

q -HYPERELLIPTIC COMPACT NONORIENTABLE KLEIN SURFACES WITHOUT BOUNDARY

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Let X be a nonorientable Klein surface (KS in short), that is a compact nonorientable surface with a dianalytic structure defined on it. A Klein surface X is said to be q -hyperelliptic if and only if there exists an involution Φ on X (a dianalytic homeomorphism of order two) such that the quotient $X/\langle\Phi\rangle$ has algebraic genus q . q -hyperelliptic nonorientable KSs without boundary (nonorientable Riemann surfaces) were characterized by means of non-Euclidean crystallographic groups. In this paper, using that characterization, we determine bounds for the order of the automorphism group of a nonorientable q -hyperelliptic Klein surface X such that $X/\langle\Phi\rangle$ has no boundary and prove that the bounds are attained. Besides, we obtain the dimension of the Teichmüller space associated to this type of surfaces.

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1. Introduction. A Klein surface X (KS in short) is a compact surface with a dianalytic structure defined on it [1]. Nonorientable KSs without boundary are also called nonorientable Riemann surfaces. A dianalytic homeomorphism of X onto itself is called an automorphism of X . We say that a Klein surface X is q -hyperelliptic if and only if there exists an involution Φ on X such that the quotient $X/\langle\Phi\rangle$ has algebraic genus q . (If $q = 0$, then X is called *hyperelliptic*, and if $q = 1$, then X is *elliptic-hyperelliptic*.)

Let X be a nonorientable q -hyperelliptic KS without boundary and $\text{Aut}(x)$ the full group of its automorphisms. It is known [11] that $|\text{Aut}(x)| \leq 84(p - 1)$, where p is the algebraic genus of X . This type of surfaces was characterized in [7] by means of non-Euclidean Crystallographic groups, and there, the bound $84(p - 1)$ was reduced to $12(p - 1)$ if the quotient $X/\langle\Phi\rangle$ has boundary.

In this paper, we improve the bound $84(p - 1)$ when $X/\langle\Phi\rangle$ has no boundary. The new bound will depend on the parity of $|\text{Aut}(X)|$. That is made in Section 3. In Section 4, we prove that the bounds are attained by showing different examples. In Section 5, the dimension of the Teichmüller space associated to this type of surfaces is obtained.

Similar characterizations for compact orientable KS without boundary (i.e., Riemann surfaces) and compact KS with boundary have been obtained in [2, 3, 4, 10].

2. Preliminaries. Denote by \mathcal{D} the hyperbolic plane, and by \mathcal{G} its group of isometries. A *non-Euclidean crystallographic group* Γ (NEC group in short) is a discrete subgroup of \mathcal{G} with compact quotient space $X = \mathcal{D}/\Gamma$. NEC groups were introduced by

Wilkie [14] and Macbeath [9] associated to each NEC group a symbol, called *signature*, that determines its algebraic structure and has the following form:

$$\sigma = \left(g; \pm; [m_1, \dots, m_r]; \{ (n_{i1}, \dots, n_{is_i}), i = 1, \dots, k \} \right), \tag{2.1}$$

where $g, m_i,$ and n_{ij} are integers satisfying $g \geq 0, m_i \geq 2,$ and $n_{ij} \geq 2;$ g is the topological genus of $X.$ The sign determines the orientability of $X.$ The numbers m_i are the *proper periods* corresponding to cone points in $X.$ The brackets $(n_{i1}, \dots, n_{is_i})$ are the *period-cycles.* The number k of period-cycles is equal to the number of boundary components of $X.$ The numbers n_{ij} are the periods of the period-cycle $(n_{i1}, \dots, n_{is_i})$ also called *link-periods,* corresponding to corner points in the boundary of $X.$ The number $p = \alpha g + k - 1$ is called the *algebraic genus* of $X,$ where $\alpha = 1$ or $2,$ if the sign of $\sigma(\Gamma)$ is “ $-$ ” or “ $+$ ”, respectively. If the sign of σ is “ $+$ ” and $k = 0,$ then Γ is a Fuchsian group.

An NEC group Γ with signature σ has the following presentation:
Generators:

$$\begin{aligned} x_i, & \quad i = 1, \dots, r; \\ e_i, & \quad i = 1, \dots, k; \\ c_{ij}, & \quad i = 1, \dots, k; j = 0, \dots, s_i; \\ a_i, b_i, & \quad i = 1, \dots, g, \text{ (if } \sigma \text{ has sign +);} \\ d_i, & \quad i = 1, \dots, g, \text{ (if } \sigma \text{ has sign -).} \end{aligned} \tag{2.2}$$

Relations:

$$\begin{aligned} x_i^{m_i}, & \quad i = 1, \dots, r; \\ c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}}, & \quad i = 1, \dots, k; j = 1, \dots, s_i; \\ e_i^{-1}c_{i0}e_ic_{is_i} = 1, & \quad i = 1, \dots, k; \\ x_1 \cdots x_r e_1 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 & \quad \text{(if } \sigma \text{ has sign +),} \\ x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_g^2 = 1 & \quad \text{(if } \sigma \text{ has sign -).} \end{aligned} \tag{2.3}$$

It is known [9] that cyclic permutations of periods in period-cycles or arbitrary permutations of proper periods in the signature τ of an NEC group Γ lead to a signature τ' of an NEC group Γ' isomorphic to $\Gamma.$ Every Γ with signature (2.1) has associated a fundamental region whose area, $|\Gamma|,$ is also called *the area of the group* [12]:

$$|\Gamma| = 2\pi \left(\alpha g + k - 2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right) + \frac{1}{2} \sum_{i=1}^k \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}} \right) \right). \tag{2.4}$$

An NEC group Γ with signature σ actually exists if and only if the right-hand side of (2.4) is greater than zero [15]. If Γ' is a subgroup of an NEC group Γ of finite index $N,$ then it is also an NEC group and the following Hurwitz-Riemann formula holds:

$$|\Gamma'| = N|\Gamma|. \tag{2.5}$$

Let X be a nonorientable Klein surface without boundary of topological genus $g \geq 3$. Then by [11], there exists an NEC group Γ with signature

$$(g; -; [-]; \{-\}) \tag{2.6}$$

such that $X = \mathcal{D}/\Gamma$.

