IJMMS 29:9 (2002) 545–553
PII. S0161171202007858
http://ijmms.hindawi.com

ON A CLASS OF DIOPHANTINE EQUATIONS

SAFWAN AKBIK

Received 10 June 2001

Cohn (1971) has shown that the only solution in positive integers of the equation $Y(Y+1)(Y+2)(Y+3) = 2X(X+1)(X+2)(X+3)$ is $X = 4$, $Y = 5$. Using this result, Jeyaratnam (1975) has shown that the equation $Y(Y+m)(Y+2m)(Y+3m) = 2X(X+m)(X+2m)(X+3m)$ has only four pairs of nontrivial solutions in integers given by $X = 4m$ or $-7m$, $Y = 5m$ or $-8m$ provided that m is of a specified type. In this paper, we show that if $m = (m_1,m_2)$ has a specific form then the nontrivial solutions of the equation $Y(Y+m_1)(Y+m_2)(Y+m_1+m_2) = 2X(X+m_1)(X+m_2)(X+m_1+m_2)$ are m times the primitive solutions of a similar equation with smaller m's. Then we specifically find all solutions in integers of the equation in the special case $m_2 = 3m_1$.

2000 Mathematics Subject Classification: 11D25, 11D45, 11D09, 11D41.

We generalize the equations of Cohn [1] and Jeyaratnam [2] by considering the Diophantine equation

$$Y(Y+m_1)(Y+m_2)(Y+m_1+m_2) = 2X(X+m_1)(X+m_2)(X+m_1+m_2). \tag{1}$$

The trivial solutions of (1) are the sixteen pairs obtained by equating both sides of the equation to zero. A nontrivial solution with $(X,Y,m_1,m_2) = 1$ is called a primitive solution.

Theorem 1. *If every prime p dividing $m = (m_1,m_2)$ is such that*

$$p \equiv 2,3,5 \pmod 8 \quad \text{or} \quad p \equiv 1 \pmod 8 \quad \text{with } 2^{(p-1)/4} \equiv -1 \pmod p, \tag{2}$$

then every nontrivial solution of (1) is m times a primitive solution of

$$Y\left(Y+\frac{m_1}{m}\right)\left(Y+\frac{m_2}{m}\right)\left(Y+\frac{m_1+m_2}{m}\right) = 2X\left(X+\frac{m_1}{m}\right)\left(X+\frac{m_2}{m}\right)\left(X+\frac{m_1+m_2}{m}\right). \tag{3}$$

Theorem 2. *If every prime p dividing N is of the form (2), then every nontrivial solution of*

$$Y(Y+N)(Y+cN)(Y+(1+c)N) = 2X(X+N)(X+cN)(X+(1+c)N) \tag{4}$$

is N times a nontrivial solution of

$$Y(Y+1)(Y+c)(Y+1+c) = 2X(X+1)(X+c)(X+1+c), \tag{5}$$

where c is a positive integer.

Theorem 3. *The equation*

$$Y(Y+1)(Y+3)(Y+4) = 2X(X+1)(X+3)(X+4) \tag{6}$$

has only four pairs of nontrivial solutions in integers given by $X = 14$ or -18, $Y = 17$ or -21.

Theorem 4. *If every prime p dividing N is of the form (2), then the equation*

$$Y(Y+N)(Y+3N)(Y+4N) = 2X(X+N)(X+3N)(X+4N) \tag{7}$$

has only four pairs of nontrivial solutions in integers given by $X = 14N$ or $-18N$, $Y = 17N$ or $-21N$.

Note that Theorem 2 follows immediately by applying Theorem 1 with $m_1 = N$, $m_2 = cN$, and $m = (N, cN) = N$. Also Theorem 4 follows easily by combining Theorem 2, in the case $c = 3$, with Theorem 3.

