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SCHRÖDINGER OPERATORS WITH A SINGULAR POTENTIAL

TOKA DIAGANA

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This note is devoted to the study of some Schrödinger operators with a singular real potential Q. The potential Q is chosen so that the algebraic sum $L = -\Delta + Q$ is not defined. Next, we define the sum form operator which will be well defined and we show that this operator verifies the well-known Kato's square root problem.

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1. Introduction. Let H be a Hilbert space and let A and B be two unbounded linear operators on H. The algebraic sum of A and B is defined by

$$D(A+B) = D(A) \cap D(B), \quad (A+B)u = Au + Bu, \quad \forall u \in D(A) \cap D(B). \tag{1.1}$$

Thus, if A and B are chosen so that $D(A) \cap D(B)$ is reduced to $\{0\}$, therefore the algebraic sum of A and B is not defined except in zero. So, the algebraic sum is not always well adapted to problems arising in mathematical analysis. This has motivated some mathematical developments where the notion of sum is weakened or relaxed, see for example [2, 3, 4]. Consider A and B be two linear operators given on $L^2(R)$ by

$$\begin{aligned} &D(A) = H^2(\mathbb{R}^n), \quad Au = -\Delta u, \\ &D(B) = \{u \in L^2(\mathbb{R}^n) : Qu \in L^2(\mathbb{R}^n)\}, \quad Bu = Qu, \end{aligned} \tag{1.2}$$

where Q is a measurable real function. Let L be the operator of Schrödinger given by $L = -\Delta + Q$. Let Φ and Ψ be the closed and densely defined sesquilinear forms associated, respectively, to $-\Delta$ and Q, given by

$$\begin{aligned} \Phi(u,v) &= \int_{\mathbb{R}^n} \nabla u \overline{\nabla v}\, dx, \quad \forall u,v \in H^1(\mathbb{R}^n), \\ \Psi(u,v) &= \int_{\mathbb{R}^n} Qu\bar{v}\, dx, \quad \forall u,v \in D(B^{1/2}). \end{aligned} \tag{1.3}$$

We will put, $\Upsilon(u,v) = (\Phi + \Psi)(u,v)$ for all $u,v \in D(\Upsilon) = D(\Phi) \cap D(\Psi)$, the sum of sesquilinear forms associated to $-\Delta$ and Q.

2. The main result

Hypothesis on Q. We assume that the potential Q verifies H_Q, given by

$$Q > 0, \quad Q \in L^1(\mathbb{R}^n), \quad Q \notin L^2_{\text{loc}}(\mathbb{R}^n). \tag{2.1}$$

Proposition 2.1. *Under hypothesis H_Q and if $n < 4$, then,*

$$D(A) \cap D(B) = \{0\}. \tag{2.2}$$

Proof. Let $u \in D(-\Delta) \cap D(Q)$ and suppose that $u \not\equiv 0$. Since $u \in H^2(\mathbb{R}^n)$ where $n < 4$, then u is a continuous function by Sobolev's theorem (see [1]). There exists an open subset Ω of $\mathbb{R}^n$ and there exists $\delta > 0$ such that $|u(x)| > \delta$ for all $x \in \Omega$. Let Ω' be a compact subset of Ω, equipped with the induced topology by Ω. Thus Ω' is also a compact subset of $\mathbb{R}^n$. It follows that

$$|Q|_{\Omega'} = \frac{(|Qu|)_{\Omega'}}{|u|_{\Omega'}} \in L^2(\Omega'), \tag{2.3}$$

because $(|Qu|)_{\Omega'} \in L^2(\Omega')$ and $1/(|u|)'_{\Omega} \in L^\infty(\Omega')$. Therefore $Q \in L^2(\Omega')$, that is impossible according to hypothesis H_Q, then $u \equiv 0$. □

Question 2.2. Find a characterization of $D(-\Delta) \cap D(Q)$, when $n \geq 4$?

Proposition 2.3. *Under hypothesis H_Q, then*
- $D(A^{1/2}) \hookrightarrow D(B^{1/2})$ *if $n = 1$,*
- $D(A^{1/2}) \cap D(B^{1/2}) \supseteq C_0^\infty(\mathbb{R}^n)$ *if $n > 1$.*

Example of potential Q verifying H_Q. Let Ω be a compact subset of $\mathbb{R}^n$ and let G be a complex function satisfying, $\mathrm{Re}\, G > 0$, $G \in L^1(\Omega)$, $G \notin L^2(\Omega)$, and $G \equiv 0$ on $\mathbb{R}^n - \Omega$. Consider the following rational sequence $\alpha_k = (\alpha_k^1, \alpha_K^2, \ldots, \alpha_k^n) \in \mathbb{Q}^n$. Then the function Q, given by

$$Q(x) = \sum_{k=1}^{+\infty} \frac{G(x - \alpha_k)}{k^2}, \tag{2.4}$$

verifies hypothesis H_Q.

3. Generalized sum of A and B.

Consider the closed sesquilinear forms given by

$$\begin{aligned} \Phi(u,v) &= \int_{\mathbb{R}^n} \nabla u \overline{\nabla v}\, dx \quad \forall u, v \in H^1(\mathbb{R}^n), \\ \Psi(u,v) &= \int_{\mathbb{R}^n} Q u \bar{v}\, dx \quad \forall u, v \in D(B^{1/2}), \end{aligned} \tag{3.1}$$

and the sum of the forms Φ and Ψ given by, $\Upsilon = \Phi + \Psi$, in other words, $\Upsilon(u,v) = \int_{\mathbb{R}^n} (\nabla u \overline{\nabla v} + Q u \bar{v}) dx$ for all $u, v \in D(A^{1/2}) \cap D(B^{1/2})$. The sesquilinear form Υ is a closed sectorial and densely defined form as sum of closed sectorial and densely defined forms, then there exists a unique m-sectorial operator $A \oplus B$, called sum form or generalized form of A and B associated to Υ (see[3, 4]) and Υ has the following representation:

$$\Upsilon(u,v) = \langle (A \oplus B)u, v \rangle \quad \forall u \in D(A \oplus B),\ v \in D(A^{1/2}) \cap D(B^{1/2}). \tag{3.2}$$

According to [3], the operator $A \oplus B$ verifies the well-known condition of Kato (see [4]), in other words,

$$D\big((A \oplus B)^{1/2}\big) = D(\Upsilon) = D\Big(\big((A \oplus)^*\big)^{1/2}\Big), \tag{3.3}$$

and the operator $A \oplus B$ is given by

$$D(A \oplus B) = \{u \in H^1(\mathbb{R}^n) : Q|u|^2 \in L^1(\mathbb{R}^n),\ -\Delta u + Qu \in L^2(\mathbb{R}^n)\},$$
$$(A \oplus B)u = -\Delta u + Qu. \tag{3.4}$$

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Toka Diagana: Department of Mathematics, Howard University, 2441 Sixth Street, NW, Washington, DC 20059, USA
E-mail address: tdiagana@howard.edu