kähler manifold and we know that a Kähler manifold is a cosymplectic manifold (see [11] or [20]).

To study the invariant Hermitian geometry of $F(n)$ it is natural to begin by studying its invariant almost complex structures. Borel and Hirzebruch [5] proved that there are $2^{\binom{n}{2}}$ $U(n)$-invariant almost complex structures on $F(n)$. This number is the same number of tournaments with n players or nodes. A tournament is a digraph in which any two nodes are joined by exactly one oriented edge (see [6] or [15]). There is a natural identification between almost complex structures on $F(n)$ and tournaments with n players (see [6] or [14]).

Tournaments can be classified in isomorphism classes. In this classification, one of these classes corresponds to the integrable structures and the other ones correspond to nonintegrable structures. Burstall and Salamon [6] proved that an almost complex structure J on $F(n)$ is integrable if and only if the tournament associated to J is isomorphic to the canonical tournament (the canonical tournament with n players, $\{1,2,\dots,n\}$, is defined by $i \to j$ if and only if $i<j$).

Borel proved the existence of an $(n-1)$-dimensional family of invariant Kähler metrics on $F(n)$ for each invariant complex structure on $F(n)$ (see [2] or [4]). Eells and Salamon [8] proved that any parabolic structure on $F(n)$ admits a $(1,2)$-symplectic metric. Mo and Negreiros [13] showed explicitly that there is an n-dimensional family of invariant $(1,2)$-symplectic metrics for each parabolic structure on $F(n)$.

In this paper, we characterize new n-parametric families of $(1,2)$-symplectic invariant metrics on $F(n)$, different to the Kähler and parabolic ones. More precisely, we obtain explicitly $n-3$ different n-dimensional families of $(1,2)$-symplectic invariant metrics, for each $n \geq 5$. Each of them corresponds to a different class of nonintegrable invariant almost complex structure on $F(n)$. These metrics are used to produce new examples of harmonic maps $\phi : M^2 \to F(n)$, using the previous result by Lichnerowicz.

2. Preliminaries. A full flag manifold is defined by

$$F(n) = \{(L_1,\dots,L_n) : L_i \text{ is a subspace of } \mathbb{C}^n, \dim_{\mathbb{C}} L_i = 1, L_i \perp L_j\}. \tag{2.1}$$

The unitary group $U(n)$ acts transitively on $F(n)$. Using this action we obtain an algebraic description for $F(n)$

$$F(n) = \frac{U(n)}{T}, \tag{2.2}$$

where $T = \underbrace{U(1)\times\cdots\times U(1)}_{n \text{ times}}$ is a maximal torus in $U(n)$.

Let $\mathfrak{p}$ be the tangent space to $F(n)$ at the point (T). An invariant almost complex structure on $F(n)$ is an $\mathrm{ad}(\mathfrak{u}(1)\oplus\cdots\oplus\mathfrak{u}(1))$-invariant linear map $J : \mathfrak{p} \to \mathfrak{p}$ such that $J^2 = -I$.

A tournament (n-tournament) $\mathcal{T}$, consists of a finite set $T = \{p_1, p_2, \dots, p_n\}$ of n players, together with a dominance relation, $\to$, that assigns to every pair of players a winner, that is, $p_i \to p_j$ or $p_j \to p_i$. If $p_i \to p_j$, then we say that p_i beats p_j. A tournament $\mathcal{T}$ may be represented by a directed graph in which T is the set of vertices and any two vertices are joined by an oriented edge.

Let $\mathcal{T}_1$ be a tournament with n players $\{1,\dots,n\}$ and $\mathcal{T}_2$ another tournament with m players $\{1,\dots,m\}$. A homomorphism between $\mathcal{T}_1$ and $\mathcal{T}_2$ is a mapping $\phi : \{1,\dots,n\} \to \{1,\dots,m\}$ such that

$$s \xrightarrow{\mathcal{T}_1} t \Longrightarrow \phi(s) \xrightarrow{\mathcal{T}_2} \phi(t) \qquad \text{or} \qquad \phi(s) = \phi(t). \tag{2.3}$$

When ϕ is bijective we said that $\mathcal{T}_1$ and $\mathcal{T}_2$ are isomorphic.

An n-tournament determines a score vector $(s_1,\dots,s_n)$, such that $\sum_{i=1}^n s_i = \binom{n}{2}$, whose components equal the number of games won by each player. Isomorphic tournaments have identical score vectors. Figure 2.1 shows the isomorphism classes of n-tournaments for $n = 2,3,4$, together with their score vectors. This figure was taken from Moon's book [15]. In Moon's notation not all of the arcs are included in the drawings. If an arc joining two nodes has not been drawn, then it is to be understood that the arc is oriented from the higher node to the lower node.

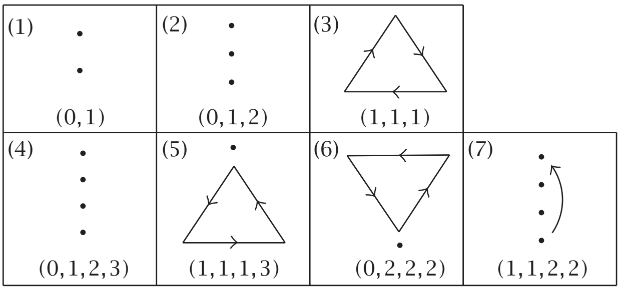


FIGURE 2.1 Isomorphism classes of n-tournaments for $n = 2,3,4$.