Lemma 5. *Every solution of (1) that is not primitive is $K = (X, Y, m_1, m_2)$ times a primitive solution of*

$$Y\left(Y+\frac{m_1}{K}\right)\left(Y+\frac{m_2}{K}\right)\left(Y+\frac{m_1+m_2}{K}\right) = 2X\left(X+\frac{m_1}{K}\right)\left(X+\frac{m_2}{K}\right)\left(X+\frac{m_1+m_2}{K}\right). \tag{8}$$

Proof. Suppose that X, Y is a solution of (1). By dividing both sides of that equation by K^4 we find

$$\begin{aligned} &\frac{Y}{K}\left(\frac{Y}{K}+\frac{m_1}{K}\right)\left(\frac{Y}{K}+\frac{m_2}{K}\right)\left(\frac{Y}{K}+\frac{m_1+m_2}{K}\right) \\ &\qquad = 2\cdot\frac{X}{K}\left(\frac{X}{K}+\frac{m_1}{K}\right)\left(\frac{X}{K}+\frac{m_2}{K}\right)\left(\frac{X}{K}+\frac{m_1+m_2}{K}\right). \end{aligned} \tag{9}$$

Thus X/K, Y/K is a solution of (8). The lemma follows since $(X/K, Y/K, m_1/K, m_2/K) = 1$. □

Lemma 6. *Equation (1) cannot have a primitive solution if the greatest common divisor $m = (m_1, m_2)$ is divisible by a prime p of the form (2).*

Proof. By completing the squares in (1) we find

$$\left[\frac{(2Y+m_1+m_2)^2-m_1^2-m_2^2}{2}\right]^2 - 2\left[\frac{(2X+m_1+m_2)^2-m_1^2-m_2^2}{2}\right]^2 = -m_1^2m_2^2. \tag{10}$$

Letting

$$y = 2Y+m_1+m_2, \tag{11}$$

$$x = 2X+m_1+m_2, \tag{12}$$

$$\begin{aligned} A &= \frac{y^2-m_1^2-m_2^2}{2} = 2Y^2+2Y(m_1+m_2)+m_1m_2, \\ B &= \frac{x^2-m_1^2-m_2^2}{2} = 2X^2+2X(m_1+m_2)+m_1m_2, \end{aligned} \tag{13}$$

we obtain the equations

$$y^2 = 2A + m_1^2 + m_2^2, \qquad x^2 = 2B + m_1^2 + m_2^2, \tag{14}$$

$$A^2 - 2B^2 = -m_1^2 m_2^2. \tag{15}$$

If $2 \mid m$, then

$$\begin{aligned} A^2 - 2B^2 = -m_1^2 m_2^2 \Longrightarrow A, B \equiv 0 \pmod 4) \underset{\text{by (13)}}{\Longrightarrow} 2X^2, 2Y^2 \equiv 0 \pmod 4) \\ \Longrightarrow X, Y \equiv 0 \pmod 2) \Longrightarrow 2 \mid (X, Y, m_1, m_2) \neq 1. \end{aligned} \tag{16}$$

Let $p \mid m$ such that $p \equiv 3, 5 \pmod 8$. Assume that $p \nmid A$, then by (15), $p \nmid B$. Also by (15), $1 = (2B^2/p) = (2/p) = -1$, a contradiction. Thus $p \mid A$ and hence $p \mid B$. By (13), $p \mid X$ and Y. Therefore $(X, Y, m_1, m_2) \neq 1$.

Suppose that $p \mid m$ such that $p \equiv 1 \pmod 8$ and $2^{(p-1)/4} \equiv -1 \pmod p$. If $p \nmid A$, then $p \nmid B$. Since $(2/p) = 1$, (13) implies that A and B are quadratic residues mod p. Thus $B^{(p-1)/2} \equiv A^{(p-1)/2} \equiv 1 \pmod p$. From (15) we find that

$$2B^2 \equiv A^2 \pmod p \Longrightarrow 2^{(p-1)/4} B^{(p-1)/2} \equiv A^{(p-1)2} \Longrightarrow 2^{(p-1)/4} \equiv 1 \pmod p, \tag{17}$$

a contradiction. Therefore $p \mid A, B$. By (13), $p \mid X, Y$ and hence $(X, Y, m_1, m_2) \neq 1$ and the lemmas follows. □

Proof of Theorem 1. By Lemmas 5 and 6 and the fact that $(m_1/K, m_2/K)$ can only have prime divisors of the form (2), a nontrivial solution of (2) is a multiple of a primitive solution of (3) with $(m_1/K, m_2/K) = 1$. This happens when $K = (m_1, m_2) = m$ and the theorem follows. □

For Theorem 3 we now prove the following lemma.

