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CERTAIN CONVEX HARMONIC FUNCTIONS

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We define and investigate a family of complex-valued harmonic convex univalent functions related to uniformly convex analytic functions. We obtain coefficient bounds, extreme points, distortion theorems, convolution and convex combinations for this family.

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1. Introduction. A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain $\mathfrak{D} \subset \mathbb{C}$ is said to be harmonic in $\mathfrak{D}$ if both u and v are real harmonic in $\mathfrak{D}$. Consider the functions U and V analytic in $\mathfrak{D}$ so that $u = \Re U$ and $v = \Im V$. Then the harmonic function f can be expressed by

$$f(z) = h(z) + \overline{g(z)}, \quad z \in \mathfrak{D}, \tag{1.1}$$

where $h = (U+V)/2$ and $g = (U-V)/2$. We call h the analytic part and g the co-analytic part of f. If the co-analytic part of f is identically zero then f reduces to the analytic case.

The mapping $z \mapsto f(z)$ is sense-preserving and locally one-to-one in $\mathfrak{D}$ if and only if the Jacobian of f is positive (see [1]), that is, if and only if

$$J_f(z) = \left|h'(z)\right|^2 - \left|g'(z)\right|^2 > 0, \quad z \in \mathfrak{D}. \tag{1.2}$$

Let $\mathcal{H}$ denote the family of functions $f = h + \bar{g}$ which are harmonic, sense-preserving, and univalent in the open unit disk $\Delta = \{z : |z| < 1\}$ with $h(0) = f(0) = f_z(0) - 1 = 0$. Thus, we may write

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad \left|b_1\right| < 1. \tag{1.3}$$

Also let $\overline{\mathcal{H}}$ denote the subclass of $\mathcal{H}$ consisting of functions $f = h + \bar{g}$ so that the functions h and g take the form

$$h(z) = z - \sum_{n=2}^{\infty} \left|a_n\right| z^n, \qquad g(z) = -\sum_{n=1}^{\infty} \left|b_n\right| z^n, \quad \left|b_1\right| < 1. \tag{1.4}$$

Recently, Kanas and Wisniowska [5] (see also Kanas and Srivastava [4]), studied the class of k-uniformly convex analytic functions, denoted by k-$\mathcal{UCV}$, $0 \le k < \infty$, so that $h \in k$-$\mathcal{UCV}$ if and only if

$$\Re\left\{1 + (z-\zeta)\frac{h''(z)}{h'(z)}\right\} \ge 0, \quad |\zeta| \le k, \ z \in \Delta. \tag{1.5}$$

For real ϕ we may let $\zeta = -kze^{i\phi}$. Then condition (1.5) can be written as

$$\Re\left\{1+(1+ke^{i\phi})\frac{zh''(z)}{h'(z)}\right\} \geq 0. \tag{1.6}$$

Now considering the harmonic functions $f = h+\bar{g}$ of the form (1.3) we define the family $\mathcal{HCV}(k,\alpha)$, $0 \leq \alpha < 1$, so that $f = h+\bar{g} \in \mathcal{HCV}(k,\alpha)$ if and only if

$$\Re\left\{1+(1+ke^{i\phi})\frac{z^2h''(z)+\overline{2zg'(z)+z^2g''(z)}}{zh'(z)-\overline{zg'(z)}}\right\} \geq \alpha, \quad 0 \leq \alpha < 1. \tag{1.7}$$

Finally, we let $\overline{\mathcal{HCV}}(k,\alpha) \equiv \mathcal{HCV}(k,\alpha) \cap \overline{\mathcal{H}}$.

Notice that if $g \equiv 0$ and $\alpha = 0$ then the family $\mathcal{HCV}(k,\alpha)$ defined by (1.7) reduces to the class k-$\mathcal{UCV}$ of k-uniformly convex analytic functions defined by (1.5). If we, further, let $k = 1$ in (1.5), we obtain the class of uniformly convex analytic functions defined by Goodman [2]. A geometric characterization of the general family $\mathcal{HCV}(k,\alpha)$ is an open question.

In Section 2, we introduce sufficient coefficient bounds for functions to be in $\mathcal{HCV}(k,\alpha)$ and show that these bounds are also necessary for functions in $\overline{\mathcal{HCV}}(k,\alpha)$. In Section 3, the inclusion relation between the classes k-$\mathcal{UCV}$ and $\mathcal{HCV}(k,\alpha)$ is examined. Extreme points and distortion bounds for $\mathcal{HCV}(k,\alpha)$ are given in Section 4. Finally, in Section 5, we show that the class $\overline{\mathcal{HCV}}(k,\alpha)$ is closed under convolution and convex combinations.