The canonical n-tournament $\mathcal{T}_n$ is defined by setting $i \to j$ if and only if $i < j$. Up to isomorphism, $\mathcal{T}_n$ is the unique n-tournament satisfying the following equivalent conditions:

- the dominance relation is transitive, that is, if $i \to j$ and $j \to k$ then $i \to k$,
- there are no 3-cycles, that is, closed paths $i_1 \to i_2 \to i_3 \to i_1$, see [15],
- the score vector is $(0,1,2,\dots,n-1)$.

For each invariant almost complex structure J on $F(n)$, we can associate an n-tournament $\mathcal{T}(J)$ in the following way: if $J(a_{ij}) = (a'_{ij})$, then $\mathcal{T}(J)$ is such that for $i < j$

$$\left(i \longrightarrow j \Longleftrightarrow a'_{ij} = \sqrt{-1}a_{ij}\right) \quad \text{or} \quad \left(i \longleftarrow j \Longleftrightarrow a'_{ij} = -\sqrt{-1}a_{ij}\right) \tag{2.4}$$

(see [14]).

An almost complex structure J on $F(n)$ is said to be integrable if $F(n)$ is a complex manifold, that is, $F(n)$ admits complex coordinate systems with holomorphic coordinate changes. Burstall and Salamon [6] proved the following result.

THEOREM 2.1. *An almost complex structure J on $F(n)$ is integrable if and only if $\mathcal{T}(J)$ is isomorphic to the canonical tournament $\mathcal{T}_n$.*

Thus, if $\mathcal{T}(J)$ contains a 3-cycle then J is not integrable. Classes (2) and (4) in Figure 2.1 correspond to the integrable almost complex structures on $F(3)$ and $F(4)$, respectively.

An invariant almost complex structure J on $F(n)$ is called parabolic if there is a permutation τ of n elements such that the associated tournament $\mathcal{T}(J)$ is given, for $i < j$, by

$$(\tau(j) \longrightarrow \tau(i), \text{ if } j-i \text{ is even}) \quad \text{or} \quad (\tau(i) \longrightarrow \tau(j), \text{ if } j-i \text{ is odd}). \tag{2.5}$$

Classes (3) and (7) in Figure 2.1 represent the parabolic structures on $F(3)$ and $F(4)$, respectively.

An n-tournament $\mathcal{T}$, for $n \geq 3$, is called irreducible or Hamiltonian if it contains an n-cycle, that is, a path $\pi(n) \to \pi(1) \to \pi(2) \to \cdots \to \pi(n-1) \to \pi(n)$, where π is a permutation of n elements.

An n-tournament $\mathcal{T}$ is transitive if, given three nodes i, j, k of $\mathcal{T}$, then $i \to j$ and $j \to k \Rightarrow i \to k$. The canonical tournament is the only transitive tournament up to isomorphisms.

We consider $\mathbb{C}^n$ equipped with the standard Hermitian inner product, that is, for $V = (v_1,\dots,v_n)$ and $W = (w_1,\dots,w_n)$ in $\mathbb{C}^n$, we have $\langle V,W\rangle = \sum_{i=1}^n v_i\overline{w_i}$. We use the convention $v_{\bar{i}} = \overline{v_i}$ and $f_{\bar{i}j} = \overline{f_{i\bar{j}}}$.

A frame consists of an ordered set of n vectors $(Z_1,\dots,Z_n)$, such that $Z_1\wedge\cdots\wedge Z_n \neq 0$, and it is called unitary if $\langle Z_i,Z_j\rangle = \delta_{i\bar{j}}$. The set of unitary frames can be identified with the unitary group $U(n)$.

If we write $dZ_i = \sum_j \omega_{i\bar{j}}Z_j$, the coefficients $\omega_{i\bar{j}}$ are the Maurer-Cartan forms of the unitary group $U(n)$. They are skew-Hermitian, that is, $\omega_{i\bar{j}}+\omega_{\bar{j}i} = 0$. For more details see [7].

We may define all left-invariant metrics on $(F(n),J)$ by (see [3] or [17])

$$ds^2_\Lambda = \sum_{i,j}\lambda_{ij}\omega_{i\bar{j}}\otimes\omega_{\bar{i}j}, \tag{2.6}$$

where $\Lambda = (\lambda_{ij})$ is a symmetric real matrix such that

$$\lambda_{ij} > 0, \quad \text{if } i \neq j, \qquad \lambda_{ij} = 0, \quad \text{if } i = j \tag{2.7}$$

and the Maurer-Cartan forms $\omega_{i\bar{j}}$ are such that

$$\omega_{i\bar{j}} \in \mathbb{C}^{1,0}((1,0) \text{ type forms}) \Longleftrightarrow i \xrightarrow{\mathcal{T}(J)} j. \tag{2.8}$$

The metrics (2.6) are called of Borel type and they are almost Hermitian for every invariant almost complex structure J, that is, $ds^2_\Lambda(JX,JY) = ds^2_\Lambda(X,Y)$ for all tangent vectors X, Y. When J is integrable, ds^2_Λ is said to be Hermitian.

