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A NOTE ON OPERATORS OF DELETION AND CONTRACTION FOR ANTICHAINS

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The operators of deletion and contraction for clutters are generalized to those for antichains of finite bounded posets. A generalization of the result by Seymour (1976), describing the relationship between the operators of deletion, contraction, and the blocker map, is considered as a comparison in the lattice of antichains of a poset.

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1. Introduction. Deletion and contraction are basic operators on clutters. Recall that for a finite nonempty set S, a family of its subsets is called a *clutter* (or a *Sperner family*) if no set from that family contains another. Denote by $\hat{0}$ the empty subset of S. The clutter $\varnothing$ (containing no sets) and the clutter $\{\hat{0}\}$ are called *trivial*.

Consider a nontrivial clutter $\mathcal{G}$ on the ground set S. Let $x \in S$. Recall that the *deletion* $\mathcal{G}\backslash x$ and *contraction* $\mathcal{G}/x$ are the clutters, defined as follows:

$$\begin{aligned}\mathcal{G}\backslash x &= \{G \in \mathcal{G} : G \not\ni x\},\\ \mathcal{G}/x &= \{\text{inclusion-wise minimal sets of the family } \{G-\{x\} : G \in \mathcal{G}\}\},\end{aligned} \tag{1.1}$$

on the ground set $S-\{x\}$.

Deletion and contraction are also defined for the trivial clutters

$$\varnothing\backslash x = \varnothing/x = \varnothing, \qquad \{\hat{0}\}\backslash x = \{\hat{0}\}/x = \{\hat{0}\}, \tag{1.2}$$

on the ground set $S-\{x\}$.

If $X = \{x_1,\dots,x_n\} \subseteq S$, $n \geq 1$, then the *deletion* $\mathcal{G}\backslash X$ and *contraction* $\mathcal{G}/X$ are defined in the following way: $\mathcal{G}\backslash X = \mathcal{G}\backslash x_1\backslash\cdots\backslash x_n$, and $\mathcal{G}/X = \mathcal{G}/x_1/\cdots/x_n$. Deletions and contractions, sequentially performed on a clutter, produce its *minors*.

The *blocker* of a nontrivial clutter $\mathcal{G}$ on the ground set S is the clutter $\mathfrak{B}(\mathcal{G})$ on S, consisting of all the inclusion-wise minimal subsets $H \subseteq S$ with the property, for each $G \in \mathcal{G}$, $|H \cap G| \geq 1$.

The *blockers* of the trivial clutters are defined as follows (see, e.g., [2]):

$$\mathfrak{B}(\{\hat{0}\}) = \varnothing, \qquad \mathfrak{B}(\varnothing) = \{\hat{0}\}. \tag{1.3}$$

Seymour described in [4] the relationship between a clutter $\mathcal{G}$ on the ground set S, deletions, contractions, and relevant blockers; if X is a nonempty subset of S then we have

$$\mathfrak{B}(\mathcal{G})\backslash X = \mathfrak{B}(\mathcal{G}/X), \qquad \mathfrak{B}(\mathcal{G})/X = \mathfrak{B}(\mathcal{G}\backslash X). \tag{1.4}$$

Recall that the clutters on S are in natural one-to-one correspondence with the antichains of the Boolean lattice of all subsets of S.

We present in this paper operators of deletion and contraction for antichains of a finite bounded poset. The main results of the paper are Theorems 2.5 and 2.6. Theorem 2.5 states that deletion and contraction for antichains are (co)closure operators on the lattice of antichains of the poset. Theorem 2.6 provides a generalization of result (1.4) to antichains of the poset.

It is a consequence of Theorem 2.6 that equalities (1.4) might be read as

$$\mathscr{B}(\mathscr{G})\backslash X = \mathscr{B}(\mathscr{G}/X) \preceq \mathscr{B}(\mathscr{G}) \preceq \mathscr{B}(\mathscr{G})/X = \mathscr{B}(\mathscr{G}\backslash X), \tag{1.5}$$

where $\preceq$ is a certain comparison that comes from the lattice of antichains of the Boolean lattice of all subsets of S.

