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SOME THEOREMS OF RANDOM OPERATOR EQUATIONS

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We investigate a class of random operator equations, generalize a famous theorem, and obtain some new results.

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Let E be a separable real Banach space, $(\mathbf{E},\mathbf{B})$ a measurable space, where $\mathbf{B}$ denotes the σ-algebra generated by all open subsets in E, let (Ω, U, γ) be a complete probability measure space, where $\gamma(\Omega) = 1$, let D be a bounded open set in X and ∂D the boundary of D in X. Let X be a cone in E, and let "$\leq$", "$<$" be partial order of E.

Lemma 1. *When $y > 1$, $\alpha > 0$, $x \in X$, and $x \neq \theta$, the following inequality holds:*

$$(y-1)^{\alpha+1}x < y^{\alpha+1}x - x. \tag{1}$$

Proof. Letting $f(y) = y^{\alpha+1} - 1 - (y-1)^{\alpha+1}$, where $\alpha > 0$, then

$$\begin{aligned} f'(y) &= (\alpha+1)y^{\alpha} - (\alpha+1)(y-1)^{\alpha} \\ &= (\alpha+1)\left[y^{\alpha} - (y-1)^{\alpha}\right] > 0 \end{aligned} \tag{2}$$

(since $y > 1$, then $0 < y-1 < y$, and $\alpha > 0$, obtaining $0 < (y-1)^{\alpha} < y^{\alpha}$, i.e., $y^{\alpha} - (y-1)^{\alpha} > 0$).

Therefore $f(y)$ is a monotonous increasing function. When $y > 1$, we have $f(y) > f(1)$, and $f(1) = 0$. Hence $f(y) > 0$, that is, $y^{\alpha+1} - 1 - (y-1)^{\alpha+1} > 0$, that is,

$$(y-1)^{\alpha+1} < y^{\alpha+1} - 1. \tag{3}$$

When $x \in X$, $x \neq \theta$, that is, $x > \theta$, we have $(y-1)^{\alpha+1}x < y^{\alpha+1}x - x$. $\square$

Lemma 2 (see [1]). *Let X be a closed convex subset of E, D a bounded open subset in X, and $\theta \in D$. Suppose that $A : \Omega \times \bar{D} \to X$ is a random semiclosed 1-set-contractive operator. Meanwhile, such that $x \neq (t/\mu)A(\omega,x)$ a.s., for every $\omega \in \Omega$, for every $x \in \partial D$, where $t \in (0,1]$, $\mu \geq 1$. Then the random operator equation $A(\omega,x) = \mu x$, (for every $(\omega,x) \in \Omega \times \bar{D}$, $\mu \geq 1$) has a random solution in D.*

Theorem 3. *Let D be a bounded open subset in X and $\theta \in D$. Suppose that $A : \Omega \times \bar{D} \to X$ is a random semiclosed 1-set-contractive operator, such that*

$$\begin{aligned} &\left[\lambda\|\mu x\| + \|A(\omega,x) - \mu x\|^{\alpha}\right]\|A(\omega,x) - \mu x\|x \\ &\quad \geq \left[\lambda\|\mu x\| + \|A(\omega,x)\|^{\alpha}\right]\|A(\omega,x)\|x - \lambda\|\mu x\|^2 x - \|\mu x\|^{\alpha+1}x \end{aligned} \tag{4}$$

for every $(\omega,x)\in\Omega\times\partial D$*,* $\lambda\geq 0$*,* $\mu\geq 1$*,* $\alpha>0$*. Then the random operator equation* $A(\omega,x)=\mu x$ *(for every* $(\omega,x)\in\Omega\times\bar{D}$*, where* $\mu\geq 1$*) has a random solution in* $\bar{D}$*.*

Proof. Assume that $A(\omega,x)=\mu x$ has no random solution on ∂D (otherwise, the theorem has obtained proof), that is, $A(\omega,x)\neq\mu x$ a.s., for every $(\omega,x)\in\Omega\times\partial D$, where $\mu\geq 1$. That is,

$$x\neq\frac{1}{\mu}A(\omega,x)\quad\text{a.s.}\tag{5}$$

We prove that

$$x\neq t\frac{1}{\mu}A(\omega,x),\tag{6}$$

where $\mu\geq 1$, $t\in(0,1)$, for every $(\omega,x)\in\Omega\times\partial D$.

