IJMMS 30:9 (2002) 569–574
PII. S016117120200786X
http://ijmms.hindawi.com

CERTAIN INTEGRAL OPERATOR AND STRONGLY STARLIKE FUNCTIONS

JIN-LIN LIU

Received 15 June 2001

Let $S^*(\rho,\gamma)$ denote the class of strongly starlike functions of order ρ and type γ and let $C(\rho,\gamma)$ be the class of strongly convex functions of order ρ and type γ. By making use of an integral operator defined by Jung et al. (1993), we introduce two novel families of strongly starlike functions $S_\beta^\alpha(\rho,\gamma)$ and $C_\beta^\alpha(\rho,\gamma)$. Some properties of these classes are discussed.

2000 Mathematics Subject Classification: 30C45, 30C75.

1. Introduction. Let A denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

which are analytic in the unit disc $E=\{z:|z|<1\}$. A function $f(z)$ belonging to A is said to be starlike of order γ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\gamma \quad (z\in E) \tag{1.2}$$

for some γ $(0\le\gamma<1)$. We denote by $S^*(\gamma)$ the subclass of A consisting of functions which are starlike of order γ in E. Also, a function $f(z)$ in A is said to be convex of order γ if it satisfies $zf'(z)\in S^*(\gamma)$, or

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\gamma \quad (z\in E) \tag{1.3}$$

for some γ $(0\le\gamma<1)$. We denote by $C(\gamma)$ the subclass of A consisting of all functions which are convex of order γ in E.

If $f(z)\in A$ satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)}-\gamma\right)\right|<\frac{\pi}{2}\rho \quad (z\in E) \tag{1.4}$$

for some γ $(0\le\gamma<1)$ and ρ $(0<\rho\le1)$, then $f(z)$ is said to be strongly starlike of order ρ and type γ in E, and denoted by $f(z)\in S^*(\rho,\gamma)$. If $f(z)\in A$ satisfies

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}-\gamma\right)\right|<\frac{\pi}{2}\rho \quad (z\in E) \tag{1.5}$$

for some γ $(0 \le \gamma < 1)$ and ρ $(0 < \rho \le 1)$, then we say that $f(z)$ is strongly convex of order ρ and type γ in E, and we denote by $C(\rho,\gamma)$ the class of such functions. It is clear that $f(z) \in A$ belongs to $C(\rho,\gamma)$ if and only if $zf'(z) \in S^*(\rho,\gamma)$. Also, we note that $S^*(1,\gamma) = S^*(\gamma)$ and $C(1,\gamma) = C(\gamma)$.

For $c > -1$ and $f(z) \in A$, we recall the generalized Bernardi-Libera-Livingston integral operator $L_c(f)$ as

$$L_c(f) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt. \tag{1.6}$$

The operator $L_c(f)$ when $c \in N = \{1,2,3,\dots\}$ was studied by Bernardi [1]. For $c = 1$, $L_1(f)$ was investigated by Libera [4].

Recently, Jung et al. [2] introduced the following one-parameter family of integral operators:

$$Q_\beta^\alpha f(z) = \binom{\alpha+\beta}{\beta}\frac{\alpha}{z^\beta}\int_0^z \left(1-\frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)dt \quad (\alpha > 0,\ \beta > -1,\ f \in A). \tag{1.7}$$

They showed that

$$Q_\beta^\alpha f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)\Gamma(\alpha+\beta+1)}{\Gamma(\beta+\alpha+n)\Gamma(\beta+1)} a_n z^n, \tag{1.8}$$

where $\Gamma(x)$ is the familiar Gamma function. Some properties of this operator have been studied (see [2, 3]). From (1.7) and (1.8), one can see that

$$z\left(Q_\beta^{\alpha+1} f(z)\right)' = (\alpha+\beta+1)Q_\beta^\alpha f(z) - (\alpha+\beta)Q_\beta^{\alpha+1} f(z). \tag{1.9}$$

It should be remarked in passing that the operator Q_β^α is related rather closely to the Beta or Euler transformation.

Using the operator Q_β^α, we now introduce the following classes:

$$\begin{aligned} S_\beta^\alpha(\rho,\gamma) &= \left\{ f(z) \in A : Q_\beta^\alpha f(z) \in S^*(\rho,\gamma), \frac{z\left(Q_\beta^\alpha f(z)\right)'}{Q_\beta^\alpha f(z)} \ne \gamma \ \forall z \in E \right\}, \\ C_\beta^\alpha(\rho,\gamma) &= \left\{ f(z) \in A : Q_\beta^\alpha f(z) \in C(\rho,\gamma), \frac{\left(z\left(Q_\beta^\alpha f(z)\right)'\right)'}{\left(Q_\beta^\alpha f(z)\right)'} \ne \gamma \ \forall z \in E \right\}. \end{aligned} \tag{1.10}$$

It is obvious that $f(z) \in C_\beta^\alpha(\rho,\gamma)$ if and only if $zf'(z) \in S_\beta^\alpha(\rho,\gamma)$.

