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ON BRANCHWISE IMPLICATIVE BCI-ALGEBRAS

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We introduce a new class of BCI-algebras, namely the class of branchwise implicative BCI-algebras. This class contains the class of implicative BCK-algebras, the class of weakly implicative BCI-algebras (Chaudhry, 1990), and the class of medial BCI-algebras. We investigate necessary and sufficient conditions for two types of BCI-algebras to be branchwise implicative BCI-algebras.

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1. Introduction. Iséki and Tanaka [10] defined implicative BCK-algebras and studied their properties. Further, Iséki [7, 8] gave the notion of a BCI-algebra which is a generalization of the concept of a BCK-algebra. Iséki [8] and Iséki and Thaheem [11] have shown that no proper class of implicative BCI-algebras exists, that is, such BCI-algebras are implicative BCK-algebras.

Thus, a natural question arises whether it is possible to generalize the notion of implicativeness in such a way that this generalization not only gives us a proper class of BCI-algebras but also contains the class of implicative BCK-algebras. In this paper, we answer this question in yes by introducing the concept of a branchwise implicative BCI-algebra. This proper class of BCI-algebras contains the class of implicative BCK-algebras, the class of weakly implicative BCI-algebras [1] and the class of medial BCI-algebras [4, 6].

2. Preliminaries. A BCI-algebra is an algebra $(X,*,0)$ of type (2,0) satisfying the following conditions:

$$(x*y)*(x*z) \le z*y, \quad \text{where } x \le y \text{ if and only if } x*y=0, \tag{2.1}$$

$$x*(x*y) \le y, \tag{2.2}$$

$$x \le x, \tag{2.3}$$

$$x \le y \text{ and } y \le x \text{ imply } x=y, \tag{2.4}$$

$$x \le 0 \text{ implies } x=0. \tag{2.5}$$

If (2.5) is replaced by $0 \le x$, then the algebra is called a BCK-algebra. It is well known that every BCK-algebra is a BCI-algebra.

In a BCI-algebra X, the following hold:

$$(x*y)*z=(x*z)*y, \tag{2.6}$$

$$x*0=x, \tag{2.7}$$

$$x \le y \text{ implies } x*z \le y*z \text{ and } z*y \le z*x, \tag{2.8}$$

$$(x*z)*(y*z) \le x*y, \tag{2.9}$$

$$x*(x*(x*y)) = x*y \quad \text{(see [8])}. \tag{2.10}$$

Definition 2.1 (see [9]). A subset I of a BCI-algebra X is called an ideal of X if it satisfies

$$0 \in I, \quad x*y \in I, \quad y \in I \text{ imply } x \in I. \tag{2.11}$$

Definition 2.2 (see [10]). If in a BCK-algebra X

$$(x*y)*z = (x*z)*(y*z) \tag{2.12}$$

holds for all $x,y,z \in X$, then it is called positive implicative.

Definition 2.3 (see [10]). If in a BCK-algebra X

$$x*(x*y) = y*(y*x) \tag{2.13}$$

holds for all $x,y \in X$, then it is called commutative.

Theorem 2.4 (see [10]). *A BCK-algebra X is positive implicative if and only if it satisfies*

$$(x*y) = (x*y)*y \quad \forall x,y \in X. \tag{2.14}$$

It has been shown in [8, 11] that no proper classes of positive implicative BCI-algebras and commutative BCI-algebras exist and such BCI-algebras are BCK-algebras of the corresponding type. That is why we generalized these notions and defined weakly positive implicative BCI-algebras [1] and branchwise commutative BCI-algebras [3] and studied some of their properties. Each class of these proper BCI-algebras contains the class of BCK-algebras of the corresponding type.

Definition 2.5 (see [1]). A BCI-algebra X satisfying

$$(x*y)*z = ((x*z)*z)*(y*z) \quad \forall x,y,z \in X \tag{2.15}$$

is called a weakly positive implicative BCI-algebra.

Theorem 2.6 (see [1]). *A BCI-algebra X is weakly positive implicative if and only if*

$$x*y = ((x*y)*y)*(0*y) \quad \forall x,y \in X. \tag{2.16}$$

A BCI-algebra satisfying $(x*y)*(z*u) = (x*z)*(y*u)$ is called a medial BCI-algebra.

Let X be a BCI-algebra and $M = \{x : x \in X \text{ and } 0*x = 0\}$. Then M is called its BCK-part. If $M = \{0\}$, then X is called p-semisimple.

