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A DISTRIBUTIONAL HARDY TRANSFORMATION

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ABSTRACT: The Hardy's F-transform

$$F(t) = \int_0^\infty F_\nu(ty)\, yf(y)\, dy$$

is extended to distributions. The corresponding inversion formula

$$f(x) = \int_0^\infty C_\nu(tx)\, t\, F(t)dt$$

is shown to be valid in the weak distributional sense. This is accomplished by transferring the inversion formula onto the testing function space for the generalized functions under consideration and then showing that the limiting

process in the resulting formula converges with respect to the topology of the testing function space.

KEY WORDS AND PHRASES. *Integral Transform, Hardy Transform, Hankel Transform, Distributions, Generalized Functions.*

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1. INTRODUCTION.

The Hardy transforms with their inversion formulae are represented by the following two integral equations:

$$f(x) = \int_0^\infty F_\nu(tx)t\,dt \int_0^\infty C_\nu(ty)yf(y)dy \tag{1}$$

and

$$f(x) = \int_0^\infty C_\nu(tx)t\,dt \int_0^\infty F_\nu(ty)y\, f(y)\, dy \tag{2}$$

where

$$C_\nu(z) = \cos p\pi\, J_\nu(z) + \sin p\pi\, Y_\nu(z) \tag{3}$$

and

$$F_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\nu+2p+2m}}{\Gamma(p+m+1)\,\Gamma(p+m+\nu+1)} \tag{4}$$

$$= 2^{2-\nu-2p}\, s_{\nu+2p-1,\ \nu}(z) / \{\Gamma(p)\,\Gamma(\nu+p)\}\ [2, p. 40].$$

The theory of the inversion formulae (1) and (2) has been given by Cooke [1]. The Hankel transform with its inversion formula can be deduced as a special case of both (1) and (2) by taking $p = 0$. The Y-transform [3, p. 93] is a special case of (1) whereas H-transform [3, p. 155] is a special case of (2) for $p = \frac{1}{2}$. Recently the inversion formula (1) was proved to be valid for the generalized function space $H'_\alpha(I)$ by Pathak and Pandey [7] in the weak distributional sense. It turns out that the kernal $y\,F_\nu(ty)$ of F_ν-transform does not belong to the space

$H_\alpha(I)$ and therefore the inversion formula (2) cannot be proved to be valid for the space of distributions directly as a corollary to theorems proved in [7]. We will therefore extend briefly the inversion formula (2) to a generalized function space essentially by following the techniques and results proved in [7].

2. <u>TESTING FUNCTION SPACE</u> $H_{\alpha,\beta}^{\nu,p}(I)$. For $-\frac{1}{2} \leq \nu \leq \frac{1}{2}$ and real p let $F_\nu(z)$ be the function defined in (4) and let α be a fixed number satisfying $\alpha + \nu + 2p \geq 0$. Assume that β is also a fixed number satisfying $\beta \geq \sigma = \max(\nu + 2p - 2, -\frac{1}{2})$. For each $k = 0, 1, 2, \ldots$ define a positive and continuous function $\xi_k(x)$ on $I = \{x;\ 0 < x < \infty\}$ satisfying

$$\xi_k(x) = \begin{matrix} x^{2k+\alpha} & 0 < x \leq 1 \\ x^{-\beta-2} & x > 1. \end{matrix}$$

An infinitely differentiable complex-valued function $\phi(x)$ defined over I is said to belong to the space $H_{\alpha,\beta}^{\nu,p}(I)$ if

$$\nu_k(\phi) \equiv \sup_{0<x<\infty} \left| \xi_k(x)\ \Delta_x^k \left(\frac{\phi(x)}{x}\right) \right| < \infty$$

for each $k = 0, 1, 2, 3, \ldots$ where Δ_x stands for the differentiation operator $\left(D_x^2 + \frac{1}{x} D_x - \frac{\nu^2}{x^2}\right)$, $D_x = \frac{d}{dx}$. It can be readily seen that $H_{\alpha,\beta}^{\nu,p}(I)$ is a vector space. The topology over $H_{\alpha,\beta}^{\nu,p}(I)$ is generated by the sequence of seminorms $\{\nu_k\}_{k=0}^{\infty}$ [9; p. 8].

A sequence $\{\phi_\nu\}$ in this space is said to converge to the element ϕ if

$\gamma_k(\phi_\nu - \phi) \to 0$ as $\nu \to \infty$ for each $k = 0, 1, 2, 3, \ldots$ A sequence ϕ_ν in $H^{\nu,p}_{\alpha,\beta}$ (I) is said to be a Cauchy sequence if $\gamma_k(\phi_m - \phi_n) \to 0$ as m, n $\to \infty$ independently of each other. It is a simple exercise to verify that the space $H^{\nu,p}_{\alpha,\beta}$ (I) is sequentially complete and so it is a Fréchet space. Since $D(I) \subset H^{\nu,p}_{\alpha,\beta}$ (I) and the topology of D(I) is stronger than that induced on D(I) by $H^{\nu,p}_{\alpha,\beta}$ (I), it follows that the restriction of any $f \in H^{\nu,p}_{\alpha,\beta}$ (I) to D(I) is in D' (I).