Let J be an invariant almost complex structure on $F(n)$, $\mathcal{T}(J)$ the associated tournament, and ds^2_Λ an invariant metric. The Kähler form with respect to J and ds^2_Λ is defined by

$$\Omega(X,Y) = ds^2_\Lambda(X,JY), \tag{2.9}$$

for any tangent vectors X, Y. For each permutation τ of n elements, the Kähler form can be written in the following way (see [13]):

$$\Omega = -2\sqrt{-1}\sum_{i<j}\mu_{\tau(i)\tau(j)}\omega_{\tau(i)\overline{\tau(j)}}\wedge\omega_{\overline{\tau(i)}\tau(j)}, \tag{2.10}$$

where

$$\mu_{\tau(i)\tau(j)} = \varepsilon_{\tau(i)\tau(j)}\lambda_{\tau(i)\tau(j)}, \qquad \varepsilon_{ij} = \begin{cases} 1 & \text{if } i \longrightarrow j, \\ -1 & \text{if } j \longrightarrow i, \\ 0 & \text{if } i = j. \end{cases} \tag{2.11}$$

Let J be an invariant almost complex structure on $F(n)$. Then $F(n)$ is said to be almost Kähler if and only if Ω is closed, that is, $d\Omega = 0$. If J is integrable and Ω is closed, then $F(n)$ is said to be a Kähler manifold.

Mo and Negreiros proved in [13] that

$$d\Omega = 4\sum_{i<j<k}C_{\tau(i)\tau(j)\tau(k)}\Psi_{\tau(i)\tau(j)\tau(k)}, \tag{2.12}$$

where

$$C_{ijk} = \mu_{ij} - \mu_{ik} + \mu_{jk}, \qquad \Psi_{ijk} = \operatorname{Im}\left(\omega_{i\bar{j}} \wedge \omega_{\bar{i}k} \wedge \omega_{j\bar{k}}\right). \tag{2.13}$$

We denote by $\mathbb{C}^{p,q}$ the space of complex forms with degree (p,q) on $F(n)$. Then, for any i, j, k, we have either $\Psi_{ijk} \in \mathbb{C}^{0,3} \oplus \mathbb{C}^{3,0}$ or $\Psi_{ijk} \in \mathbb{C}^{1,2} \oplus \mathbb{C}^{2,1}$. An invariant almost Hermitian metric ds^2_Λ is said to be $(1,2)$-symplectic if and only if $(d\Omega)^{1,2} = 0$. If $d^*\Omega = 0$ then the metric is said to be cosymplectic.

The following result due to Mo and Negreiros [13] is very useful to study $(1,2)$-symplectic metrics on $F(n)$.

Theorem 2.2. *If J is a $U(n)$-invariant almost complex structure on $F(n)$, $n \geq 4$, such that $\mathcal{T}(J)$ contains one of the 4-tournaments (5) or (6) in Figure 2.1; then J does not admit any invariant $(1,2)$-symplectic metric.*

3. Main theorem. It is known that on $F(3)$ there are a 2-parametric family of Kähler metrics and a 3-parametric family of $(1,2)$-symplectic metrics corresponding to the nonintegrable almost complex structures class (the parabolic class). Then, each invariant almost complex structure on $F(3)$ admits a $(1,2)$-symplectic metric (see [4, 8]). Barros and Urbano in [1] considered a family of almost Hermitian structures on $F(3)$.

On $F(4)$, there are four isomorphism classes of 4-tournaments or equivalently almost complex structures. Theorem 2.2 shows that two of them do not admit any $(1,2)$-symplectic metric. The other two classes correspond to the Kähler and parabolic cases. $F(4)$ has a 3-parametric family of Kähler metrics and a 4-parametric family of $(1,2)$-symplectic metrics which are not Kähler (see [13]).

On $F(5)$, $F(6)$, and $F(7)$ we have the following families of $(1,2)$-symplectic invariant metrics, different to the Kähler and parabolic ones: on $F(5)$, two 5-parametric families; on $F(6)$, four 6-parametric families, two of them generalizing the two families on $F(5)$ and, on $F(7)$ there are eight 7-parametric families, four of them generalizing the four ones on $F(6)$ (see [19] or [18]).

In this paper we prove the following result.