In this note, we investigate some properties of the classes $S_\beta^\alpha(\rho,\gamma)$ and $C_\beta^\alpha(\rho,\gamma)$. The basic tool for our investigation is the following lemma which is due to Nunokawa [5].

Lemma 1.1. *Let a function $p(z) = 1 + c_1z + c_2z^2 + \cdots$ be analytic in E and $p(z) \ne 0$ $(z \in E)$. If there exists a point $z_0 \in E$ such that*

$$|\arg p(z)| < \frac{\pi}{2}\rho \quad (|z| < |z_0|), \qquad |\arg p(z_0)| = \frac{\pi}{2}\rho \quad (0 < \rho \le 1), \tag{1.11}$$

then

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\rho, \tag{1.12}$$

where

$$\begin{aligned} k &\geq \frac{1}{2}\left(a+\frac{1}{a}\right) \quad \left(\text{when } \arg p(z_0) = \frac{\pi}{2}\rho\right), \\ k &\leq -\frac{1}{2}\left(a+\frac{1}{a}\right) \quad \left(\text{when } \arg p(z_0) = -\frac{\pi}{2}\rho\right), \end{aligned} \tag{1.13}$$

and $p(z_0)^{1/\rho} = \pm ia \ (a > 0)$.

2. Main results.

Our first inclusion theorem is stated as follows.

Theorem 2.1. *The class* $S_\beta^\alpha(\rho,\gamma) \subset S_\beta^{\alpha+1}(\rho,\gamma)$ *for* $\alpha > 0$, $\beta > -1$, $0 \leq \gamma < 1$ *and* $\alpha+\beta \geq -\gamma$.

Proof. Let $f(z) \in S_\beta^\alpha(\rho,\gamma)$. Then we set

$$\frac{z(Q_\beta^{\alpha+1} f(z))'}{Q_\beta^{\alpha+1} f(z)} = (1-\gamma)p(z)+\gamma, \tag{2.1}$$

where $p(z) = 1+c_1 z+c_2 z^2+\cdots$ is analytic in E and $p(z) \neq 0$ for all $z \in E$. Using (1.9) and (2.1), we have

$$(\alpha+\beta+1)\frac{Q_\beta^\alpha f(z)}{Q_\beta^{\alpha+1} f(z)} = (\alpha+\beta+\gamma)+(1-\gamma)p(z). \tag{2.2}$$

Differentiating both sides of (2.2) logarithmically, it follows from (2.1) that

$$\frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)} - \gamma = (1-\gamma)p(z)+\frac{(1-\gamma)zp'(z)}{(\alpha+\beta+\gamma)+(1-\gamma)p(z)}. \tag{2.3}$$

Suppose that there exists a point $z_0 \in E$ such that

$$|\arg p(z)| < \frac{\pi}{2}\rho \quad (|z| < |z_0|), \qquad |\arg p(z_0)| = \frac{\pi}{2}\rho. \tag{2.4}$$

Then, by applying Lemma 1.1, we can write that $z_0 p'(z_0)/p(z_0) = ik\rho$ and that $(p(z_0))^{1/\rho} = \pm ia \ (a > 0)$.

Therefore, if $\arg p(z_0) = -(\pi/2)\rho$, then

$$\begin{aligned} \frac{z_0(Q_\beta^\alpha f(z_0))'}{Q_\beta^\alpha f(z_0)} - \gamma &= (1-\gamma)p(z_0)\left[1+\frac{z_0 p'(z_0)/p(z_0)}{(\alpha+\beta+\gamma)+(1-\gamma)p(z_0)}\right] \\ &= (1-\gamma)a^\rho e^{-i\pi\rho/2}\left[1+\frac{ik\rho}{(\alpha+\beta+\gamma)+(1-\gamma)a^\rho e^{-i\pi\rho/2}}\right]. \end{aligned} \tag{2.5}$$