In view of the fact that

$$\Delta_x^k \left[F_\nu\ (xt) \right] = (-1)^k\ t^{2k}\ F_\nu(xt) - P(x,t)$$

where

$$P(x,t) = t^{\nu+2p} \sum_{i=1}^{k} a_i\ x^{\nu+2p-2i}\ t^{2k-2i} \tag{5}$$

a_k being certain constants depending on ν and p, and the asymptotic orders [9, p. 345]

$$F_\nu(z) = \begin{cases} 0 \quad |z|^{\nu+2p} & |z| \to 0 \\ 0 \quad |z|^{\sigma} & |z| \to \infty \end{cases} \tag{6}$$

where

$$\sigma = \max\ (\nu + 2p - 2, -\tfrac{1}{2}) \qquad [8, \text{pp. } 347, 351]$$

it follows that for fixed $t > 0$, $x F_\nu\ (tx)$ belongs to the space $H^{\nu,p}_{\alpha,\beta}$ (I) when treated as a function of x. Therefore, Hardy's F_ν-transform F(y) of a generalized function $f \in H^{\nu,p}_{\alpha,\beta}$ (I) can be defined by

$$F(y) = \langle f(x),\ x F_\nu\ (xy) \rangle, \qquad y > 0. \tag{7}$$

By following the technique as used in [7] it can be shown that F(y) is differentiable for each $y > 0$ and that

$$F'(y) = \langle f(x),\ \frac{\partial}{\partial y} \left\{ x F_\nu\ (xy) \right\} \rangle \tag{8}$$

Note that

$$\frac{\partial}{\partial y}[x\, F_\nu(xy)] \text{ also belongs to } H^{\nu,p}_{\alpha,\beta}(I).$$

We now state some results which will be used in the sequel.

Define

$$H_N(t,x) = \int_0^N C_\nu(tx)\, C_\nu(xy)\, y\, dy$$

$$= \frac{N}{x^2 - t^2}\left[x\, C_{\nu+1}(xN)\, C_\nu(tN) - t\, C_{\nu+1}(tN)\, C_\nu(xN)\right] - Q(x,t)$$

where

$$Q(x,t) = \frac{2 \sin p\pi}{\pi \sin \nu\pi} \sin(p+\nu)\pi \; \frac{x^{2\nu} - t^{2\nu}}{x^\nu t^\nu (x^2 - t^2)}, \quad x \neq t \tag{10}$$

$$= \frac{2 \sin p\pi \sin(p+\nu)\pi}{\pi \sin \nu\pi} \; \frac{\nu}{x^2} \quad \text{when } t = x$$

[8, p. 466].

Using the technique employed in proving Lemma 2 in [7] it can be proved that for fixed t, $x.Q(x,t) \in H^{\nu,p}_{\alpha,\beta}(I)$.

It is now a simple exercise to prove for $\alpha \geq |\nu|$, $\beta \leq \nu - 4$ and $\phi \in D(I)$ that

$$x \int_a^b Q(x,y)\, \phi(y)\, y\, dy \text{ also belongs to } H^{\nu,p}_{\alpha,\beta}(I).$$

LEMMA 2. Let $\xi_k(t)$ <u>be defined as in section 2</u>. <u>Then</u> for $0 < y < 1$

$$\sup_{0 < t < \infty} \left| \frac{\xi_k(t)}{\xi_k(ty)} \right| = y^{\min(\beta + 2,\; -\alpha - 2k)}, \quad k = 0, 1, 2, \ldots$$

PROOF. The result follows by dividing the t- line into three parts $0 < t < 1$, $1 < t < 1/y$, $1/y < t < \infty$ and considering the corresponding

$$\sup_{0 < t < \infty} \left| \frac{\xi_k(t)}{\xi_k(ty)} \right|$$

LEMMA 3. Let $C_\nu(z)$ be the function as defined in (3) and let $-\frac{1}{2} \le \nu \le \frac{1}{2}$, $-\nu - 2p \le \alpha \le 3/2$, $\beta \ge \max(\nu + 2p - 2, -\frac{1}{2})$. Then for fixed $x > 0$,

$$t \int_0^\eta F_\nu(ty)\, C_\nu(xy)\, y dy \to 0 \text{ in } H^{\nu,p}_{\alpha,\beta}(I) \text{ as } \eta \to 0+$$

PROOF. The lemma can be proved by using lemma 2 and a variation of the technique used in proving lemma 4 of [7].

LEMMA 4. Let α, β, ν and p be restricted as in Lemma 3 and let $f \in H^{\nu,p}_{\alpha,\beta}$, then

$$\int_0^N \langle f(t), t F_\nu(ty) \rangle\, C_\nu(xy)\, y dy = \langle f(t), t \int_0^N F_\nu(ty)\, C_\nu(xy)\, y dy \rangle.$$

PROOF. The result follows in view of Lemma 3. The details of the technique to be used can be found in [7, Lemma 5].