Theorem 3.1. *Let J be an invariant almost complex structure on $F(n)$ such that the associated tournament $\mathcal{T}(J)$ is one of the tournaments in Figure 3.1. An invariant metric $ds^2_{\Lambda^k}$ is $(1,2)$-symplectic with respect to J if and only if the matrix $\Lambda^k = (\lambda_{ij})$ satisfies*

$$\lambda_{ij} = \lambda_{i(i+1)} + \lambda_{(i+1)(i+2)} + \cdots + \lambda_{(j-1)j}, \tag{3.1}$$

for $i = 1, \ldots, n-1$ and $j = 2, \ldots, n$, except for $\lambda_{1n}, \lambda_{2n}, \ldots, \lambda_{kn}$, which satisfy the following relations:

$$\begin{aligned} \lambda_{2n} &= \lambda_{12} + \lambda_{1n}, \\ \lambda_{3n} &= \lambda_{12} + \lambda_{23} + \lambda_{1n}, \\ &\vdots \\ \lambda_{kn} &= \lambda_{12} + \lambda_{23} + \cdots + \lambda_{(k-1)k} + \lambda_{1n}. \end{aligned} \tag{3.2}$$

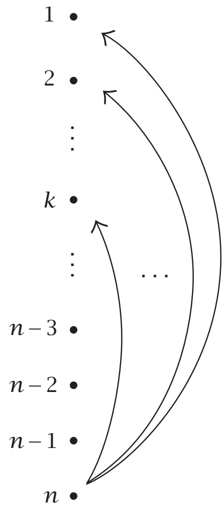


FIGURE 3.1 Tournaments in Theorem 3.1.

This theorem provides an n-family of $(1,2)$-symplectic metrics on $F(n)$, for each $1 \le k \le n-3$. These families are different to the family described by Mo and Negreiros in [13] and correspond to nonintegrable almost complex structures. All of the studied families are n-parametric.

None of these families contains the normal metric. This fact is in accordance with the result in [21] proved by Wolf and Gray, that the normal metric on $F(n)$ is $(1,2)$-symplectic if and only if $n \le 3$.

The score vector of these families can be written as

$$(1,2,\dots,k,k,\dots,n-k-1,n-k-1,\dots,n-3,n-2), \tag{3.3}$$

for $n \ge 2k+1$.

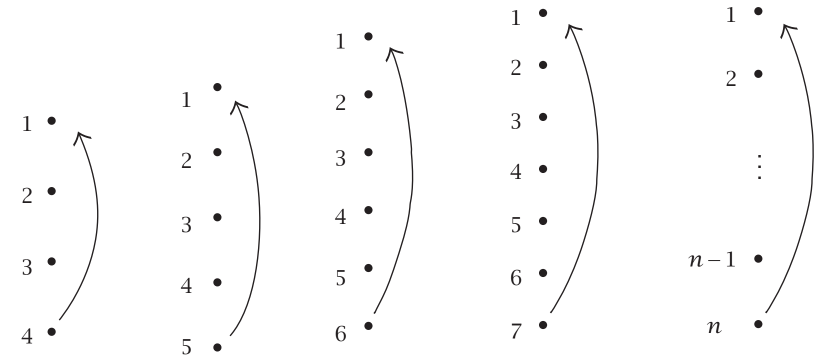
In order to prove this theorem we prove, in the following section, some preliminary results.

4. The families for $k = 1,2,3,4$

PROPOSITION 4.1. *Let J be an invariant almost complex structure on $F(n)$, $n \ge 4$, such that the associated tournament $\mathcal{T}(J)$ is the last tournament in Figure 4.1. An invariant metric ds^2_Λ is $(1,2)$-symplectic with respect to J if and only if the matrix $\Lambda = (\lambda_{ij})$ satisfies*

$$\lambda_{ik} = \lambda_{i(i+1)} + \lambda_{(i+1)(i+2)} + \cdots + \lambda_{(k-1)k}, \tag{4.1}$$

for $i = 1,\dots,n-1$ and $k = 2,\dots,n$, except for λ_{1n}.



(1,1,2,2) (1,1,2,3,3) (1,1,2,3,4,4) (1,1,2,3,4,5,5)

FIGURE 4.1 Tournaments of the family for $k = 1$.

PROOF. The proof will follow using induction over n. First, we prove the result for $n = 4$. In this case, the tournament $\mathcal{T}(J)$ is isomorphic to the first tournament in Figure 4.1. From (2.12) we obtain

$$\begin{aligned} d\Omega &= C_{123}\Psi_{123} + C_{124}\Psi_{124} + C_{134}\Psi_{134} + C_{234}\Psi_{234} \\ &= (\lambda_{12} - \lambda_{13} + \lambda_{23})\Psi_{123} + (\lambda_{12} + \lambda_{14} + \lambda_{24})\Psi_{124} \\ &\quad + (\lambda_{13} + \lambda_{14} + \lambda_{34})\Psi_{134} + (\lambda_{23} - \lambda_{24} + \lambda_{34})\Psi_{234} \end{aligned} \tag{4.2}$$

and $d\Omega^{(1,2)} = (\lambda_{12} - \lambda_{13} + \lambda_{23})\Psi_{123} + (\lambda_{23} - \lambda_{24} + \lambda_{34})\Psi_{234}$. Then ds^2_Λ is $(1,2)$-symplectic if and only if

$$\begin{aligned} \lambda_{12} - \lambda_{13} + \lambda_{23} = 0 &\iff \lambda_{13} = \lambda_{12} + \lambda_{23}, \\ \lambda_{23} - \lambda_{24} + \lambda_{34} = 0 &\iff \lambda_{24} = \lambda_{23} + \lambda_{34}. \end{aligned} \tag{4.3}$$