Suppose that (6) is not true, that is, there exists a $t_0\in(0,1)$, an $\omega_0\in\Omega$, and an $x_0\in\partial D$, such that $x_0=t_0(1/\mu)A(\omega_0,x_0)$. That is, $A(\omega_0,x_0)=(\mu/t_0)x_0$, where $\mu\geq 1$, $t_0\in(0,1)$, $\omega_0\in\Omega$, and $x_0\in\partial D$.

Inserting $A(\omega_0,x_0)=(\mu/t_0)x_0$ into (4), obtaining

$$\begin{aligned}&\left[\lambda\|\mu x_0\|+\left\|\frac{\mu}{t_0}x_0-\mu x_0\right\|^{\alpha}\right]\left\|\frac{\mu}{t_0}x_0-\mu x_0\right\|x_0\\&\quad\geq\left[\lambda\|\mu x_0\|+\left\|\frac{\mu}{t_0}x_0\right\|^{\alpha}\right]\left\|\frac{\mu}{t_0}x_0\right\|x_0-\lambda\|\mu x_0\|^2x_0-\|\mu x_0\|^{\alpha+1}x_0,\end{aligned}\tag{7}$$

where $\lambda\geq 0$, $\mu\geq 1$, $\alpha>0$, $t_0\in(0,1)$, and $x_0\in\partial D$. This implies that

$$\begin{aligned}&\lambda\|\mu x_0\|\left\|\frac{\mu}{t_0}x_0-\mu x_0\right\|x_0+\left\|\frac{\mu}{t_0}x_0-\mu x_0\right\|^{\alpha+1}x_0\\&\quad\geq\lambda\|\mu x_0\|\left\|\frac{\mu}{t_0}x_0\right\|x_0+\left\|\frac{\mu}{t_0}x_0\right\|^{\alpha+1}x_0-\lambda\|\mu x_0\|^2x_0-\|\mu x_0\|^{\alpha+1}x_0,\end{aligned}\tag{8}$$

that is,

$$\begin{aligned}&\lambda\left(\frac{1}{t_0}-1\right)\|\mu x_0\|^2x_0+\left(\frac{1}{t_0}-1\right)^{\alpha+1}\|\mu x_0\|^{\alpha+1}x_0\\&\quad\geq\lambda\left(\frac{1}{t_0}-1\right)\|\mu x_0\|^2x_0+\frac{1}{t_0^{\alpha+1}}\|\mu x_0\|^{\alpha+1}x_0-\|\mu x_0\|^{\alpha+1}x_0\end{aligned}\tag{9}$$

since $\mu\geq 1$, $x_0\in\partial D$, thus $\mu x_0\neq 0$.

Therefore $\|\mu x_0\|^{\alpha+1}\neq 0$, by (9), we obtain

$$\left(\frac{1}{t_0}-1\right)^{\alpha+1}x_0\geq\frac{1}{t_0^{\alpha+1}}x_0-x_0.\tag{10}$$

Letting $y = 1/t_0$, by (10), we have

$$(y-1)^{\alpha+1}x_0 \geq y^{\alpha+1}x_0 - x_0, \tag{11}$$

where $y > 1$, $\alpha > 0$, $x_0 \in X$, and $x_0 \neq \theta$.

This is in contradiction with Lemma 1. Hence

$$x \neq t\frac{1}{\mu}A(\omega,x) \tag{12}$$

for every $(\omega,x) \in \Omega\times\partial D$, where $t \in (0,1)$, $\mu \geq 1$.

By (5) and (12), we know that

$$x \neq t\frac{1}{\mu}A(\omega,x) \quad \text{a.s.}, \tag{13}$$

where $\mu \geq 1$, $t \in (0,1)$, for every $(\omega,x) \in \Omega\times\partial D$.