It has been shown in [4, 5, 6, 13] that in a BCI-algebra X the following are equivalent:

$$\begin{gathered} X \text{ is medial}, \quad x*(x*y) = y \quad \forall x,y \in X, \\ 0*(0*x) = x \quad \forall x \in X, \quad X \text{ is } p\text{-semisimple}. \end{gathered} \tag{2.17}$$

We now describe the notions of branches of a BCI-algebra and branchwise commutative BCI-algebras defined and investigated in [2, 3].

Definition 2.7 (see [3]). Let X be a BCI-algebra, then the set $\mathrm{Med}(X) = \{x : x \in X$ and $0 * (0 * x) = x\}$ is called medial part of X.

Obviously, $0 \in \mathrm{Med}(X)$ and thus $\mathrm{Med}(X)$ is nonempty. In what follows the elements of $\mathrm{Med}(X)$ will be denoted by $x_0, y_0, \ldots$. It is known that $\mathrm{Med}(X)$ is a medial subalgebra of X and for each $x \in X$, there is a unique $x_0 = 0 * (0 * x) \in \mathrm{Med}(X)$ such that $x_0 \leq x$ (see [3]). Further, $\mathrm{Med}(X)$, in general, is not an ideal of X. Obviously, for a BCK-algebra X, $\mathrm{Med}(X) = \{0\}$ and hence is an ideal of X.

Definition 2.8 (see [3]). Let X be a BCI-algebra and $x_0 \in \mathrm{Med}(X)$, then the set $B(x_0) = \{x : x \in X$ and $x_0 * x = 0\}$ is called a branch of X determined by the element x_0.

The following theorem (proved in [2, 3]) shows that the branches of a BCI-algebra X are pairwise disjoint and form its partition. So the study of branches of a BCI-algebra X plays an important role in investigation of the properties of X. Obviously, a BCK-algebra X is a one-branch BCI-algebra and in this case $X = B(0)$.

Theorem 2.9 (see [2, 3]). *Let X be a BCI-algebra with medial part* $\mathrm{Med}(X)$*, then*

(i) $X = \cup\{B(x_0) : x_0 \in \mathrm{Med}(X)\}$.

(ii) $B(x_0) \cap B(y_0) = \phi$, $x_0, y_0 \in \mathrm{Med}(X)$, *and* $x_0 \neq y_0$.

(iii) *If* $x, y \in B(x_0)$, *then* $0 * x = 0 * y = 0 * x_0 = 0 * y_0$ *and* $x * y, y * x \in M$.

Definition 2.10 (see [3]). A BCI-algebra X is said to be branchwise commutative if and only if for $x_0 \in \mathrm{Med}(X)$, $x, y \in B(x_0)$, the following equality holds:

$$x * (x * y) = y * (y * x). \tag{2.18}$$

Since a BCK-algebra is a one-branch BCI-algebra, therefore, it is commutative if and only if it is branchwise commutative.

Theorem 2.11 (see [3]). *A BCI-algebra X is branchwise commutative if and only if*

$$x * (x * y) = y * (y * (x * (x * y))) \quad \forall x, y \in X. \tag{2.19}$$

3. Branchwise implicative BCI-algebras. In this section, we define branchwise implicative BCI-algebras. We show that this proper class of BCI-algebras contains the class of implicative BCK-algebras [10], the class of weakly implicative BCI-algebras [1] and the class of medial BCI-algebras. We also find necessary and sufficient conditions for two types of BCI-algebras to be branchwise implicative.

Definition 3.1 (see [10]). A BCK-algebra X is said to be implicative if and only if

$$x * (y * x) = x \quad \forall x, y \in X. \tag{3.1}$$

It has been shown in [8, 11] that no proper class of implicative BCI-algebras exists. Due to this reason we generalized the notion of implicativeness to weak implicativeness [1] mentioned below.

Definition 3.2 (see [1]). A BCI-algebra X is said to be weakly implicative if and only if

$$x = (x * (y * x)) * (0 * (y * x)) \quad \forall x, y \in X. \tag{3.2}$$

We further generalize this concept and find a generalization of the following well-known result of Iséki [10].

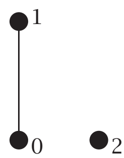
Theorem 3.3. *An implicative BCK-algebra is a positive implicative and commutative BCK-algebra.*

Definition 3.4. A BCI-algebra X is said to be a branchwise implicative BCI-algebra if and only if

$$x*(y*x)=x \quad \forall x,y\in B(x_0) \text{ and } x_0\in \operatorname{Med}(X). \tag{3.3}$$

Example 3.5. Let $X=\{0,1,2,\}$ in which $*$ is defined by

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0



Then X is a branchwise implicative BCI-algebra. This shows that proper branchwise implicative BCI-algebras exist.

