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SOME RESULTS ON COMPOSITION OPERATORS ON ℓ^2

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ABSTRACT. A necessary and sufficient condition for a bounded operator to be a composition operator is investigated in this paper. Normal, quasi-hyponormal, paranormal composition operators are characterised.

KEY WORDS AND PHRASES. Invertible, normal, quasi-normal, hyponormal, quasi-hyponormal, paranormal composition operators.

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1. PRELIMINARIES.

Let N be the set of all non-zero positive integers and ℓ^2 be the Hilbert space of all square-summable sequences. Let ϕ be a mapping from N into itself. Then we define a composition transformation C_ϕ from ℓ^2 into the space of all complex valued sequences by

$$C_\phi f = f o \phi \quad \text{for every} \quad f \varepsilon \ell^2 .$$

In the case C_ϕ is bounded and the range of C_ϕ is in ℓ^2, we call it a composition operator. The symbol $B(\ell^2)$ denotes the Banach algebra of all bounded linear operators on ℓ^2 .

In the first section of this paper, a criterion for a bounded operator to be a composition operator is given. In latter sections, Normal, Quasi-hyponormal, Paranormal composition operators are characterized.

2. CRITERION FOR A BOUNDED OPERATOR TO BE A COMPOSITION OPERATOR.

In this section we obtain a necessary and sufficient condition for a bounded operator A to be a composition operator.

THEOREM 2.1. Let $A \in B(\ell^2)$. Then A is a composition operator if and only if for every $n \in N$, there is an $m \in N$ such that $A^* e^{(n)} = e^{(m)}$, where $e^{(n)}$ is the sequence defined by $e^{(n)}(p) = \delta_{np}$ (the Kronecker delta).

PROOF. Let A be a composition operator on ℓ^2 . Then $A = C_\phi$ for some ϕ . Let $n \in N$. Then $A^* e^{(n)} = C_\phi^* e^{(n)} = e^{(\phi(n))}$ by definition of C_ϕ^* [6] .

Conversely suppose $A^* e^{(n)} = e^{(m)}$. Then define $\phi(n) = m$. ϕ is well defined, since m is unique. Thus $A^* e^{(n)} = e^{(\phi(n))} = C_\phi^* e^{(n)}$ for every $n \in N$. This shows that $A^* = C_\phi^*$ and hence $A = C_\phi$.

3. NORMAL COMPOSITION OPERATORS.

An operator $A \in B(H)$ is normal if A commutes with its adjoint. It is not true in general that every invertible operator is normal. It is true in case of composition operators and is shown in the following theorem.

THEOREM 3.1. Let $C_\phi \in B(\ell^2)$. Then C_ϕ is invertible if and only if C_ϕ is normal.

PROOF. Suppose that C_ϕ is invertible. Then by Theorem 2.3 of [6] C_ϕ is unitary and therefore, it is normal.

Conversely, if C_ϕ is not invertible, then by theorem 2.2 of [6] , ϕ is not invertible, and so either ϕ is not one-to-one or ϕ is not onto.

If ϕ is not one-to-one, then $||C_\phi^* e^{(n)}|| = 1$ and $||C_\phi e^{(n)}|| \geq \sqrt{2}$ for some $n \in N$. And if ϕ is not onto, then $||C_\phi^* e^{(n)}|| = 1$ and $||C_\phi e^{(n)}|| = 0$ for $n \in N \setminus \phi(N)$, where $\phi(N)$ is the range of ϕ. Thus in both the cases, C_ϕ is not normal. This proves the sufficient part.

COROLLARY. Let $n \in N$. Then C_ϕ^n is normal if and only if C_ϕ is normal.

4. QUASI-HYPONORMAL COMPOSITION OPERATORS.

Let $C_\phi \in B(\ell^2)$. Then the measure $\lambda\phi^{-1}$ is absolutely continuous with respect to the measure λ [4]. It is clear that the measure $\lambda(\phi o \phi)^{-1}$ is absolutely continuous with respect to the measure $\lambda\phi^{-1}$.

Let $\frac{d\lambda\phi^{-1}}{d\lambda} = f_o$ (the Radon-Nikodym derivative of the measure $\lambda\phi^{-1}$ with respect to the measure λ.),

$$\frac{d\lambda\ (\phi o \phi)^{-1}}{d\lambda\phi^{-1}} = g_o\ , \quad \text{and} \quad \frac{d\lambda\ (\phi o \phi)^{-1}}{d\lambda} = h_o\ .$$

Then by Theorem A [1, p. 133], $h_o = g_o \cdot f_o$

An operator $A \in B(\ell^2)$ is quasi-hyponormal if $||A^*A\, x|| \leq ||AA\, x||$ for all $x \in \ell^2$. A is called paranormal if $||A\, x||^2 \leq ||A^2 x||$ for all unit vectors x in ℓ^2. It is shown in this section that the class of quasi-hyponormal composition operators coincides with the class of paranormal composition operators.

THEOREM 4.1. Let $C_\phi \in B(\ell^2)$. Then C_ϕ is quasi-hyponormal if and only if $f_o \leq g_o$.

PROOF. Suppose C_ϕ is quasi-hyponormal. Then $||C_\phi^* C_\phi X_E|| \leq ||C_\phi C_\phi X_E||^2$, where X_E is the characteristic function of the set E.

or $$||M_{f_o} X_E||^2 \leq ||X_E o \phi o \phi||^2 \text{ by proof of Th. 3 [3],}$$

or $$\int_N f_o^2 \chi_E d\lambda \le \int_N |\chi_E o\phi o\phi|^2 d\lambda = \int_N \chi_E d\lambda\ (\phi o\phi)^{-1}$$

or $$\int_N f_o^2 \chi_E d\lambda \le \int_N \chi_E h_o d\lambda$$

or $$\int_E (h_o - f_o^2)\ d\lambda \ge 0\ .$$

Since this is true for all $E \subseteq N$, therefore, $h_o \ge f_o^2$. This shows that $f_o \le g_o$.