From (2.5) we have

$$\begin{aligned}
&\arg\left\{\frac{z_0(Q_\beta^\alpha f(z_0))'}{Q_\beta^\alpha f(z_0)}-\gamma\right\}\\
&\quad=-\frac{\pi}{2}\rho+\arg\left\{1+\frac{ik\rho}{(\alpha+\beta+\gamma)+(1-\gamma)a^\rho e^{-i\pi\rho/2}}\right\}\\
&\quad=-\frac{\pi}{2}\rho+\tan^{-1}\left\{\left(k\rho\left[(\alpha+\beta+\gamma)+(1-\gamma)a^\rho\cos\frac{\pi\rho}{2}\right]\right)\right.\\
&\qquad\qquad\times\left((\alpha+\beta+\gamma)^2+2(\alpha+\beta+\gamma)(1-\gamma)a^\rho\cos\frac{\pi\rho}{2}\right.\\
&\qquad\qquad\quad\left.\left.+(1-\gamma)^2a^{2\rho}-k\rho(1-\gamma)a^\rho\sin\frac{\pi\rho}{2}\right)^{-1}\right\}\\
&\quad\le-\frac{\pi}{2}\rho,
\end{aligned}\tag{2.6}$$

where $k\le-(1/2)(a+1/a)\le-1$, $\alpha+\beta\ge-\gamma$, which contradicts the condition $f(z)\in S_\beta^\alpha(\rho,\gamma)$.

Similarly, if $\arg p(z_0)=(\pi/2)\rho$, then we have

$$\arg\left\{\frac{z_0(Q_\beta^\alpha f(z_0))'}{Q_\beta^\alpha f(z_0)}-\gamma\right\}\ge\frac{\pi}{2}\rho,\tag{2.7}$$

which also contradicts the hypothesis that $f(z)\in S_\beta^\alpha(\rho,\gamma)$.

Thus the function $p(z)$ has to satisfy $|\arg p(z)|<(\pi/2)\rho$ $(z\in E)$, which leads us to the following:

$$\left|\arg\left\{\frac{z(Q_\beta^{\alpha+1}f(z))'}{Q_\beta^{\alpha+1}f(z)}-\gamma\right\}\right|<\frac{\pi}{2}\rho\quad(z\in E).\tag{2.8}$$

This evidently completes the proof of Theorem 2.1. □

We next state the following theorem.

Theorem 2.2. *The class $C_\beta^\alpha(\rho,\gamma)\subset C_\beta^{\alpha+1}(\rho,\gamma)$ for $\alpha>0$, $\beta>-1$, $0\le\gamma<1$, and $\alpha+\beta\ge-\gamma$.*

Proof. By definition (1.10), we have

$$\begin{aligned}
f(z)\in C_\beta^\alpha(\rho,\gamma)&\Longleftrightarrow Q_\beta^\alpha f(z)\in C(\rho,\gamma)\Longleftrightarrow z(Q_\beta^\alpha f(z))'\in S^*(\rho,\gamma)\\
&\Longleftrightarrow Q_\beta^\alpha(zf'(z))\in S^*(\rho,\gamma)\Longleftrightarrow zf'(z)\in S_\beta^\alpha(\rho,\gamma)\\
&\Longrightarrow zf'(z)\in S_\beta^{\alpha+1}(\rho,\gamma)\Longleftrightarrow Q_\beta^{\alpha+1}(zf'(z))\in S^*(\rho,\gamma)\\
&\Longleftrightarrow z(Q_\beta^{\alpha+1}f(z))'\in S^*(\rho,\gamma)\Longleftrightarrow Q_\beta^{\alpha+1}f(z)\in C(\rho,\gamma)\\
&\Longleftrightarrow f(z)\in C_\beta^{\alpha+1}(\rho,\gamma).
\end{aligned}\tag{2.9}$$

□

The following theorem involves the generalized Bernardi-Libera-Livingston integral operator $L_c(f)$ given by (1.6).

Theorem 2.3. *Let $c > -\gamma$ and $0 \le \gamma < 1$. If $f(z) \in A$ and $z(Q_\beta^\alpha L_c f(z))'/Q_\beta^\alpha L_c f(z) \neq \gamma$ for all $z \in E$, then $f(z) \in S_\beta^\alpha(\rho,\gamma)$ implies that $L_c(f) \in S_\beta^\alpha(\rho,\gamma)$.*

Proof. Let $f(z) \in S_\beta^\alpha(\rho,\gamma)$. Put

$$\frac{z(Q_\beta^\alpha L_c f(z))'}{Q_\beta^\alpha L_c f(z)} = \gamma + (1-\gamma)p(z), \tag{2.10}$$

where $p(z)$ is analytic in E, $p(0) = 1$ and $p(z) \neq 0$ $(z \in E)$. From (1.6) we have

$$z(Q_\beta^\alpha L_c f(z))' = (c+1)Q_\beta^\alpha f(z) - cQ_\beta^\alpha L_c f(z). \tag{2.11}$$

Using (2.10) and (2.11), we get

$$(c+1)\frac{Q_\beta^\alpha f(z)}{Q_\beta^\alpha L_c f(z)} = (c+\gamma) + (1-\gamma)p(z). \tag{2.12}$$

Differentiating both sides of (2.12) logarithmically, we obtain

$$\frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)} - \gamma = (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{(c+\gamma)+(1-\gamma)p(z)}. \tag{2.13}$$

Suppose that there exists a point $z_0 \in E$ such that

$$|\arg p(z)| < \frac{\pi}{2}\rho \quad (|z| < |z_0|), \qquad |\arg p(z_0)| = \frac{\pi}{2}\rho. \tag{2.14}$$

Then, applying Lemma 1.1, we can write that $z_0 p'(z_0)/p(z_0) = ik\rho$ and $(p(z_0))^{1/\rho} = \pm ia$ $(a > 0)$.