Here we state a result due to Jahangiri [3], which we will use throughout this paper.

Theorem 1.1. *Let $f = h+\bar{g}$ with h and g of the form (1.3). If*

$$\sum_{n=2}^{\infty}\frac{n(n-\alpha)}{1-\alpha}|a_n| + \sum_{n=1}^{\infty}\frac{n(n+\alpha)}{1-\alpha}|b_n| \leq 1, \quad 0 \leq \alpha < 1, \tag{1.8}$$

then f is harmonic, sense-preserving, univalent in Δ, and f is convex harmonic of order α denoted by $\mathcal{HK}(\alpha)$. Condition (1.8) is also necessary if $f \in \overline{\mathcal{HK}}(\alpha) \equiv \mathcal{HK}(\alpha) \cap \overline{\mathcal{H}}$.

2. Coefficient bounds.

First we state and prove a sufficient coefficient bound for the class $\mathcal{HCV}(k,\alpha)$.

Theorem 2.1. *Let $f = h+\bar{g}$ be of the form (1.3). If $0 \leq k < \infty$, $0 \leq \alpha < 1$, and*

$$\sum_{n=2}^{\infty}\frac{n(n+nk-k-\alpha)}{1-\alpha}|a_n| + \sum_{n=1}^{\infty}\frac{n(n+nk+k+\alpha)}{1-\alpha}|b_n| \leq 1, \tag{2.1}$$

then f is harmonic, sense-preserving, univalent in Δ, and $f \in \mathcal{HCV}(k,\alpha)$.

Proof. Since $n-\alpha \leq n+nk-k-\alpha$ and $n+\alpha \leq n+nk+k+\alpha$ for $0 \leq k < \infty$, it follows from Theorem 1.1 that $f \in \mathcal{HK}(\alpha)$ and hence f is sense-preserving and convex univalent in Δ. Now, we only need to show that if (2.1) holds then

$$\Re\left\{\frac{zh'(z)+(1+ke^{i\phi})z^2h''(z)+(1+2ke^{i\phi})\overline{zg'(z)}+(1+ke^{i\phi})\overline{z^2g''(z)}}{zh'(z)-\overline{zg'(z)}}\right\} = \Re\frac{A(z)}{B(z)} \geq \alpha. \tag{2.2}$$

Using the fact that $\Re(w) \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$ it suffices to show that

$$\left|A(z)+(1-\alpha)B(z)\right| - \left|A(z)-(1+\alpha)B(z)\right| \geq 0, \tag{2.3}$$

where $A(z) = zh'(z)+(1+ke^{i\phi})z^2h''(z)+(1+2ke^{i\phi})\overline{zg'(z)}+(1+ke^{i\phi})\overline{z^2g''(z)}$ and $B(z) = zh'(z)-\overline{zg'(z)}$. Substituting for $A(z)$ and $B(z)$ in (2.3), we obtain

$$\begin{aligned}
&\left|A(z)+(1-\alpha)B(z)\right| - \left|A(z)-(1+\alpha)B(z)\right| \\
&\quad = \left|(2-\alpha)z+\sum_{n=2}^{\infty} n\left[n+1-\alpha+k(n-1)e^{i\phi}\right]a_n z^n \right.\\
&\qquad \left. +\sum_{n=1}^{\infty} n\left[n-1+\alpha+k(n+1)e^{i\phi}\right]\bar{b}_n \bar{z}^n\right| \\
&\qquad - \left|-\alpha z+\sum_{n=2}^{\infty} n\left[n-1-\alpha+k(n-1)e^{i\phi}\right]a_n z^n \right.\\
&\qquad\quad \left. +\sum_{n=1}^{\infty} n\left[n+1+\alpha+k(n+1)e^{i\phi}\right]\bar{b}_n \bar{z}^n\right| \\
&\quad \geq (2-\alpha)|z|-\sum_{n=2}^{\infty} n\left[n(k+1)+1-k-\alpha\right]\left|a_n\right||z|^n \\
&\qquad -\sum_{n=1}^{\infty} n\left[n(k+1)-1+k+\alpha\right]\left|b_n\right||z|^n \\
&\qquad -\alpha|z|-\sum_{n=2}^{\infty} n\left[n(k+1)-1-k-\alpha\right]\left|a_n\right||z|^n \\
&\qquad -\sum_{n=1}^{\infty} n\left[n(k+1)+1+k+\alpha\right]\left|b_n\right||z|^n \\
&\quad \geq 2(1-\alpha)|z|\left\{1-\sum_{n=2}^{\infty}\frac{n\left[n(k+1)-k-\alpha\right]}{1-\alpha}\left|a_n\right|\right. \\
&\qquad\qquad \left. -\sum_{n=1}^{\infty}\frac{n\left[n(k+1)+k+\alpha\right]}{1-\alpha}\left|b_n\right|\right\} \geq 0, \quad \text{by (2.1)}.
\end{aligned} \tag{2.4}$$