2. Deletion and contraction. We refer the reader to [5, Chapter 3] for basic information and terminology in the theory of posets.

We use $\min Q$ to denote the set of all minimal elements of a poset Q. If Q has a least element then it is denoted $\hat{0}_Q$; if Q has a greatest element then it is denoted $\hat{1}_Q$.

Throughout this note, P stands for a finite bounded poset with $|P| > 1$; P^{a} denotes its atom set, that is the set of all elements covering $\hat{0}_P$. $\mathfrak{I}(A)$ and $\mathfrak{F}(A)$ denote the order ideal and filter of P generated by an antichain A of P, respectively.

We denote the distributive lattice of all antichains of P by $\mathfrak{A}(P)$. If $A_1, A_2 \in \mathfrak{A}(P)$ then we set

$$A_1 \le A_2 \quad \text{iff } \mathfrak{F}(A_1) \subseteq \mathfrak{F}(A_2). \tag{2.1}$$

The least and greatest elements $\hat{0}_{\mathfrak{A}(P)}$ and $\hat{1}_{\mathfrak{A}(P)}$ of $\mathfrak{A}(P)$ are the *trivial antichains* $\varnothing \subset P$ and $\{\hat{0}_P\}$, respectively. For the rest of the paper, we denote by $\wedge$ and $\vee$ the operations of meet and join in the lattice $\mathfrak{A}(P)$; if $A_1, A_2 \in \mathfrak{A}(P)$ then

$$\begin{aligned} A_1 \vee A_2 &= \min(A_1 \cup A_2), \\ A_1 \wedge A_2 &= \min(\mathfrak{F}(A_1) \cap \mathfrak{F}(A_2)), \end{aligned} \tag{2.2}$$

respectively.

Let A be a *nontrivial antichain* of P, that is $A \in \mathfrak{A}(P) - \{\hat{0}_{\mathfrak{A}(P)}, \hat{1}_{\mathfrak{A}(P)}\}$. The *blocker* $\mathfrak{b}(A)$ of A, defined in [3], is the antichain

$$\min\{b \in P : |\mathfrak{I}(b) \cap \mathfrak{I}(a) \cap P^{\mathrm{a}}| \ge 1 \, \forall a \in A\}. \tag{2.3}$$

The *blockers* of the trivial antichains are defined as follows:

$$\mathfrak{b}(\hat{0}_{\mathfrak{A}(P)}) = \hat{1}_{\mathfrak{A}(P)}, \qquad \mathfrak{b}(\hat{1}_{\mathfrak{A}(P)}) = \hat{0}_{\mathfrak{A}(P)}. \tag{2.4}$$

For a one-element antichain $\{a\}$ of P, we write $\mathfrak{b}(a)$ instead of $\mathfrak{b}(\{a\})$. If $a \ne \hat{0}_P$ then $\mathfrak{b}(a) = \mathfrak{I}(a) \cap P^{\mathrm{a}}$, and we have $\{a\} \le \mathfrak{b}(\mathfrak{b}(a)) \le \mathfrak{b}(a)$.

If A is a nontrivial antichain of P then its blocker $\mathfrak{b}(A)$ is determined, in particular, by the equality $\mathfrak{b}(A) = \bigwedge_{a \in A} \mathfrak{b}(a)$.

The map $\mathfrak{b} : \mathfrak{A}(P) \to \mathfrak{A}(P)$, reflecting an antichain to its blocker, is the *blocker map* on $\mathfrak{A}(P)$; it is antitone. The composite map $\mathfrak{b} \circ \mathfrak{b}$ is a closure operator on $\mathfrak{A}(P)$. The image $\mathfrak{b}(\mathfrak{A}(P))$ is called in [3] the *lattice of blockers* in P and it is denoted $\mathfrak{B}(P)$. The restriction $\mathfrak{b}\,|_{\mathfrak{B}(P)}$ of the blocker map is an anti-automorphism of $\mathfrak{B}(P)$. The lattice $\mathfrak{B}(P)$ is a meet-subsemilattice of $\mathfrak{A}(P)$. For every blocker $B \in \mathfrak{B}(P)$, its preimage $\mathfrak{b}^{-1}(B)$ is a convex join-subsemilattice of $\mathfrak{A}(P)$; the greatest element of $\mathfrak{b}^{-1}(B)$ is $\mathfrak{b}(B)$.