If $\arg p(z_0) = (\pi/2)\rho$, then

$$\begin{aligned} \frac{z_0(Q_\beta^\alpha f(z_0))'}{Q_\beta^\alpha f(z_0)} - \gamma &= (1-\gamma)p(z_0)\left[1 + \frac{z_0 p'(z_0)/p(z_0)}{(c+\gamma)+(1-\gamma)p(z_0)}\right] \\ &= (1-\gamma)a^\rho e^{i\pi\rho/2}\left[1 + \frac{ik\rho}{(c+\gamma)+(1-\gamma)a^\rho e^{i\pi\rho/2}}\right]. \end{aligned} \tag{2.15}$$

This shows that

$$\begin{aligned} &\arg\left\{\frac{z_0(Q_\beta^\alpha f(z_0))'}{Q_\beta^\alpha f(z_0)} - \gamma\right\} \\ &\quad = \frac{\pi}{2}\rho + \arg\left\{1 + \frac{ik\rho}{(c+\gamma)+(1-\gamma)a^\rho e^{i\pi\rho/2}}\right\} \\ &\quad = \frac{\pi}{2}\rho + \tan^{-1}\left\{\left(k\rho\left[(c+\gamma)+(1-\gamma)a^\rho\cos\frac{\pi\rho}{2}\right]\right)\right. \\ &\qquad\qquad \times\left((c+\gamma)^2 + 2(c+\gamma)(1-\gamma)a^\rho\cos\frac{\pi\rho}{2}\right. \\ &\qquad\qquad\qquad \left.\left. + (1-\gamma)^2a^{2\rho} + k\rho(1-\gamma)a^\rho\sin\frac{\pi\rho}{2}\right)^{-1}\right\} \\ &\quad \ge \frac{\pi}{2}\rho, \end{aligned} \tag{2.16}$$

where $k \ge (1/2)(a+1/a) \ge 1$, which contradicts the condition $f(z) \in S_\beta^\alpha(\rho,\gamma)$.

Similarly, we can prove the case $\arg p(z_0) = -(\pi/2)\rho$. Thus we conclude that the function $p(z)$ has to satisfy $|\arg p(z)| < (\pi/2)\rho$ for all $z \in E$. This shows that

$$\left| \arg\left\{ \frac{z\left(Q_\beta^\alpha L_c f(z)\right)'}{Q_\beta^\alpha L_c f(z)} - \gamma \right\} \right| < \frac{\pi}{2}\rho \quad (z \in E). \tag{2.17}$$

The proof is complete. □

Theorem 2.4. *Let $c > -\gamma$ and $0 \le \gamma < 1$. If $f(z) \in A$ and $(z(Q_\beta^\alpha L_c f(z))')'/(Q_\beta^\alpha L_c f(z))' \ne \gamma$ for all $z \in E$, then $f(z) \in C_\beta^\alpha(\rho,\gamma)$ implies that $L_c(f) \in C_\beta^\alpha(\rho,\gamma)$.*

Proof. Using the same method as in Theorem 2.2 we have

$$\begin{aligned} f(z) \in C_\beta^\alpha(\rho,\gamma) &\Longleftrightarrow zf'(z) \in S_\beta^\alpha(\rho,\gamma) \Longrightarrow L_c\left(zf'(z)\right) \in S_\beta^\alpha(\rho,\gamma) \\ &\Longleftrightarrow z\left(L_c f(z)\right)' \in S_\beta^\alpha(\rho,\gamma) \Longleftrightarrow L_c f(z) \in C_\beta^\alpha(\rho,\gamma). \end{aligned} \tag{2.18}$$

□

References

[1] S. D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc. **135** (1969), 429–446.

[2] I. B. Jung, Y. C. Kim, and H. M. Srivastava, *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*, J. Math. Anal. Appl. **176** (1993), no. 1, 138–147.

[3] J. L. Li, *Some properties of two integral operators*, Soochow J. Math. **25** (1999), no. 1, 91–96.

[4] R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc. **16** (1965), 755–758.

[5] M. Nunokawa, *On properties of non-Carathéodory functions*, Proc. Japan Acad. Ser. A Math. Sci. **68** (1992), no. 6, 152–153.

Jin-Lin Liu: Department of Mathematics, Yangzhou University, Yangzhou 225002, Jiangsu, China