The harmonic functions

$$f(z) = z+\sum_{n=2}^{\infty}\frac{1-\alpha}{n(nk+n-k-\alpha)}x_n z^n+\sum_{n=1}^{\infty}\frac{1-\alpha}{n(nk+n+k+\alpha)}\bar{y}_n\bar{z}^n, \tag{2.5}$$

where $\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n| = 1$, show that the coefficient bound given in Theorem 2.1 is sharp.

The functions of the form (2.5) are in $\mathcal{HCV}(k,\alpha)$ because

$$\sum_{n=2}^{\infty}\frac{n(n+nk-k-\alpha)}{1-\alpha}\left|a_n\right|+\sum_{n=1}^{\infty}\frac{n(n+nk+k+\alpha)}{1-\alpha}\left|b_n\right| = \sum_{n=2}^{\infty}\left|x_n\right|+\sum_{n=1}^{\infty}\left|y_n\right| = 1. \tag{2.6}$$

$\square$

Next we show that the bound (2.1) is also necessary for functions in $\overline{\mathcal{HCV}}(k,\alpha)$.

Theorem 2.2. *Let $f = h+\bar{g}$ with h and g of the form (1.4). Then $f \in \overline{\mathcal{HCV}}(k,\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{n(n+nk-k-\alpha)}{1-\alpha}|a_n| + \sum_{n=1}^{\infty} \frac{n(n+nk+k+\alpha)}{1-\alpha}|b_n| \le 1. \tag{2.7}$$

Proof. In view of Theorem 2.1, we only need to show that $f \notin \overline{\mathcal{HCV}}(k,\alpha)$ if condition (2.7) does not hold. We note that a necessary and sufficient condition for $f = h+\bar{g}$ given by (1.4) to be in $\overline{\mathcal{HCV}}(k,\alpha)$ is that the coefficient condition (1.7) to be satisfied. Equivalently, we must have

$$\Re \frac{(1-\alpha)zh'(z)+(1+ke^{i\phi})z^2h''(z)+(1+\alpha+2ke^{i\phi})\overline{zg'(z)}+(1+ke^{i\phi})\overline{z^2g''(z)}}{zh'(z)-\overline{zg'(z)}} \ge 0. \tag{2.8}$$

Upon choosing the values of z on the positive real axis where $0 \le z = r < 1$, the above inequality reduces to

$$\frac{1-\alpha-\{\sum_{n=2}^{\infty} n(nk+n-k-\alpha)|a_n| + \sum_{n=1}^{\infty} n(nk+n+k+\alpha)|b_n|\}r^{n-1}}{1-\sum_{n=2}^{\infty} n|a_n|r^{n-1}+\sum_{n=1}^{\infty} n|b_n|r^{n-1}} \ge 0. \tag{2.9}$$

If condition (2.7) does not hold then the numerator in (2.9) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in $(0,1)$ for which the quotient (2.9) is negative. This contradicts the required condition for $f \in \overline{\mathcal{HCV}}(k,\alpha)$ and so the proof is complete. □

3. Inclusion relations.

As mentioned earlier in the proof of Theorem 2.1, the functions in $\overline{\mathcal{HCV}}(k,\alpha)$ are convex harmonic in Δ. In the following example we show that this inclusion is proper.