Conversely, if $f_o \le g_o$, then $f_o^2 \le f_o \cdot g_o = h_o$.

Hence, $$||C_\phi^* C_\phi f||^2 = \int_N |M_{f_o} f|^2\ d\lambda \le \int_N |f|^2 h_o d\lambda$$
$$= \int_N |f|^2 d\lambda\ (\phi o\phi)^{-1} = \int_N |fo\phi o\phi|^2 d\lambda$$
$$= \int_N |C_\phi C_\phi f|^2 d\lambda = ||C_\phi C_\phi f||^2\ .$$

This completes the proof.

THEOREM 4.2. Let $C_\phi \in B(\ell^2)$. Then C_ϕ is quasi-hyponormal if and only if C_ϕ is paranormal.

PROOF. Necessity is true for any bounded operator A . For the sufficiency, if C_ϕ is paranormal and $n \in N$, then

$$||C_\phi \chi_{\{n\}}||^2 \le ||C_\phi C_\phi \chi_{\{n\}}|| \quad \text{for all } n \in N\ .$$

or $$\int |\chi_{\{n\}} o\phi|^2\ d\lambda \le (\int |\chi_{\{n\}}|^2\ d\lambda\ (\phi o\phi)^{-1})^{\frac{1}{2}}$$

or $$\int |\chi_{\{n\}}|^2\ d\lambda\phi^{-1} \le (\int |\chi_{\{n\}}|^2\ d\lambda\ (\phi o\phi)^{-1})^{\frac{1}{2}}$$

or $$\int_{\{n\}} f_o\ d\lambda \le (\int_{\{n\}} h_o\ d\lambda)^{\frac{1}{2}}$$

or $$f_o(n) \le (h_o(n))^{\frac{1}{2}}$$

or $$(f_o(n))^2 \le h_o(n) = g_o(n) \cdot f_o(n) \quad \text{for all } n \in N\ .$$

Hence $f_o \le g_o$. Thus C_ϕ is quasi-hyponormal in view of the previous theorem.

THEOREM 4.3. Let $C_\phi \in B(\ell^2)$ and $\phi : N \to N$ be one-to-one. Then the following are equivalent.

(i) C_ϕ is normal

(ii) C_ϕ is hyponormal

(iii) C_ϕ is quasi-hyponormal .

PROOF. The implications (i) implies (ii), (ii) implies (iii) are true for any bounded operator A . Here we show that (iii) implies (i). For this let C_ϕ be quasi-hyponormal. Then ϕ is onto, because if ϕ is not onto, then taking $X_{\{\phi(n)\}}$ to be the characteristic function of the singleton set $\{\phi(n)\}$ for $n \in N\backslash\phi(N)$ we have

$$||C_\phi^* C_\phi X_{\{\phi(n)\}}|| = 1 \quad \text{and} \quad ||C_\phi C_\phi X_{\{\phi(n)\}}|| = 0$$

which is a contradiction. Since ϕ is one-to-one by hypothesis, therefore ϕ is invertible. By theorem 2.2 [6] C_ϕ is invertible and so by theorem 3.1 C_ϕ is normal.

REMARK. One has the following inclusion relation for classes of operators.

Normal $\subseteq$ Quasi-normal $\subseteq$ Hyponormal $\subseteq$ Quasi-hyponormal .

All the inclusions are proper [2]. We show with the help of examples that these inclusions are also proper for composition operators.

EXAMPLE 1. Quasi-normal but not normal.

Let $\phi : N \to N$ be defined by

$$\phi(n) = \begin{cases} (n+1)/2 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases}$$

Then from theorem 3 of [3] it follows that C_ϕ is quasi-normal since $C_\phi^* C_\phi = M_{f_o} = 2I$, where I is the identity operator. But C_ϕ is not normal in view of theorem 3.1.

EXAMPLE 2. Hyponormal but not quasi-normal.

Let $\phi : N \to N$ be the mapping such that $\phi(1)$ and $\phi(2)$ equal 1 and $\phi(3n+m) = n+1$ for $m = 0,1,2$ and $n \in N$. Then $f_o(n) = 2$ if $n = 1$ and $f_o(n) = 3$ if $n \neq 1$. It is clear that $f_o o\phi \leq f_o$. Hence C_ϕ is hyponormal. But $(f_o o\phi)(2) \neq f_o(2)$, hence C_ϕ is not quasi-normal.

EXAMPLE 3. Quasi-hyponormal but not hyponormal.

Let $\phi : N \to N$ be the mapping such that $\phi(1)$ and $\phi(3)$ equal 2, $\phi(2)$ equals 1 and $\phi(3n+m) = n+2$ for $m = 1,2,3$ and $n \in N$. Then C_ϕ is quasi-hyponormal in view of theorem 4.1. But C_ϕ is not hyponormal because if $x \in \ell^2$ is such that $x(1) = 2$. $x(3) = 1$ and $x(n) = 0$ otherwise, then $||C_\phi^* x|| = 3$ and $||C_\phi x|| = \sqrt{7}$. Thus $||C_\phi^* x|| \not\leq ||C_\phi x||$.

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