Remark 3.6. (i) Since a BCK-algebra is a one-branch BCI-algebra, therefore, it is implicative if and only if it is branchwise implicative.

(ii) Let X be weakly implicative and let $x,y\in B(x_0)$, $x_0\in \operatorname{Med}(X)$, then using Theorem 2.9(iii), we get $y*x\in M$. Thus $0*(y*x)=0$. So $x=(x*(y*x))*(0*(y*x))$ reduces to $x=x*(y*x)$. Hence every weakly implicative BCI-algebra is branchwise implicative BCI-algebra. But the branchwise implicative BCI-algebra X of Example 3.5 is not weakly implicative because $(1*(2*1))*(0*(2*1))=(1*2)*(0*2)=2*2=0\neq 1$.

(iii) It is known that each branch of a medial BCI-algebra X is a singleton. Let X be a medial BCI-algebra and $x_0\in \operatorname{Med}(X)$. Then $B(x_0)=\{x_0\}$. Hence $x_0*(x_0*x_0)=x_0*0=x_0$, which implies that X is branchwise implicative.

Thus the class of branchwise implicative BCI-algebras contains the class of implicative BCK-algebras, the class of weakly implicative BCI-algebras, and the class of medial BCI-algebras. We now prove the following results.

Lemma 3.7. *Let X be a BCI-algebra. If $x,y\in X$ and $x\le y$, then $x,y\in B(x_0)$ for $x_0\in \operatorname{Med}(X)$.*

Proof. Let $x\in X$, then there is a unique $x_0=0*(0*x)\in \operatorname{Med}(X)$ such that $x\in B(x_0)$. Now $x_0*y=(0*(0*x))*y=(0*y)*(0*x)\le x*y=0$. Hence $x_0*y=0$, which implies $y\in B(x_0)$. □

Theorem 3.8. *If X is a branchwise implicative BCI-algebra, then it is branchwise commutative.*

Proof. Let $x,y\in X$, then $x*(x*y)\le y$ and Lemma 3.7 imply that $x*(x*y)$ and $y\in B(y_0)$ for some $y_0\in \operatorname{Med}(X)$. Since X is branchwise implicative, therefore using (3.3), we get

$$(x*(x*y))*(y*(x*(x*y))) = x*(x*y). \tag{3.4}$$

Using (2.2) and (2.8), we get

$$\begin{aligned} x*(x*y) &= (x*(x*y))*(y*(x*(x*y))) \\ &\le y*(y*(x*(x*y))) \le x*(x*y). \end{aligned} \tag{3.5}$$

Thus

$$x*(x*y) = y*(y*(x*(x*y))), \tag{3.6}$$

which along with Theorem 2.11 implies that X is branchwise commutative. □

Theorem 3.9. *If X is a branchwise implicative BCI-algebra, then it satisfies*

$$(x*y)*(0*y) = (((x*y)*y)*(0*y))*(0*y). \tag{3.7}$$

Proof. Since X is branchwise implicative, therefore Theorem 3.8 implies that X is branchwise commutative. Let $x, y \in X$. Since $(x*y)*(0*y) \le x$, therefore Lemma 3.7 implies that $(x*y)*(0*y), x \in B(x_0)$. Now branchwise implicativeness of X implies

$$((x*y)*(0*y))*(x*((x*y)*(0*y))) = (x*y)*(0*y), \tag{3.8}$$

which, using (2.6) twice, gives

$$((x*(x*((x*y)*(0*y))))*y)*(0*y) = (x*y)*(0*y). \tag{3.9}$$

Using branchwise commutativeness of X, from (3.9) we get

$$((((x*y)*(0*y))*(((x*y)*(0*y))*x))*y)*(0*y) = (x*y)*(0*y), \tag{3.10}$$

which implies

$$(((x*y)*(0*y))*y)*(0*y) = (x*y)*(0*y), \tag{3.11}$$

so

$$(((x*y)*y)*(0*y))*(0*y) = (x*y)*(0*y). \tag{3.12}$$

□

Remark 3.10. Since a BCK-algebra is a one-branch BCI-algebra, therefore an implicative BCK-algebra is commutative. Further, for a BCK-algebra $0*y = 0$ and thus (3.7) reduces to $x*y = (x*y)*y$, which implies X is positive implicative. So we get Theorem 3.3, a well-known result of Iséki [10], as a corollary from Theorems 3.8 and 3.9.

We now investigate necessary and sufficient conditions for two types of BCI-algebras to be branchwise implicative.