According to Lemma 2, we obtain that the random operator equation $A(\omega,x) = \mu x$ (where $\mu \geq 1$, for every $(\omega,x) \in \Omega\times\bar{D}$) has a random solution in D. □

Remark 4. In Theorem 3, when $\lambda = 0$, $\alpha = 1$, $\mu = 1$, and $A(\omega,\cdot) = A$, (4) is that $\|Ax-x\|^2 \geq \|Ax\|^2 - \|x\|^2$. Thus, Theorem 3 is a generalization of the famous Altman theorem.

We can see that Lemma 5 holds easily.

Lemma 5. *When $y > 1$, $\alpha > 0$, $x \in X$, and $x \neq \theta$, the following inequality holds:*

$$(y+1)^{\alpha+1}x > y^{\alpha+1}x + x. \tag{14}$$

Theorem 6. *Let D be a bounded open subset in X and $\theta \in D$. Suppose that $A : \Omega\times\bar{D} \to X$ is a random semiclosed 1-set-contractive operator, such that*

$$\begin{aligned}&[\lambda\|\mu x\| + ||A(\omega,x)+\mu x||^{\alpha}]||A(\omega,x)+\mu x||x \\ &\quad \leq [\lambda\|\mu x\| + ||A(\omega,x)||^{\alpha}]||A(\omega,x)||x + \lambda\|\mu x\|^2 x + \|\mu x\|^{\alpha+1}x,\end{aligned} \tag{15}$$

where $\lambda \geq 0$, $\mu \geq 1$, $\alpha > 0$, for every $(\omega,x) \in \Omega\times\partial D$. Then the random operator equation $A(\omega,x) = \mu x$ (where $\mu \geq 1$, for every $(\omega,x) \in \Omega\times\bar{D}$) has a random solution in D.

Proof. From (15), we can easily prove that $A(\omega,x) = \mu x$ has no random solution on ∂D, by virtue of Lemma 5, see Theorem 3 for other section. □

Lemma 7. *When $y > 1$, $\alpha > 0$, $x \in X$, and $x \neq \theta$, the following inequality holds:*

$$(y+1)^{\alpha+1}x - (y-1)^{\alpha+1}x > 2x. \tag{16}$$

Proof. By Lemmas 1 and 5, we have

$$(y-1)^{\alpha+1}x < y^{\alpha+1}x - x, \tag{17}$$

$$y^{\alpha+1}x + x < (y+1)^{\alpha+1}x, \tag{18}$$

summing (17) and (18) we obtain

$$(y-1)^{\alpha+1}x + y^{\alpha+1}x + x < y^{\alpha+1}x - x + (y+1)^{\alpha+1}x. \tag{19}$$

That is,

$$(y+1)^{\alpha+1}x - (y-1)^{\alpha+1}x > 2x, \tag{20}$$

where $\alpha > 0$, $y > 1$, $x \in X$, and $x \neq \theta$. □

Theorem 8. *Let D be a bounded open subset in X and $\theta \in D$. Suppose that $A : \Omega \times \bar{D} \to X$ is a random semiclosed 1-set-contractive operator, such that*

$$\|A(\omega,x) + \mu x\|^{\alpha+1}x - \|A(\omega,x) - \mu x\|^{\alpha+1}x \leq 2\|\mu x\|^{\alpha+1}x, \tag{21}$$

where $\alpha > 0$, $\mu \geq 1$, for every $(\omega,x) \in \Omega \times \partial D$. Then the random operator equation $A(\omega,x) = \mu x$ (where $\mu \geq 1$, for every $(\omega,x) \in \Omega \times \bar{D}$) has a random solution in D.

Proof. The theorem can be proved using Lemma 7, see also Theorems 3 and 6. □

Remark 9. Since X is a cone in E, then X is a closed convex subset of E.

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References

[1] C. X. Zhu, *Some theorems on random operator equations of 1-set-contractive type*, Adv. in Math. (China) **27** (1998), no. 5, 464–468.

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