Example 3.1. Consider the harmonic functions

$$f_n(z) = z - \frac{1}{2}\bar{z} - \frac{1}{2n^2}\bar{z}^n, \quad z \in \Delta, \ n = 2,3,\ldots. \tag{3.1}$$

For $a_n \equiv 0$ and $b_n = -1/2n^2$, we observe that

$$\sum_{n=2}^{\infty} n^2|a_n| + \sum_{n=1}^{\infty} n^2|b_n| = \frac{1}{2} + n^2\left(\frac{1}{2n^2}\right) = \frac{1}{2}+\frac{1}{2} = 1. \tag{3.2}$$

Therefore, by Theorem 1.1, $f_n \in \overline{\mathcal{HK}}(0)$.

On the other hand,

$$\frac{2k+1+\alpha}{1-\alpha}\left|-\frac{1}{2}\right| + \frac{n(nk+n+k+\alpha)}{1-\alpha}\left|-\frac{1}{2n}\right| = \frac{2k+1+\alpha}{2(1-\alpha)} + \frac{nk+n+k+\alpha}{2n(1-\alpha)} > 1. \tag{3.3}$$

Thus, by Theorem 2.2, $f \notin \overline{\mathcal{HCV}}(k,\alpha)$.

More generally, we can prove the following theorem.

Theorem 3.2. *Let $0 \le k < \infty$, $0 \le \alpha < 1$, and $0 \le \beta < 1$. If $k > \beta/(1-\beta)$ then the proper inclusion relation $\overline{\mathcal{HCV}}(k,\alpha) \subset \overline{\mathcal{HK}}(\beta)$.*

Proof. Let $f \in \overline{\mathcal{HCV}}(k,\alpha)$, then, by Theorem 2.2,

$$\sum_{n=2}^{\infty} \frac{n(nk+n-k-\alpha)}{1-\alpha}|a_n| + \sum_{n=1}^{\infty} \frac{n(nk+n+k+\alpha)}{1-\alpha}|b_n| \leq 1. \tag{3.4}$$

Since $(n-\beta)/(1-\beta) < (nk+n-k-\alpha)/(1-\alpha)$ and $(n+\beta)/(1-\beta) < (nk+n+k+\alpha)/(1-\alpha)$, by Theorem 1.1, we conclude that $f \in \overline{\mathcal{HK}}(\beta)$.

To show that the inclusion is proper, consider the harmonic functions

$$f_n(z) = z - \frac{1-\beta}{2(1+\beta)}\bar{z} - \frac{1-\beta}{2n(n+\beta)}\bar{z}^n, \quad z \in \Delta,\ n = 2,3,\dots. \tag{3.5}$$

By Theorem 1.1, $f_n \in \overline{\mathcal{HK}}(\beta)$, because

$$\sum_{n=2}^{\infty} \frac{n(n-\beta)}{1-\beta}|a_n| + \sum_{n=1}^{\infty} \frac{n(n+\beta)}{1-\beta}|b_n| = \frac{1+\beta}{1-\beta}\frac{1-\beta}{2(1+\beta)} + \frac{n(n+\beta)}{1-\beta}\frac{1-\beta}{2n(n+\beta)} = 1. \tag{3.6}$$

On the contrary, by Theorem 2.2, $f_n \notin \overline{\mathcal{HCV}}(k,\alpha)$, because

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n(nk+n+k+\alpha)}{1-\alpha}|b_n| &= \frac{1+\alpha+2k}{1-\alpha}\frac{1-\beta}{2(1+\beta)} + \frac{n(n+\alpha+(n+1)k)}{1-\alpha}\frac{1-\beta}{2n(n+\beta)} \\
&= \frac{1-\beta}{2(1-\alpha)}\left\{\frac{1+\alpha+2k}{1+\beta} + \frac{n+\alpha+(n+1)k}{n+\beta}\right\} \\
&> \frac{1-\beta}{2(1-\alpha)}\left\{\frac{1+\alpha+2\beta/(1-\beta)}{1+\beta} + \frac{n+\alpha+(n+1)\beta/(1-\beta)}{n+\beta}\right\} \\
&= \frac{1}{2(1-\alpha)}\left\{2 + \frac{\alpha(1-\beta)(n+1+2\beta)}{(1+\beta)(n+\beta)}\right\} \geq 1.
\end{aligned} \tag{3.7}$$

□

4. Extreme points and distortion bounds. Using definition (1.7), and according to the arguments given in [3], we obtain the following extreme points of the closed convex hulls of $\overline{\mathcal{HCV}}(k,\alpha)$ denoted by $\overline{\text{clco}}\,\overline{\mathcal{HCV}}(k,\alpha)$.