Suppose that the result is true to $n-1$. For n we must consider two cases:
(a) $i < j < k$, $i \neq 1$, or $k \neq n$. Then $\varepsilon_{ij} = \varepsilon_{ik} = \varepsilon_{jk} = 1$, and $C_{ijk} = \lambda_{ij} - \lambda_{ik} + \lambda_{jk} \neq 0$.
(b) $1 < j < n$. Then $\varepsilon_{1j} = \varepsilon_{jn} = 1$, $\varepsilon_{1n} = -1$, and $C_{1jn} = \lambda_{1j} + \lambda_{1n} + \lambda_{jn} \neq 0$.

$$\begin{aligned} &(\text{a}) \Longrightarrow (d\Omega)^{2,1} + (d\Omega)^{1,2} = \sum_{i<j<k} C_{ijk}\Psi_{ijk}, \quad i \neq 1,\ k \neq n. \\ &(\text{b}) \Longrightarrow (d\Omega)^{3,0} + (d\Omega)^{0,3} = \sum_{j=2}^{n-1} C_{1jn}\Psi_{1jn} \neq 0. \end{aligned} \tag{4.4}$$

Then ds^2_Λ is $(1,2)$-symplectic if and only if $\Lambda = (\lambda_{ij})$ satisfies the linear system

$$\begin{aligned} \lambda_{12} - \lambda_{13} + \lambda_{23} &= 0, \\ \lambda_{12} - \lambda_{14} + \lambda_{24} &= 0, \\ &\vdots \\ \lambda_{12} - \lambda_{1(n-1)} + \lambda_{2(n-1)} &= 0, \end{aligned}$$

$$
\begin{aligned}
\lambda_{13}-\lambda_{14}+\lambda_{34}&=0,\\
&\vdots\\
\lambda_{13}-\lambda_{1(n-1)}+\lambda_{3(n-1)}&=0,\\
\lambda_{14}-\lambda_{15}+\lambda_{45}&=0,\\
&\vdots\\
\lambda_{1(n-2)}-\lambda_{1(n-1)}+\lambda_{(n-2)(n-1)}&=0,\\
\lambda_{23}-\lambda_{24}+\lambda_{34}&=0,\\
&\vdots\\
\lambda_{23}-\lambda_{2n}+\lambda_{3n}&=0,\\
&\vdots\\
\lambda_{(n-3)(n-2)}-\lambda_{(n-3)n}+\lambda_{(n-2)n}&=0,\\
\lambda_{(n-2)(n-1)}-\lambda_{(n-2)n}+\lambda_{(n-1)n}&=0.
\end{aligned}
\tag{4.5}
$$

This system contains all of the equations corresponding to the system for $n-1$. Then all the elements of Λ for $n-1$ are equal to the matrix for n, except $\lambda_{1(n-1)}$. Using the system above we see how to write $\lambda_{1(n-1)},\lambda_{2n},\lambda_{3n},\dots,\lambda_{(n-2)n}$:

$$
\begin{aligned}
\lambda_{12}-\lambda_{1(n-1)}+\lambda_{2(n-1)}=0 &\Longrightarrow \lambda_{1(n-1)}=\lambda_{12}+\lambda_{2(n-1)}\\
&\Longrightarrow \lambda_{1(n-1)}=\lambda_{12}+\lambda_{23}+\cdots+\lambda_{(n-2)(n-1)},\\
\lambda_{(n-2)(n-1)}-\lambda_{(n-2)n}+\lambda_{(n-1)n}=0 &\Longrightarrow \lambda_{(n-2)n}=\lambda_{(n-2)(n-1)}+\lambda_{(n-1)n},\\
\lambda_{(n-3)(n-2)}-\lambda_{(n-3)n}+\lambda_{(n-2)n}=0 &\Longrightarrow \lambda_{(n-3)n}=\lambda_{(n-3)(n-2)}+\lambda_{(n-2)n}\\
&\Longrightarrow \lambda_{(n-3)n}=\lambda_{(n-3)(n-2)}+\lambda_{(n-2)(n-1)}+\lambda_{(n-1)n},\\
&\vdots\\
\lambda_{23}-\lambda_{2n}+\lambda_{3n}=0 &\Longrightarrow \lambda_{2n}=\lambda_{23}+\lambda_{3n}\\
&\Longrightarrow \lambda_{2n}=\lambda_{23}+\lambda_{34}+\cdots+\lambda_{(n-1)n}.
\end{aligned}
\tag{4.6}
$$

□

In order to use induction to prove Theorem 3.1 we denote the symmetric matrix Λ for this family by Λ^1. Then,

$$
\Lambda^1=\begin{pmatrix}
0 & \lambda_{12} & \lambda_{12}+\lambda_{23} & \cdots & \lambda_{12}+\cdots+\lambda_{(n-2)(n-1)} & \lambda_{1n}\\
\lambda_{12} & 0 & \lambda_{23} & \cdots & \lambda_{23}+\cdots+\lambda_{(n-2)(n-1)} & \lambda_{23}+\cdots+\lambda_{(n-1)n}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
* & * & * & \cdots & \lambda_{(n-2)(n-1)} & \lambda_{(n-2)(n-1)}+\lambda_{(n-1)n}\\
* & * & * & \cdots & 0 & \lambda_{(n-1)n}\\
* & * & * & \cdots & \lambda_{(n-1)n} & 0
\end{pmatrix}.
\tag{4.7}
$$

For $F(4)$, this family is the same as the family obtained by Mo and Negreiros [13], because the corresponding 4-tournament is the parabolic one. Any tournament of this family in $F(n)$, $n \geq 4$, is irreducible and such that each of its 4-subtournaments are transitive, class (4) in Figure 2.1, or irreducible, class (7) in Figure 2.1.