We start with generalizing the notions of deletion and contraction.

Definition 2.1. Let $X \subseteq P^{\mathrm{a}}$, $|X| \geq 1$.

(i) If $\{a\}$ is a nontrivial one-element antichain of P then the *deletion* $\{a\}\backslash X$ and *contraction* $\{a\}/X$ are the antichains

$$\{a\}\backslash X = \begin{cases} \{a\}, & \text{if } |\mathfrak{b}(a) \cap X| = 0, \\ \hat{0}_{\mathfrak{A}(P)}, & \text{if } |\mathfrak{b}(a) \cap X| \geq 1, \end{cases}$$
$$\{a\}/X = \begin{cases} \{a\}, & \text{if } |\mathfrak{b}(a) \cap X| = 0, \\ \mathfrak{b}(\mathfrak{b}(a) - X), & \text{if } |\mathfrak{b}(a) \cap X| \geq 1,\ \mathfrak{b}(a) \not\subseteq X, \\ \hat{1}_{\mathfrak{A}(P)}, & \text{if } \mathfrak{b}(a) \subseteq X. \end{cases} \tag{2.5}$$

(ii) If A is a nontrivial antichain of P then the *deletion* $A\backslash X$ and *contraction* A/X are the antichains

$$A\backslash X = \bigvee_{a \in A} (\{a\}\backslash X), \qquad A/X = \bigvee_{a \in A} (\{a\}/X). \tag{2.6}$$

(iii) The *deletion* and *contraction* for the trivial antichains of P are

$$\begin{aligned} \hat{0}_{\mathfrak{A}(P)}\backslash X &= \hat{0}_{\mathfrak{A}(P)}/X = \hat{0}_{\mathfrak{A}(P)}, \\ \hat{1}_{\mathfrak{A}(P)}\backslash X &= \hat{1}_{\mathfrak{A}(P)}/X = \hat{1}_{\mathfrak{A}(P)}. \end{aligned} \tag{2.7}$$

If we take into consideration the empty subset $\varnothing^{\mathrm{a}}$ of the atom set P^{a}, then for every $A \in \mathfrak{A}(P)$ we define the antichains $A\backslash\varnothing^{\mathrm{a}}$ and $A/\varnothing^{\mathrm{a}}$ to be equal to A.

The following observations are an immediate consequence of Definition 2.1: if $\{a\}$ is a one-element antichain of P, then we have

$$\{a\}\backslash X \leq \{a\} \leq \{a\}/X, \tag{2.8}$$

$$\mathfrak{b}(a)\backslash X \leq \mathfrak{b}(\{a\}/X) \leq \mathfrak{b}(a) \leq \mathfrak{b}(a)/X = \mathfrak{b}(\{a\}\backslash X). \tag{2.9}$$

Another observation is the following lemma.

Lemma 2.2. *Let $X \subseteq P^{\mathrm{a}}$, $|X| \geq 1$. If $A_1, A_2 \in \mathfrak{A}(P)$ and $A_1 \leq A_2$, then*

$$A_1\backslash X \leq A_2\backslash X, \qquad A_1/X \leq A_2/X. \tag{2.10}$$

If $A \in \mathfrak{A}(P)$ then the elements $A\backslash X$, A, and A/X of the lattice $\mathfrak{A}(P)$ are comparable.