Lemma 7. *The only solution in positive integers of (6) is $X = 14$, $Y = 17$.*

Proof. Note that (6) can be obtained from (1) by letting $m_1 = 1$ and $m_2 = 3$. Then (11), (12), (13), (14), and (15) become

$$y = 2Y + 4, \qquad x = 2X + 4, \tag{18}$$

$$A = 2Y^2 + 8Y + 3, \qquad B = 2X^2 + 8X + 3, \tag{19}$$

$$y^2 = 2A + 10, \qquad x^2 = 2B + 10, \tag{20}$$

$$A^2 - 2B^2 = -9. \tag{21}$$

All solutions in positive integers of (21) are given by

$$A = V_n, \qquad B = U_n, \tag{22}$$

where

$$V_n + \sqrt{2} U_n = (3 + 3\sqrt{2})(3 + 2\sqrt{2})^n = 3(1 + \sqrt{2})^{2n+1}, \quad n = 0, 1, 2, \ldots. \tag{23}$$

Thus

$$V_n = \frac{3(1+\sqrt{2})^{2n+1} + 3(1-\sqrt{2})^{2n+1}}{2}, \qquad U_n = \frac{3(1+\sqrt{2})^{2n+1} - 3(1-\sqrt{2})^{2n+1}}{-2\sqrt{2}}. \tag{24}$$

Let $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$, then

$$\alpha+\beta = 2, \qquad \alpha-\beta = -2\sqrt{2}, \qquad \alpha\beta = -1, \qquad V_n = 3\left(\frac{\alpha^{2n+1}+\beta^{2n+1}}{\alpha+\beta}\right), \qquad U_n = 3\left(\frac{\alpha^{2n+1}-\beta^{2n+1}}{\alpha-\beta}\right). \tag{25}$$

From (20) and (22), we must have

$$y^2 = 2V_n + 10, \tag{26}$$

$$x^2 = 2U_n + 10. \tag{27}$$

Using (25), we can easily find that

$$V_{-n} = -V_{n-1}, \tag{28}$$

$$U_{-n} = U_{n-1}, \tag{29}$$

$$U_{n+2} = 6U_{n+1} - U_n, \tag{30}$$

$$V_{n+2} = 6V_{n+1} - V_n. \tag{31}$$

Let

$$\eta_r = \frac{\alpha^r+\beta^r}{\alpha+\beta}, \qquad \xi_r = \frac{\alpha^r-\beta^r}{\alpha-\beta}, \tag{32}$$

then we easily find that

$$V_n = 3\eta_{2n+1}, \qquad U_n = 3\xi_{2n+1}, \tag{33}$$

$$\xi_{2r} = 2\xi_r\eta_r, \tag{34}$$

$$\eta_{2r} = 2\eta_r^2 + (-1)^{r+1} = 4\xi_r^2 + (-1)^r, \tag{35}$$

$$\eta_{m+n} = \eta_m\eta_n + 2\xi_m\xi_n, \tag{36}$$

$$\xi_{m+n} = \xi_m\eta_n + \xi_n\eta_m. \tag{37}$$

Using relations (33), (34), (35), (36), and (37), we get

$$V_{n+r} \equiv (-1)^{r+1}V_n \pmod{\eta_r}, \tag{38}$$

$$V_{n+2r} \equiv V_n \pmod{\eta_r}, \tag{39}$$

$$U_{n+r} \equiv (-1)^{r+1}U_n \pmod{\eta_r}, \tag{40}$$

$$U_{n+2r} \equiv U_n \pmod{\eta_r}, \tag{41}$$

$$\eta_{3r} = \eta_r\left[4\eta_r^2 + 3(-1)^{r+1}\right], \tag{42}$$

$$\xi_{3r} = \xi_r\left[4\eta_r^2 + (-1)^{r+1}\right]. \tag{43}$$

Let

$$\theta_t = \xi_{2^t}, \qquad \phi_t = \eta_{2^t}, \tag{44}$$

then we get

$$\theta_{t+1} = 2\theta_t \phi_t, \tag{45}$$

$$\phi_{t+1} = 2\phi_t^2 - 1 = 4\theta_t^2 + 1 = \phi_t^2 + 2\theta_t^2, \tag{46}$$