The following propositions are presented without proof. They are proved in a similar way as Proposition 4.1.

Proposition 4.2. *Let J be an invariant almost complex structure on $F(n)$, $n \geq 5$, such that the associated tournament $\mathcal{T}(J)$ is the tournament (1) in Figure 4.2. An invariant metric ds^2_Λ is $(1,2)$-symplectic with respect to J if and only if the matrix $\Lambda = (\lambda_{ij})$ satisfies*

$$\lambda_{ik} = \lambda_{i(i+1)} + \lambda_{(i+1)(i+2)} + \cdots + \lambda_{(k-1)k}, \tag{4.8}$$

for $i = 1,\dots,n-1$ and $k = 2,\dots,n$, except for λ_{1n} and λ_{2n}, which satisfy $\lambda_{2n} = \lambda_{12} + \lambda_{1n}$.

In this case, the corresponding symmetric matrix Λ^2 is

$$\Lambda^2 = \begin{pmatrix} 0 & \lambda_{12} & \lambda_{12}+\lambda_{23} & \cdots & \lambda_{12}+\cdots+\lambda_{(n-2)(n-1)} & \lambda_{1n} \\ \lambda_{12} & 0 & \lambda_{23} & \cdots & \lambda_{23}+\cdots+\lambda_{(n-2)(n-1)} & \lambda_{12}+\lambda_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & \lambda_{(n-2)(n-1)} & \lambda_{(n-2)(n-1)}+\lambda_{(n-1)n} \\ * & * & * & \cdots & 0 & \lambda_{(n-1)n} \\ * & * & * & \cdots & \lambda_{(n-1)n} & 0 \end{pmatrix}. \tag{4.9}$$

Proposition 4.3. *Let J be an invariant almost complex structure on $F(n)$, $n \geq 6$, such that the associated tournament $\mathcal{T}(J)$ is the tournament (2) in Figure 4.2. An invariant metric ds^2_Λ is $(1,2)$-symplectic with respect to J if and only if the matrix $\Lambda = (\lambda_{ij})$ satisfies*

$$\lambda_{ik} = \lambda_{i(i+1)} + \lambda_{(i+1)(i+2)} + \cdots + \lambda_{(k-1)k}, \tag{4.10}$$

for $i = 1,\dots,n-1$ and $k = 2,\dots,n$, except for λ_{1n}, λ_{2n}, and λ_{3n}, which satisfy $\lambda_{2n} = \lambda_{12} + \lambda_{1n}$ and $\lambda_{3n} = \lambda_{12} + \lambda_{23} + \lambda_{1n}$.

Proposition 4.4. *Let J be an invariant almost complex structure on $F(n)$, $n \geq 7$, such that the associated tournament $\mathcal{T}(J)$ is the tournament (3) in Figure 4.2. An invariant metric ds^2_Λ is $(1,2)$-symplectic with respect to J if and only if the matrix $\Lambda = (\lambda_{ij})$ satisfies*

$$\lambda_{ik} = \lambda_{i(i+1)} + \lambda_{(i+1)(i+2)} + \cdots + \lambda_{(k-1)k}, \tag{4.11}$$

for $i = 1,\dots,n-1$ and $k = 2,\dots,n$, except for λ_{1n}, λ_{2n}, λ_{3n} and λ_{4n}, which satisfy $\lambda_{2n} = \lambda_{12} + \lambda_{1n}$, $\lambda_{3n} = \lambda_{12} + \lambda_{23} + \lambda_{1n}$, and $\lambda_{4n} = \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{1n}$.

Any tournament of these families is irreducible and such that any 4-subtournament of it is transitive, class (4) in Figure 2.1, or irreducible, class (7) in Figure 2.1.

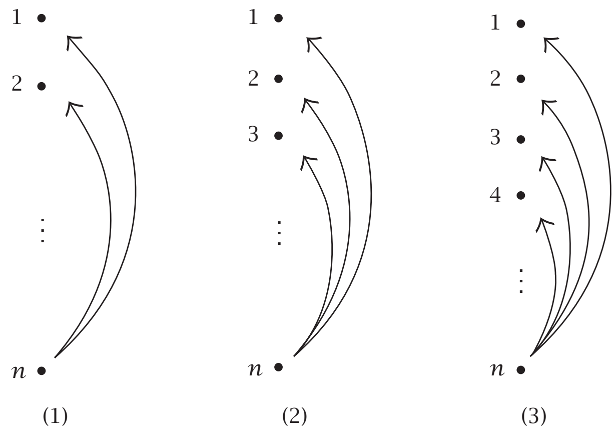


FIGURE 4.2 Tournaments in Propositions 4.2, 4.3, and 4.4.