Lemma 2.3. *If* $A \in \mathfrak{A}(P)$ *and* $X \subseteq P^{\mathrm{a}}$, $|X| \geq 1$, *then*

$$A\backslash X \leq A \leq A/X. \tag{2.11}$$

Proof. There is nothing to prove if A is a trivial antichain. Suppose that A is nontrivial. With the help of (2.6) and (2.8), we see that $A\backslash X = \bigvee_{a\in A}(\{a\}\backslash X) \leq \bigvee_{a\in A}\{a\} = A \leq \bigvee_{a\in A}(\{a\}/X) = A/X$. □

If $\{a\}$ is a nontrivial one-element antichain of P and $X \subseteq P^{\mathrm{a}}$, $|X| \geq 1$, then we obviously have $\{a\}\backslash X = (\{a\}\backslash X)\backslash X$. The antichain $\{a\}/X$ has an analogous property. Indeed, if $|\mathfrak{b}(a) \cap X| = 0$ or if $\mathfrak{b}(a) \subseteq X$ then $(\{a\}/X)/X = \{a\}/X$, due to the definition of contraction. Further, if $|\mathfrak{b}(a) \cap X| \geq 1$ and $\mathfrak{b}(a) \nsubseteq X$ then, on one hand, we have $(\{a\}/X)/X \geq \{a\}/X$, by Lemma 2.3. On the other hand, for every $b \in \{a\}/X = \mathfrak{b}(\mathfrak{b}(a) - X)$ we have $\mathfrak{b}(b) - X \geq \mathfrak{b}(a) - X$ and, as a consequence, we have $(\{a\}/X)/X = \bigvee_{b\in\{a\}/X}(\{b\}/X) \leq \mathfrak{b}(\mathfrak{b}(a) - X) = \{a\}/X$. We conclude that $(\{a\}/X)/X = \{a\}/X$. In view of (2.6), we can formulate the following lemma.

Lemma 2.4. *If* $A \in \mathfrak{A}(P)$ *and* $X \subseteq P^{\mathrm{a}}$, $|X| \geq 1$, *then*

$$(A\backslash X)\backslash X = A\backslash X, \qquad (A/X)/X = A/X. \tag{2.12}$$

Altogether, Lemmas 2.2, 2.3, and 2.4 describe the connection of the maps $(\backslash X) : \mathfrak{A}(P) \to \mathfrak{A}(P)$ and $(/X) : \mathfrak{A}(P) \to \mathfrak{A}(P)$ with (co)closure operators (see, e.g., [1, Chapter IV]).

Theorem 2.5. *Let* $X \subseteq P^{\mathrm{a}}$, $|X| \geq 1$. *The map* $(\backslash X)$ *is a coclosure operator on* $\mathfrak{A}(P)$. *The map* $(/X)$ *is a closure operator on* $\mathfrak{A}(P)$.

Given a nonempty atom subset X, we denote, slightly abusing denotations, the images $(\backslash X)(\mathfrak{A}(P)) = \{A\backslash X : A \in \mathfrak{A}(P)\}$ and $(/X)(\mathfrak{A}(P)) = \{A/X : A \in \mathfrak{A}(P)\}$ by $\mathfrak{A}(P)\backslash X$ and $\mathfrak{A}(P)/X$, respectively. We can interpret well-known properties of (semi)lattice maps and (co)closure operators on lattices in case of maps $(\backslash X)$ and $(/X)$.

Definition 2.1 implies that the maps $(\backslash X), (/X) : \mathfrak{A}(P) \to \mathfrak{A}(P)$ are upper $\{\hat{0}_{\mathfrak{A}(P)}, \hat{1}_{\mathfrak{A}(P)}\}$-homomorphisms, that is for all $A_1, A_2 \in \mathfrak{A}(P)$, we have $(A_1 \vee A_2)\backslash X = (A_1\backslash X) \vee (A_2\backslash X)$, $(A_1 \vee A_2)/X = (A_1/X) \vee (A_2/X)$ and, moreover, we have $\hat{0}_{\mathfrak{A}(P)}\backslash X = \hat{0}_{\mathfrak{A}(P)}/X = \hat{0}_{\mathfrak{A}(P)}$ and $\hat{1}_{\mathfrak{A}(P)}\backslash X = \hat{1}_{\mathfrak{A}(P)}/X = \hat{1}_{\mathfrak{A}(P)}$.