Theorem 3.11. *A BCI-algebra X, with $\mathrm{Med}(X)$ as an ideal of X, is a branchwise implicative BCI-algebra if and only if it is branchwise commutative and satisfies*

$$(x*y)*(0*y)=(((x*y)*y)(0*y))*(0*y)\quad \forall x,y\in X. \tag{3.13}$$

Proof. $(\Rightarrow)$ Sufficiency follows from Theorems 3.8 and 3.9.

$(\Leftarrow)$ For necessity we consider $x,y\in X$ such that $x,y\in B(x_0)$ for some $x_0\in\mathrm{Med}(X)$. Now from Theorem 2.9(iii), we get $x*y$ and $y*x\in M$. So $0*(x*y)=0*(y*x)=0$. Further, $(x*(y*x))*x=(x*x)*(y*x)=0*(y*x)=0$, so

$$x*(y*x)\le x. \tag{3.14}$$

Now (3.14) along with Lemma 3.7 implies $x*(y*x)$ and x belong to the branch determined by x, that is, $B(x_0)$. Hence x,y and $x*(y*x)\in B(x_0)$. Since X is branchwise commutative, therefore,

$$\begin{aligned}
&(x*(x*(y*x)))*(0*x)\\
&\quad=[(y*x)*((y*x)*(x*(x*(y*x))))]*(0*x)\\
&\quad=[(y*x)*(0*x)]*[(y*x)*(x*(x*(y*x)))]\quad(\text{using }(2.6))\\
&\quad=[(((y*x)*x)*(0*x))*(0*x)]*[(y*x)*(x*(x*(y*x)))]\quad(\text{using }(3.13)).
\end{aligned}\tag{3.15}$$

Now by using (2.6) three times, we get

$$\begin{aligned}
&(x*(x*(y*x)))*(0*x)\\
&\quad=[[[(y*x)*((y*x)(x*(x*(y*x))))]*x]*(0*x)]*(0*x).
\end{aligned}\tag{3.16}$$

Since x,y and $x*(y*x)\in B(x_0)$, therefore $x*y,y*x,x*(x*(y*x))\in M=B(0)$. Since X is branchwise commutative, therefore,

$$\begin{aligned}
&(x*(x*(y*x)))*(0*x)\\
&\quad=[[[(x*(x*(y*x)))*((x*(x*(y*x)))*(y*x))]*x]*(0*x)]*(0*x)\\
&\quad=((((x*(x*(y*x)))*0)*x)*(0*x))*(0*x)\\
&\quad=(((x*(x*(y*x)))*x)*(0*x))*(0*x)\\
&\quad=((0*(x*(y*x)))*(0*x))*(0*x)\\
&\quad=(((0*x)*(0*(y*x)))*(0*x))*(0*x)\\
&\quad=(((0*x)*(0*x))*(0*(y*x)))*(0*x)\\
&\quad=(0*(0*(y*x)))*(0*x)\\
&\quad=(0*0)*(0*x)=0*(0*x).
\end{aligned}\tag{3.17}$$

Hence

$$(x*(x*(y*x)))*(0*x)=0*(0*x)\in\mathrm{Med}(X). \tag{3.18}$$

But (2.10) implies $0*(0*(0*x))=0*x$. So $0*x\in \mathrm{Med}(X)$. Since $\mathrm{Med}(X)$ is an ideal of X, therefore, $x*(x*(y*x))\in \mathrm{Med}(X)$. Hence

$$x*(x*(y*x))=0*(0*(x*(x*(y*x)))). \tag{3.19}$$

Since $x*(x*(y*x))\in M=B(0)$, therefore, $0*(x*(x*(y*x)))=0$. Thus $x*(x*(y*x))=0$, which gives

$$x\le x*(y*x). \tag{3.20}$$

Using (3.14) and (3.20), we get

$$x=x*(y*x)\quad \forall x,y\in B(x_0). \tag{3.21}$$

Hence X is branchwise implicative. This completes the proof. □

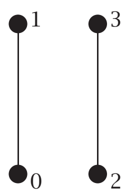
Remark 3.12. Since in a BCK-algebra X, $\mathrm{Med}(X)=\{0\}$ is always an ideal of X, therefore the following well-known result regarding BCK-algebra follows as a corollary from Theorem 3.11.

Corollary 3.13. *A BCK-algebra is implicative if and only if it is positive implicative and commutative.*

Remark 3.14. The following example shows that there exist proper BCI-algebras in which $\mathrm{Med}(X)$ is an ideal. Thus the condition, $\mathrm{Med}(X)$ is an ideal of X, in Theorem 3.11 is not unnatural.

Example 3.15 (see [12, Example 2])**.** The set $X=\{0,1,2,3\}$ with the operation $*$ defined as

$*$	0	1	2	3
0	0	0	2	2
1	1	0	3	2
2	2	2	0	0
3	3	2	1	0



is a proper BCI-algebra. Here $\mathrm{Med}(X)=\{0,2\}$ is an ideal of X. Further, X is branchwise implicative but is not medial.