$$\phi_t^2 = 2\theta_t^2 + 1. \tag{47}$$

Using (42), (43), and (44), we find that for $k = 2^t$ we have

$$\eta_{6k} = \phi_{t+1}[4\phi_{t+1}^2 - 3], \tag{48}$$

$$\xi_{6k} = \theta_{t+1}[4\phi_{t+1}^2 - 1]. \tag{49}$$

We will need some of the entries in Tables 1 and 2.

TABLE 1

n	U_n	V_n
1	15	21
3	507	717
4	2955	4179
11	675176043	954843117
8	3410067	4822563
23	1037608383669414483	1467399848617311837
24	6047624848242867123	8552633080529593443

TABLE 2

k	η_k
2	3
3	7
4	17
6	$3^2 \cdot 11$
8	577
12	$17 \cdot 1153$
24	$97 \cdot 577 \cdot 13729$
48	$193 \cdot 9188923201 \cdot 665857$

Now we consider the following cases.

(a) Equation (26) is impossible if $n \equiv 1 \pmod 3$. Let $n = 1 + 3r$ where $r \geq 0$, then using (38) we get

$$\begin{aligned} V_n &\equiv \pm V_1 \pmod{\eta_3}, \\ V_n &\equiv \pm 21 \equiv 0 \pmod 7. \end{aligned} \tag{50}$$

Hence $2V_n + 10 \equiv 10 \equiv 3 \pmod 7$. Since $(3/7) = -1$, (26) is impossible.

(b) Equation (27) is impossible if $n \equiv 1, 2 \pmod 4$. Using (40), we get

$$\begin{aligned} U_n &\equiv \pm U_1, \pm U_2 \pmod{\eta_4}, \\ U_n &\equiv \pm 15, \pm 87 \equiv \pm 2 \pmod{17}. \end{aligned} \tag{51}$$

Hence $2U_n + 10 \equiv \pm 4 + 10 \equiv 6, -3 \pmod{17}$. Since $(6/17) = (-3/17) = -1$, (27) is impossible.

(c) Equation (26) is impossible if $n \equiv 8 \pmod{12}$. Using (39) and (28) we get

$$\begin{aligned} V_n &\equiv V_{-4} = -V_3 \pmod{\eta_6}, \\ V_n &\equiv -717 \equiv -2 \pmod{11} \quad \text{since } 11 \mid \eta_6. \end{aligned} \tag{52}$$

Hence $2V_n + 10 \equiv 6 \pmod{11}$. Since $(6/11) = -1$, (26) is impossible.

(d) Equation (26) is impossible if $n \equiv 11 \pmod{16}$. Using (39) and (28) we get

$$\begin{aligned} V_n &\equiv V_{-5} = -V_4 \pmod{\eta_8}, \\ V_n &\equiv -4179 \equiv -140 \pmod{577}. \end{aligned} \tag{53}$$

Hence $2V_n + 10 \equiv -270 \pmod{577}$. Since $(-270/577) = -1$, (26) is impossible.

(e) Equation (26) is impossible if $n \equiv 11, 12 \pmod{24}$. Using (38) and (28) we get

$$\begin{aligned} V_n &\equiv \pm V_{11}, \pm V_{-12} = \pm V_{11}, \mp V_{11} \pmod{\eta_{24}}, \\ V_n &\equiv \pm 954843117 \equiv \pm 46 \pmod{97} \quad \text{since } 97 \mid \eta_{24}. \end{aligned} \tag{54}$$

Hence $2V_n + 10 \equiv \pm 102 + 10 \equiv 5, 15 \pmod{97}$. Since $(5/97) = (15/97) = -1$, (26) is impossible.