The posets $\mathfrak{A}(P)\backslash X$ and $\mathfrak{A}(P)/X$, with the partial orders induced by the partial order on $\mathfrak{A}(P)$, are lattices.

The lattice $\mathfrak{A}(P)\backslash X$ is a join-subsemilattice of $\mathfrak{A}(P)$. Denote by $\wedge_{\mathfrak{A}(P)\backslash X}$ the operation of meet in $\mathfrak{A}(P)\backslash X$. If $D_1, D_2 \in \mathfrak{A}(P)\backslash X$, then we have $D_1 \wedge_{\mathfrak{A}(P)\backslash X} D_2 = (D_1 \wedge D_2)\backslash X$.

The lattice $\mathfrak{A}(P)/X$ is a sublattice of $\mathfrak{A}(P)$.

If $D \in \mathfrak{A}(P)\backslash X$, then the preimage $(\backslash X)^{-1}(D)$ of D under the map $(\backslash X)$ is the closed interval $[D, D \vee X]$ of $\mathfrak{A}(P)$.

If $D \in \mathfrak{A}(P)/X$, then the preimage $(/X)^{-1}(D)$ of D under the map $(/X)$ is a convex join-subsemilattice of the lattice $\mathfrak{A}(P)$, with the greatest element D.

Equalities (1.4) may be generalized in the context of an arbitrary finite bounded poset. Indeed, let $A \in \mathfrak{A}(P)$ and $X \subseteq P^{\mathrm{a}}$, $|X| \geq 1$. We can deduce from Lemma 2.3 that the comparisons

$$\mathfrak{b}(A)\backslash X \le \mathfrak{b}(A) \le \mathfrak{b}(A)/X, \qquad \mathfrak{b}(A/X) \le \mathfrak{b}(A) \le \mathfrak{b}(A\backslash X) \tag{2.13}$$

hold, and we make an additional conclusion in the following theorem.

Theorem 2.6. *If $A \in \mathfrak{A}(P)$ and $X \subseteq P^{\mathrm{a}}$, $|X| \ge 1$, then*

$$\mathfrak{b}(A)\backslash X \le \mathfrak{b}(A/X) \le \mathfrak{b}(A) \le \mathfrak{b}(A)/X \le \mathfrak{b}(A\backslash X). \tag{2.14}$$

Proof. There is nothing to prove if the antichain A is trivial.

If A_1, A_2 are arbitrary antichains of P and $X \subseteq P^{\mathrm{a}}$, $|X| \ge 1$, then we can check with the help of routine machinery that

$$(A_1 \wedge A_2)\backslash X \le (A_1\backslash X) \wedge (A_2\backslash X), \tag{2.15}$$

$$(A_1 \wedge A_2)/X \le (A_1/X) \wedge (A_2/X). \tag{2.16}$$

Suppose that A is nontrivial. We prove that $\mathfrak{b}(A)\backslash X \le \mathfrak{b}(A/X)$. Using comparisons (2.15) and (2.9), we see that

$$\mathfrak{b}(A)\backslash X = \Big(\bigwedge_{a\in A} \mathfrak{b}(a)\Big)\backslash X \le \bigwedge_{a\in A} (\mathfrak{b}(a)\backslash X) \le \bigwedge_{a\in A} \mathfrak{b}(\{a\}/X) = \mathfrak{b}\Big(\bigvee_{a\in A} (\{a\}/X)\Big) = \mathfrak{b}(A/X). \tag{2.17}$$

We prove that $\mathfrak{b}(A)/X \le \mathfrak{b}(A\backslash X)$. With the help of (2.16) and (2.9), we see that

$$\mathfrak{b}(A)/X = \Big(\bigwedge_{a\in A} \mathfrak{b}(a)\Big)/X \le \bigwedge_{a\in A} (\mathfrak{b}(a)/X) = \bigwedge_{a\in A} \mathfrak{b}(\{a\}\backslash X) = \mathfrak{b}\Big(\bigvee_{a\in A} (\{a\}\backslash X)\Big) = \mathfrak{b}(A\backslash X). \tag{2.18}$$

□

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