Definition 3.16. Let X be a BCI-algebra. Two elements x,y of X are said to be comparable if and only if either $x*y=0$ or $y*x=0$, that is, either $x\le y$ or $y\le x$.

Definition 3.17. Let X be a BCI-algebra. If $x_0\in \mathrm{Med}(X)$ and $x_0\ne 0$, then $B(x_0)$, the branch of X determined by x_0, is called a proper BCI-branch of X.

Theorem 3.18. *Let X be a BCI-algebra such that any two elements of a proper BCI-branch of X are comparable. Then X is branchwise implicative if and only if X is branchwise commutative and satisfies*

$$(x*y)*(0*y)=(((x*y)*y)*(0*y))*(0*y)\quad \forall x,y\in X. \tag{3.22}$$

Proof. (⇒) Sufficiency follows from Theorems 3.8 and 3.9.

(⇐) For necessity we consider the following two cases.

Case 1. Let $x, y \in B(0) = M$. Then $0 * y = 0 * x = 0$ and hence (3.22) becomes $x * y = (x * y) * y$. Further, $(x * (y * x)) * x = (x * x) * (y * x) = 0 * (y * x) = 0$. Hence

$$x * (y * x) \leq x. \tag{3.23}$$

Since $x * y \in M = B(0)$ and X is branchwise commutative, therefore,

$$x * (x * (y * x)) = (y * x) * ((y * x) * x) = (y * x) * (y * x) = 0. \tag{3.24}$$

Thus

$$x * \leq x * (y * x). \tag{3.25}$$

From (3.23) and (3.25), we get $x = x * (y * x)$ for all $x, y \in B(0)$.

Case 2. Let $x, y \in B(x_0)$, where $x_0 \in \mathrm{Med}(X)$ and $x_0 \neq 0$. Thus $x * y \in M$ and $y * x \in M$. So $0 * (x * y) = 0$ and $0 * (y * x) = 0$. Further, taking $y = x * y$ in (3.22), we get

$$x * (x * y) = (x * (x * y)) * (x * y) \quad \forall x, y \in B(x_0). \tag{3.26}$$

Interchanging x and y in (3.26), we get

$$y * (y * x) = (y * (y * x)) * (y * x) \quad \forall x, y \in B(x_0). \tag{3.27}$$

Since x, y are comparable, therefore, either $y * x = 0$ or $x * y = 0$. If $y * x = 0$, then

$$x * (y * x) = x * 0 = x. \tag{3.28}$$

If $x * y = 0$, then branchwise commutativeness of X gives

$$y * (y * x) = x * (x * y) = x * 0 = x. \tag{3.29}$$

Using (3.27) and (3.29), we get

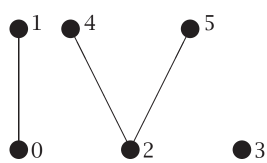
$$x = x * (y * x). \tag{3.30}$$

Thus X is branchwise implicative. □

Remark 3.19. The following example shows that the conditions $\mathrm{Med}(X)$ is an ideal of X and any two elements of a proper BCI-branch of X are comparable cannot be removed from Theorems 3.11 and 3.18, respectively.

Example 3.20. Let $X = \{0, 1, 2, 3, 4, 5\}$ in which $*$ is defined by

$*$	0	1	2	3	4	5
0	0	0	3	2	3	3
1	1	0	3	2	3	3
2	2	2	0	3	0	0
3	3	3	2	0	2	2
4	4	2	1	3	0	1
5	5	2	1	3	1	0



Routine calculations give that X is a BCI-algebra, which is branchwise commutative and satisfies (3.22). But we note that

(1) $\mathrm{Med}(X) = \{0,2,3\}$ is not an ideal of X because $4*3 = 3 \in \mathrm{Med}(X)$, $3 \in \mathrm{Med}(X)$ but $4 \notin \mathrm{Med}(X)$. Further, X is not branchwise implicative because $4,5 \in B(2)$ and $4*(5*4) = 4*1 = 2 \neq 4$;

(2) the elements 4 and 5 of $B(2)$ are not comparable and also X is not branchwise implicative.

Combining Theorems 3.11 and 3.18, we get the following theorem.

Theorem 3.21. *Let X be a BCI-algebra such that either* $\mathrm{Med}(X)$ *is an ideal of X or every pair of elements of a proper BCI-branch of X are comparable, then X is branchwise implicative if and only if X is branchwise commutative and satisfies (3.22).*

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