(f) Equation (26) is impossible if $n \equiv 15 \pmod{24}$. Using (38) and (28) we get

$$\begin{aligned} V_n &\equiv \pm V_{-9} = \mp V_8 \pmod{\eta_{24}}, \\ V_n &\equiv \mp 4822563 \equiv \pm 504289 \pmod{1331713} \quad \text{since } 1331713 \mid \eta_{24}. \end{aligned} \tag{55}$$

Hence $2V_n + 10 \equiv 323145, 1008588 \pmod{1331713}$. Since $(323145/1331713) = (1008588/1331713) = -1$, (26) is impossible.

(g) Equation (26) is impossible if $n \equiv 23, 24 \pmod{48}$. Using (38) and (28) we get

$$V_n \equiv \pm V_{23}, \pm V_{-24} = \pm V_{23}, \mp V_{23} \pmod{\eta_{48}}. \tag{56}$$

Since $V_{23} = 1467399848617311837$ and $\tau = 9188923201 \mid \eta_{48}$, we have $2V_n + 10 \equiv 11299978, -11299958 \pmod{\tau}$. Since $(11299978/\tau) = (-11299958/\tau) = -1$, (26) is impossible.

(h) Equation (27) is impossible if $n \equiv 3 \pmod{48}$, $n \neq 3$. That is, $n = 3 + 3 \cdot 2^t \cdot r$, where $t \geq 4$ and r is an odd positive integer. Using (40) we get $U_n \equiv -U_3 = -507 \pmod{\eta_{3\cdot 2^t}}$. Hence

$$2U_n + 10 \equiv -1004 \pmod{\eta_{3\cdot 2^t}}. \tag{57}$$

From (48) we get $\eta_{3\cdot 2^t} = \eta_{6\cdot 2^{t-1}} = \phi_t[4\phi_t^2 - 3]$. Using this in (57) we simultaneously get

$$\begin{aligned} 2U_n + 10 &= -1004 \pmod{\phi_t}, \\ 2U_n + 10 &= -1004 \pmod{4\phi_t^2 - 3}. \end{aligned} \tag{58}$$

Since $\phi_{t+1} = 2\phi_t^2 - 1$ and $\phi_3 = 577$ we can easily show, by induction, the following for $t \ge 3$

$$\phi_t \equiv 1 \pmod 8, \tag{59}$$

$$\phi_t \equiv 81, 69, -17, 75, -46, -36 \pmod{251}, \tag{60}$$

when

$$t \equiv 0,1,2,3,4,5 \pmod 6, \tag{61}$$

respectively. By (59) we get

$$\left(\frac{-1004}{\phi_t}\right) = \left(\frac{-1}{\phi_t}\right)\left(\frac{4}{\phi_t}\right)\left(\frac{251}{\phi_t}\right) = (1)(1)\left(\frac{\phi_t}{251}\right) = \left(\frac{\phi_t}{251}\right). \tag{62}$$

Similarly $(-1004/(4\phi_t^2-3)) = ((4\phi_t^2-3)/251)$. Using (60) we find that $(\phi_t/251) = -1$ if $t \equiv 2,5 \pmod 6$ and $((4\phi_t^2-3)/251) = -1$ if $t \equiv 0,1,3,4 \pmod 6$. Therefore (27) is always impossible in this case.

Note that for $n = 3$ we have $U_3 = 507$ and $V_3 = 717$. Now (22) and (19) imply that $X = 14$, $Y = 17$, a nontrivial solution of (6).

(i) Equation (27) is impossible if $n \equiv \delta \pmod{48}$ and $n > 0$, where $\delta = 0, -1$. That is $n = \delta + 3k(2r+1) = \delta + 6kr + 3k$, where $k = 2^t$, $t \ge 4$, and $r \ge 0$. Using (40) and (33) we get

$$U_n \equiv \pm U_{3k+\delta} = \pm 3\xi_{6k+2\delta+1} \pmod{\eta_{6k}}. \tag{63}$$