5. Proof of the main theorem. We use induction over n, beginning with $n = 4$. Proposition 4.1 shows that the result is true for $n = 4$. Suppose that the result is true for $n-1$.

We need to calculate the coefficients C_{ijk} in (2.12). Then, we have three types of 3-subtournaments of $\mathcal{T}(J)$ to consider:

(a) for the 3-cycles we have that

$$C_{ijn} = \lambda_{ij} + \lambda_{in} + \lambda_{jn} \neq 0, \tag{5.1}$$

for $k < j < n$ and $i = 1,\dots,k$. It implies that $(d\Omega)^{3,0} \neq 0$;

(b) for the 3-subtournaments, (ijn), such that $i < j \leq k$ and $i = 1,2,\dots,k-1$, we have that

$$C_{ijn} = \lambda_{ij} + \lambda_{in} - \lambda_{jn}; \tag{5.2}$$

(c) for the 3-subtournaments which neither satisfy (a) nor (b), we have that

$$C_{ijl} = \lambda_{ij} - \lambda_{il} + \lambda_{jl}, \quad i < j < l. \tag{5.3}$$

(b) and (c) give us the information to calculate $(d\Omega)^{1,2}$. Then, the metric ds^2_Λ is $(1,2)$-symplectic if and only if the matrix $\Lambda = (\lambda_{ij})$ satisfies

(d)

$$\lambda_{ij} + \lambda_{in} - \lambda_{jn} = 0; \quad i < j \leq k,\ i = 1,2,\dots,k-1, \tag{5.4}$$

(e)

$$\lambda_{ij} - \lambda_{il} + \lambda_{jl} = 0; \quad i < j < l, \text{ do not satisfy (a) and (b).} \tag{5.5}$$

(d) and (e) include all of the equations corresponding to the case for $n-1$, except the equations given by the following 3-subtournaments

$$(ij(n-1)), \quad \text{with } i = 1,\dots,k-1,\ j = 2,\dots k,\ i < j. \tag{5.6}$$

Therefore, by the hypothesis of induction, all the elements of the matrix Λ^k corresponding to $n-1$ are equal to the matrix for n, except the elements $\lambda_{1(n-1)}, \lambda_{2(n-1)}, \dots, \lambda_{k(n-1)}$. Then we must calculate $\lambda_{1(n-1)}, \dots, \lambda_{k(n-1)}, \lambda_{2n}, \dots, \lambda_{(n-2)n}$.

(i) We take $i = k$, $j = k+1$, and $l = n-1$ in (e). Then

$$\lambda_{k(k+1)} - \lambda_{k(n-1)} + \lambda_{(k+1)(n-1)} = 0, \tag{5.7}$$

hence

$$\begin{aligned} \lambda_{k(n-1)} &= \lambda_{k(k+1)} + \lambda_{(k+1)(n-1)} \\ &= \lambda_{k(k+1)} + \lambda_{(k+1)(k+2)} + \cdots + \lambda_{(n-2)(n-1)}. \end{aligned} \tag{5.8}$$

Using (e) again, with $i = k-1$, $j = k$, and $l = n-1$, we obtain

$$\lambda_{(k-1)k} - \lambda_{(k-1)(n-1)} + \lambda_{k(n-1)} = 0, \tag{5.9}$$

hence

$$\begin{aligned} \lambda_{(k-1)(n-1)} &= \lambda_{(k-1)k} + \lambda_{k(n-1)} \\ &= \lambda_{(k-1)k} + \lambda_{k(k+1)} + \cdots + \lambda_{(n-2)(n-1)}. \end{aligned} \tag{5.10}$$

If we continue using (e) for the rest of values: $i = k-2, \dots, 2, 1$, $j = k-1, \dots, 2, 1$, and $l = n-1$, we arrive at the following equations:

$$\begin{aligned} \lambda_{23} - \lambda_{2(n-1)} + \lambda_{3(n-1)} &= 0, \\ \lambda_{12} - \lambda_{1(n-1)} + \lambda_{2(n-1)} &= 0, \end{aligned} \tag{5.11}$$

which imply

$$\begin{aligned} \lambda_{2(n-1)} &= \lambda_{23} + \lambda_{3(n-1)} \\ &= \lambda_{23} + \lambda_{34} + \cdots + \lambda_{(n-2)(n-1)}, \\ \lambda_{1(n-1)} &= \lambda_{12} + \lambda_{2(n-1)} \\ &= \lambda_{12} + \lambda_{23} + \cdots + \lambda_{(n-2)(n-1)}. \end{aligned} \tag{5.12}$$

Hence (e) implies

$$\lambda_{i(n-1)} = \lambda_{i(i+1)} + \lambda_{(i+1)(i+2)} + \cdots + \lambda_{(n-2)(n-1)}, \tag{5.13}$$

for $i = 1, 2, \dots, k$.

(ii) If $i = 1$ and $j = 2$ in (d) then $\lambda_{12} + \lambda_{1n} - \lambda_{2n} = 0$, and $\lambda_{2n} = \lambda_{12} + \lambda_{1n}$. Using again (d) with $i = 1$ and $j = 3$ we obtain $\lambda_{3n} = \lambda_{12} + \lambda_{23} + \lambda_{1n}$. We use (d) repeatedly up to obtain

$$\lambda_{in} = \lambda_{12} + \lambda_{23} + \cdots + \lambda_{(i-1)i} + \lambda_{1n}, \tag{5.14}$$

for $i = 2, 3, \dots, k$.