Theorem 4.1. *Let f be the form of (1.4). Then $f \in \overline{\text{clco}}\,\overline{\mathcal{HCV}}(k,\alpha)$ if and only if $f(z) = \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n)$ where $h_1(z) = z$, $h_n(z) = z - ((1-\alpha)/n(n+nk-k-\alpha))z^n$ $(n = 2,3,\dots)$, $g_n(z) = z - ((1-\alpha)/n(n+nk+k+\alpha))\bar{z}^n$ $(n = 1,2,3,\dots)$, $\sum_{n=1}^{\infty}(X_n+Y_n) = 1$, $X_n \geq 0$ and $Y_n \geq 0$. In particular, the extreme points of $\overline{\mathcal{HCV}}(k,\alpha)$ are $\{h_n\}$ and $\{g_n\}$.*

Similarly, follows the distortion bounds for functions in $\overline{\mathcal{HCV}}(k,\alpha)$.

Theorem 4.2. *If $f \in \overline{\mathcal{HCV}}(k,\alpha)$ then*

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r + \frac{1}{2}\left(\frac{1-\alpha}{2+k-\alpha} - \frac{1+2k+\alpha}{2+k-\alpha}|b_1|\right)r^2, \quad |z| = r < 1, \\
|f(z)| &\geq (1-|b_1|)r - \frac{1}{2}\left(\frac{1-\alpha}{2+k-\alpha} - \frac{1+2k+\alpha}{2+k-\alpha}|b_1|\right)r^2, \quad |z| = r < 1.
\end{aligned} \tag{4.1}$$

If we let $r \to 1$ in the left-hand inequality of Theorem 4.2 and collect the like terms, we obtain the following theorem.

Theorem 4.3. *If* $f \in \overline{\mathcal{HCV}}(k,\alpha)$ *then* $\{w : |w| < (3+2k-\alpha)/2(2+k-\alpha) - 3(1-\alpha)/2(2+k-\alpha)|b_1|\} \subset f(\Delta)$.

5. Convolutions and convex combinations. For harmonic functions $f(z) = z - \sum_{n=2}^{\infty}|a_n|z^n - \sum_{n=1}^{\infty}|b_n|\bar{z}^n$ and $F(z) = z - \sum_{n=2}^{\infty}|A_n|z^n - \sum_{n=1}^{\infty}|B_n|\bar{z}^n$, we define the convolution of f and F as

$$(f*F)(z) = f(z)*F(z) = z - \sum_{n=2}^{\infty}|a_n||A_n|z^n - \sum_{n=1}^{\infty}|b_n||B_n|\bar{z}^n. \tag{5.1}$$

In the following theorem we examine the convolution properties of the class $\overline{\mathcal{HCV}}(k,\alpha)$.

Theorem 5.1. *For* $0 \le \alpha \le \beta < 1$, *let* $f \in \overline{\mathcal{HCV}}(k,\beta)$ *and* $F \in \overline{\mathcal{HCV}}(k,\alpha)$ *then*

$$f*F \in \overline{\mathcal{HCV}}(k,\beta) \subset \overline{\mathcal{HCV}}(k,\alpha). \tag{5.2}$$

Proof. Express the convolution of f and F as that given by (5.1) and note that $|A_n| \le 1$ and $|B_n| \le 1$. Now the theorem follows upon the application of the required condition (2.7). □

The convex combination properties of the class $\overline{\mathcal{HCV}}(k,\alpha)$ is given in the following theorem.

Theorem 5.2. *The class* $\overline{\mathcal{HCV}}(k,\alpha)$ *is closed under convex combinations.*

Proof. For $i = 1,2,\dots$, suppose that $f_i \in \overline{\mathcal{HCV}}(k,\alpha)$ where f_i is given by $f_i(z) = z - \sum_{n=2}^{\infty}|a_{i_n}|z^n - \sum_{n=1}^{\infty}|b_{i_n}|\bar{z}^n$. For $\sum_{i=1}^{\infty}t_i = 1$, $0 \le t_i \le 1$, the convex combinations of f_i may be written as

$$\sum_{i=1}^{\infty}t_if_i(z) = z - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i|a_{i_n}|\right)z^n - \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i|b_{i_n}|\right)\bar{z}^n. \tag{5.3}$$

Now, the theorem follows by (2.7) upon noting that $\sum_{i=1}^{\infty}t_i = 1$. □

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