The upper and the lower signs depend on whether r is even or odd. Using (37), we get

$$\xi_{6k+2\delta+1} = \xi_{6k}\eta_{2\delta+1} + \xi_{2\delta+1}\eta_{6k}, \tag{64}$$

where $\eta_{2\delta+1} = 1, -1$ for $\delta = 0, -1$ and $\xi_{2\delta+1} = 1$ for $\delta = 0, 1$. Now (64) becomes $\xi_{6k+2\delta+1} = \pm\xi_{6k} + \eta_{6k}$, where the upper and lower signs depend on whether $\delta = 0$ or $\delta = 1$, respectively. Using this in (63) we get

$$U_n \equiv \pm 3\xi_{6k} \pmod{\eta_{6k}}. \tag{65}$$

For $\delta = 0$, the upper sign holds if r is even and the lower sign holds if r is odd. For $\delta = -1$, upper sign holds if r is odd and the lower sign holds if r is even. Using (48) and (49) in (65) we get

$$U_n \equiv \pm 3\theta_{t+1}\left[4\phi_{t+1}^2 - 1\right] = \pm 3\theta_{t+1}\left[4\phi_{t+1}^2 - 3 + 2\right]\left(\bmod \phi_{t+1}\left[4\phi_{t+1}^2 - 3\right]\right). \tag{66}$$

Therefore we simultaneously get $U_n \equiv \pm 6\theta_{t+1} \pmod{4\phi_{t+1}^2 - 3}$ and $U_n \equiv \mp 3\theta_{t+1} \pmod{\phi_{t+1}}$. Thus

$$\begin{aligned} 2U_n + 10 &\equiv 10 \pm 12\theta_{t+1} \pmod{4\phi_{t+1}^2 - 3}, \\ 2U_n + 10 &\equiv 10 \mp 6\theta_{t+1} \pmod{\phi_{t+1}}. \end{aligned} \tag{67}$$

In what follows we need the fact that

$$\theta_t \equiv 0 \pmod 8, \quad \text{for } t \ge 3, \tag{68}$$

which follows by induction using (45) and $\theta_3 = 408$. Now we show that

$$\left(\frac{10 \pm 12\theta_{t+1}}{4\phi_{t+1}^2 - 3}\right) = \left(\frac{5 \pm 6\theta_{t+1}}{59}\right), \tag{69}$$

$$\left(\frac{10 \mp 6\theta_{t+1}}{\phi_{t+1}}\right) = \pm\left(\frac{10\theta_t \pm 3\phi_t}{59}\right). \tag{70}$$

For (69) we have

$$\begin{aligned}
\left(\frac{10 \pm 12\theta_{t+1}}{4\phi_{t+1}^2 - 3}\right) &= \left(\frac{2}{4\phi_{t+1}^2 - 3}\right)\left(\frac{5 \pm 6\theta_{t+1}}{4\phi_{t+1}^2 - 3}\right) \\
&= \left(\frac{5 \pm 6\theta_{t+1}}{4\phi_{t+1}^2 - 3}\right), \quad \text{using (59)} \\
&= \left(\frac{5 \pm 6\theta_{t+1}}{8\theta_{t+1}^2 + 1}\right), \quad \text{using (47)} \\
&= \left(\frac{8\theta_{t+1}^2 + 1}{5 \pm 6\theta_{t+1}}\right), \quad \text{since } \theta_t \equiv 0 \pmod 4 \\
&= \left(\frac{36(8\theta_{t+1}^2 + 1)}{5 \pm 6\theta_{t+1}}\right) = \left(\frac{236}{5 \pm 6\theta_{t+1}}\right) \\
&= \left(\frac{59}{5 \pm 6\theta_{t+1}}\right), \quad \text{since } 36\theta_{t+1}^2 \equiv 25 \pmod{5 \pm 6\theta_{t+1}}.
\end{aligned} \tag{71}$$