LEMMA 5. Let $b > a > 0$ and $H_N(t,x)$, $Q(t,x)$ be the functions as defined by (9) and (10). Then

$$\lim_{N\to\infty} \int_a^b [\, H_N(t,x) + Q(t,x) \,]\, x dx = \begin{matrix} 1 & t \in [a,b] \\ 0 & t \notin [a,b]. \end{matrix}$$

PROOF. See Lemma 6 in [7].

LEMMA 6. Let the support of $\varphi \in D(I)$ be contained in (a,b) where $b > a > 0$. Let $H_N(t,x)$, $Q(t,x)$ be the functions as defined in (9) and (10). Assume that $-\frac{1}{2} \le \nu \le \frac{1}{2}$, $\max(-\nu - 2p, \nu) \le \alpha \le 3/2$ and $\beta \ge \max(\nu + 2p - 2, -\frac{1}{2})$. Then

$$t \int_a^b \left[H_N(t,x) + Q(x,t) \right] \phi(x)\, x dx \to t\, \phi(t) \text{ in } H^{\nu,p}_{\alpha,\beta}(I) \text{ as } N \to \infty.$$

PROOF. The proof can be given only by using Lemma 3 and a simple variations of the techniques used in proving [7, Lemma 7] and so the details are omitted.

3. INVERSION OF THE DISTRIBUTIONAL

F_ν - transform: We now state and prove our main result.

THEOREM. Let $-\frac{1}{2} \leq \nu \leq \frac{1}{2}$, $\max(-\nu - 2p, |\nu|) \leq \alpha \leq 3/2$ and $\beta \geq \max(\nu + 2p - 2, -\frac{1}{2})$.

Assume that $F(y)$ is the *distributional* F_ν-transform of $f \in H^{\nu,p}_{\alpha,\beta}(I)$ *as defined by* (7). Then

$$\lim_{N \to \infty} \left\langle \int_0^N F(y)\, C_\nu(x,y)\, y\, dy,\ \phi(x) \right\rangle = \langle f, \phi \rangle \quad \text{for each } \phi \in D(I).$$

PROOF. Let the support of ϕ be $[a,b]$ where $b > a > 0$.

Since $F(y)\, C_\nu(xy)\, y$ generates a regular distribution we have

$$\int_0^N \left\langle F(y)\, C_\nu(xy)\, y\, dy,\ \phi(x) \right\rangle = \int_a^b \phi(x)\, dx \int_0^N F(y)\, C_\nu(xy)\, y\, dy$$

$$= \int_a^b \left\langle f(t),\ t \int_0^N F_\nu(ty)\, C_\nu(xy)\, y\, dy \right\rangle \phi(x)\, dx$$

[Lemma 4]

$$= \int_a^b \left\langle f(t),\ t \left\{ H_N(t,x) + Q_N(t,x) \right\} \right\rangle \phi(x)\, dx$$

[7, Lemma 8]

$$= \int_a^b \left\langle f(t),\ t \left\{ H_N(t,x) + Q(t,x) \right\} \right\rangle \phi(x)\, dx$$

for $N > b > 0$ [1, Lemma ε p. 394]

$$= \left\langle f(t), \; t \int_a^b \left\{ H_N(t,x) + Q(t,x) \right\} \right\rangle \frac{\phi(x)}{x} \; x \, dx$$

by Riemann Sums technique [9, p. 148]

$$\to \left\langle f(t), \quad t. \; \frac{\phi(t)}{t} \right\rangle = \langle f, \phi \rangle$$

[Lemma 6]

This completes the proof of the theorem.

Taking $p = 0$ and $p = \frac{1}{2}$ in the above theorem we derive

COROLLARY 1. Let $f \in H^{\nu,0}_{\alpha,\beta}$ (I) *where* $-\frac{1}{2} \leq \nu \leq \frac{1}{2}$, $|\nu| \leq \alpha \leq 3/2$ and $\beta \geq -\frac{1}{2}$. *Define the distributional Hankel transform* of f by

$$F(y) = \left\langle f(t), \; t \, J_\nu \, (ty) \right\rangle$$

then

$$\lim_{N \to \infty} \left\langle \int_0^N F(y) \, y \; J_\nu(xy) \, dy, \; \phi(x) \right\rangle = \langle f, \phi \rangle$$

for all $\phi \in D(I)$.

COROLLARY 2. *Let* $f \in H^{\nu,\frac{1}{2}}_{\alpha,\beta}$ (I) *where* $-\frac{1}{2} \leq \nu \leq \frac{1}{2}$, $|\nu| \leq \alpha \leq 3/2$ and $\beta \geq -\frac{1}{2}$. *Define the distributional Struve transform* (H_ν-transform) of f by

$$F(y) = \left\langle f(t), \; t \;\; H_\nu \, (ty) \right\rangle$$

then

$$\lim_{N \to \infty} \left\langle \int_0^N F(y) \, Y_\nu \, (xy) \, y \, dy, \; \phi \, (x) \right\rangle = \langle f, \phi \rangle$$

for all $\phi \in D(I)$.

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