(iii) In order to calculate $\lambda_{(k+1)n},\dots,\lambda_{(n-2)n}$, we use (e) with $i=k+1,\dots,n-2$. We obtain

$$\begin{aligned}
\lambda_{(n-2)(n-1)}-\lambda_{(n-2)n}+\lambda_{(n-1)n}=0 &\Longrightarrow \lambda_{(n-2)n}=\lambda_{(n-2)(n-1)}+\lambda_{(n-1)n}\\
\lambda_{(n-3)(n-2)}-\lambda_{(n-3)n}+\lambda_{(n-2)n}=0 &\Longrightarrow \lambda_{(n-3)n}=\lambda_{(n-3)(n-2)}+\lambda_{(n-2)n}\\
&\Longrightarrow \lambda_{(n-3)n}=\lambda_{(n-3)(n-2)}+\lambda_{(n-2)(n-1)}+\lambda_{(n-1)n}\\
&\vdots\\
\lambda_{(k+1)(k+2)}-\lambda_{(k+1)n}+\lambda_{(k+2)n}=0 &\Longrightarrow \lambda_{(k+1)n}=\lambda_{(k+1)(k+2)}+\lambda_{(k+2)n}\\
&\Longrightarrow \lambda_{(k+1)n}=\lambda_{(k+1)(k+2)}+\lambda_{(k+2)(k+3)}+\cdots+\lambda_{(n-1)n}.
\end{aligned} \tag{5.15}$$

6. Harmonic maps. In this section we construct new examples of harmonic maps using the following result due to Lichnerowicz [12].

Theorem 6.1. *Let $\phi\colon (M,g,J_1)\to(N,h,J_2)$ be a $\pm$ holomorphic map between almost Hermitian manifolds where M is cosymplectic and N is $(1,2)$-symplectic. Then ϕ is harmonic.*

In order to construct harmonic maps $\phi\colon M^2\to F(n)$ using the theorem above, we need to know examples of holomorphic maps. Then, we use the following construction due to Eells and Wood [9].

Let $h\colon M^2\to\mathbb{CP}^{n-1}$ be a full holomorphic map (h is full if $h(M)$ is not contained in any $\mathbb{CP}^k$, for all $k<n-1$). We can lift h to $\mathbb{C}^n$, that is, for every $p\in M$ we can find a neighborhood of p, $U\subset M$, such that $h_U=(u_0,\dots,u_{n-1}):M^2\supset U\to\mathbb{C}^n-0$ satisfies $h(z)=[h_U(z)]=[(u_0(z),\dots,u_{n-1}(z))]$.

We define the kth associated curve of h by

$$\mathbb{O}_k:M^2\longrightarrow \mathbb{G}_{k+1}(\mathbb{C}^n),\qquad z\longmapsto h_U(z)\wedge\partial h_U(z)\wedge\cdots\wedge\partial^k h_U(z), \tag{6.1}$$

for $0\le k\le n-1$. And we consider

$$h_k:M^2\longrightarrow\mathbb{CP}^{n-1},\qquad z\longmapsto \mathbb{O}_k^{\perp}(z)\cap\mathbb{O}_{k+1}(z), \tag{6.2}$$

for $0\le k\le n-1$.

The following theorem, by Eells and Wood [9], is very important because it gives the classification of the harmonic maps from $S^2\sim\mathbb{CP}^1$ into a projective space $\mathbb{CP}^{n-1}$.

Theorem 6.2. *For each $k\in\mathbb{N}$, $0\le k\le n-1$, h_k is harmonic. Furthermore, given $\phi:(\mathbb{CP}^1,g)\to(\mathbb{CP}^{n-1}$, Killing metric$)$ a full harmonic map, then there are unique k and h such that $\phi=h_k$.*

This theorem provides in a natural way the following holomorphic maps:

$$\Psi:M^2\longrightarrow F(n),\qquad z\longmapsto (h_0(z),\dots,h_{n-1}(z)), \tag{6.3}$$

called Eells-Wood's maps. (See [16].)

We can write the set of $(1,2)$-symplectic metrics on $F(n)$, characterized in the sections above, in the following way:

$$\mathfrak{M}_n=\{g^k=ds^2_{\Lambda^k}:1\le k\le n-3\}. \tag{6.4}$$

Using Theorem 6.1 we obtain the following results.

Proposition 6.3. *Let $\phi: M^2 \to (F(n), g)$, $g \in \mathfrak{M}_n$ be a holomorphic map. Then ϕ is harmonic.*

A known fact, necessary to the following proposition, is that a $(1,2)$-symplectic manifold is cosymplectic.

Proposition 6.4. *Let $\phi: (F(l), g) \to (F(n), \tilde{g})$ be a holomorphic map with $g \in \mathfrak{M}_l$ and $\tilde{g} \in \mathfrak{M}_n$. Then ϕ is harmonic.*

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