Equation (69) follows since $\theta_t \equiv 0 \pmod 4$. For (70) we have

$$\begin{aligned}
\left(\frac{10 \mp 6\theta_{t+1}}{\phi_{t+1}}\right) &= \left(\frac{5 \mp 3\theta_{t+1}}{\phi_{t+1}}\right) \\
&= \left(\frac{5(\phi_t^2 - 2\theta_t^2) \mp 3\theta_{t+1}}{\phi_t^2 + 2\theta_t^2}\right), \quad \text{using (46) and (47)} \\
&= \left(\frac{-20\theta_t^2 \mp 6\theta_t\phi_t}{\phi_t^2 + 2\theta_t^2}\right), \quad \text{since } \phi_t^2 \equiv -2\theta_t^2 \pmod{\phi_t^2 + 2\theta_t^2} \\
&= \left(\frac{-1}{\phi_t^2 + 2\theta_t^2}\right)\left(\frac{2}{\phi_t^2 + 2\theta_t^2}\right)\left(\frac{\theta_t}{\phi_t^2 + 2\theta_t^2}\right)\left(\frac{10\theta_t \pm 3\phi_t}{\phi_t^2 + 2\theta_t^2}\right) \\
&= (1)(1)(1)\left(\frac{10\theta_t \pm 3\phi_t}{\phi_t^2 + 2\theta_t^2}\right) \\
&= \left(\frac{\phi_t^2 + 2\theta_t^2}{10\theta_t \pm 3\phi_t}\right) = \left(\frac{9\phi_t^2 + 18\theta_t^2}{10\theta_t \pm 3\phi_t}\right) \\
&= \left(\frac{118\theta_t^2}{10\theta_t \pm 3\phi_t}\right), \quad \text{since } 9\phi_t^2 \equiv 100\theta_t^2 \pmod{10\theta_t \pm 3\phi_t} \\
&= \left(\frac{2}{10\theta_t \pm 3\phi_t}\right)\left(\frac{59}{10\theta_t \pm 3\phi_t}\right) = -\left(\frac{59}{10\theta_t \pm 3\phi_t}\right).
\end{aligned} \tag{72}$$

Equation (70) follows using (59) and (68).

Since $\theta_3 = 408$, $\phi_3 = 577$, $\phi_{t+1} = 2\phi_t^2 - 1$, and $\theta_{t+1} = 2\theta_t\phi_t$, we can inductively show the following:

$$\begin{aligned} \theta_t &\equiv 12, 5, -12, -5 \pmod{59} \quad \text{if } t \equiv 0,1,2,3 \pmod 4, \\ \phi_t &\equiv -17, -13 \pmod{59} \quad \text{if } t \equiv 0,1, \pmod 2, \text{ respectively.} \end{aligned} \tag{73}$$

Using (73) and taking the upper signs in (69) and (70), we get

$$\begin{aligned} \left(\frac{5+6\theta_{t+1}}{59}\right) &= -1 \quad \text{if } t \equiv 2,3 \pmod 4, \\ \left(\frac{10\theta_t + 3\phi_t}{59}\right) &= -1 \quad \text{if } t \equiv 0,1,2 \pmod 4. \end{aligned} \tag{74}$$

Thus this case is always impossible. Using the lower signs in (69) and (70) we get

$$\begin{aligned} \left(\frac{5-6\theta_{t+1}}{59}\right) &= -1 \quad \text{if } t \equiv 0,1 \pmod 4, \\ -\left(\frac{10\theta_t - 3\phi_t}{59}\right) &= -1 \quad \text{if } t \equiv 0,2,3 \pmod 4, \end{aligned} \tag{75}$$

and this case is also impossible. Therefore (27) is always impossible.

The only remaining case is $n = 0$. Then $U_0 = V_0 = 0$ and so $X = Y = 0$, a trivial solution and Lemma 7 is proved. □

Proof of Theorem 3. First note that if the pair (X,Y) is a solution of (6), so are $(-X-4, Y)$, $(X, -Y-4)$, and $(-X-4, -Y-4)$. Note also that $-X-4 < -4$ if and only if $X > 0$ and $-Y-4 < -4$ if and only if $Y > 0$. Since $(14, 17)$ is the only solution in positive integers of (6), $(-18, 17)$, $(14, -21)$, $(-18, -21)$ are the only solutions where each of X and Y is either positive or less than -4. The only remaining possibilities for more solutions are where X or $Y \in \{-4, -3, -2, -1, 0\}$ where there are no nontrivial solutions and the proof is completed. □

Finally note that (6) has 16 trivial solutions and 4 nontrivial solutions of a total of only 20 solutions.

References

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Safwan Akbik: Department of Mathematics, Hofstra University, Hempstead, NY 11550, USA