An NEC group with signature (2.6) is said to be a *nonorientable surface group*. In [11] it was proved that if G is a group of automorphisms of a nonorientable Klein surface without boundary $X = \mathcal{D}/\Gamma$, then G can be presented as a quotient Γ'/Γ , for some NEC group Γ' such that $\Gamma \triangleleft \Gamma'$. The full group of automorphisms of X is $\text{Aut}(X) = N_{\mathcal{G}}(\Gamma)/\Gamma$, where $N_{\mathcal{G}}(\Gamma)$ is the normalizer of Γ in \mathcal{G} .

In general, it is very difficult to decide whether a group of automorphisms $G = \Gamma'/\Gamma$ of a Klein surface $X = \mathcal{D}/\Gamma$ equals $\text{Aut}(X)$ or not. This problem is equivalent to the problem of deciding whether Γ' equals the normalizer of Γ in \mathcal{G} or not.

From now on, KS is assumed to have algebraic genus not smaller than 2.

The following results, that appear in [7], will be used through this paper.

THEOREM 2.1. *Let $X = D/\Gamma$ be a nonorientable KS without boundary of algebraic genus p . Then, X is q -hyperelliptic if and only if there exists an NEC group Γ_1 , containing Γ as a subgroup of index 2 and having the signature of one of the following types:*

- (a) $(h; +; [2, \binom{p+1}{2} - 2q, 2]; \{(-)^{q-2h+1}\})$, $0 \leq h \leq q/2$;
- (b) $(h; -; [2, \binom{p+1}{2} - 2q, 2]; \{(-)^{q-h+1}\})$, $0 \leq h \leq q$ if p is even, $1 \leq h \leq q + 1$ if p is odd.

Moreover,

- (1) for each $0 \leq h \leq q/2$, there exists a nonorientable q -hyperelliptic KS without boundary D/Γ whose q -hyperellipticity group has the signature (a);
- (2) for each $0 \leq h \leq q$ (if p is even) and for each $1 \leq h \leq q + 1$ (if p is odd), there exists a nonorientable q -hyperelliptic KS without boundary D/Γ whose q -hyperellipticity group has the signature (b).

THEOREM 2.2. *Let $X = D/\Gamma$ be a nonorientable q -hyperelliptic KS without boundary of algebraic genus $p > 4q + 1$, then the q -hyperelliptic involution Φ is central and unique in $\text{Aut}(X)$.*

3. Bounds for the order of the automorphism group. Let $X = \mathcal{D}/\Gamma$ be a nonorientable q -hyperelliptic KS without boundary, and let Φ be the q -hyperellipticity involution. Then $\langle \Phi \rangle \simeq \Gamma_1/\Gamma$ for a certain NEC group Γ_1 and, by Theorem 2.1, $X/\langle \Phi \rangle \simeq \mathcal{D}/\Gamma_1$ is an orientable KS with boundary, or a nonorientable KS (with or without boundary).

A group of automorphisms of X can be represented as a quotient Γ'/Γ where $\Gamma \triangleleft \Gamma'$. Since Φ is central and unique, we have $\Gamma \triangleleft \Gamma_1 \triangleleft \Gamma'$.

In the next theorem we prove that the bound $84(g - 1)$, for the order of an automorphism group of a nonorientable KS without boundary X , can be improved when X is q -hyperelliptic and $X/\langle \Phi \rangle$ has empty boundary.

THEOREM 3.1. *Let $X = D/\Gamma$ be a nonorientable q -hyperelliptic KS without boundary of algebraic genus $p > 4q + 1$. Let Φ be the q -hyperelliptic involution and $X/\langle \Phi \rangle$ a surface without boundary. Then,*

- (1) if $|\text{Aut}(X)| = 2N$, N is odd, then $|\text{Aut}(X)| \leq 6(q - 1)$,

- (2) if $|\text{Aut}(X)| = 2N$, N is even, and
 - (a) signature of Γ' has sign “-”, then $|\text{Aut}(X)| \leq 12(q-1)$,
 - (b) signature of Γ' has sign “+”, then $|\text{Aut}(X)| \leq 24(q-1)$.

PROOF. Since $\sigma(\Gamma) = (p+1; -; [-]\{-\})$, $p > 4q+1$, and $X/\langle\Phi\rangle$ has no boundary, by [Theorem 2.1](#) there exists a unique NEC group Γ_1 with the signature

$$\sigma(\Gamma_1) = (q+1; -; [2, \overset{(p+1)}{\dots}^{-2q}, 2]), \quad p \text{ is odd.} \tag{3.1}$$

(I) Suppose $[\Gamma' : \Gamma_1] = N$ is odd. We know by [\[5, Chapter 2\]](#) that the signature of Γ' is

$$\sigma(\Gamma') = (g'; -; [m_1, \dots, m_t]). \tag{3.2}$$

Let $x_i \in \Gamma'$, $i = 1, \dots, t$, be the generators of proper-periods such that p_i is the smallest integer satisfying $x_i^{p_i} \in \Gamma_1$ ($p_i \neq 0$). As the only elements of finite order in Γ_1 have order two, then (see [\[5, Section 2.2\]](#))

$$p_i = \frac{m_i}{2} \quad \text{or} \quad p_i = m_i. \tag{3.3}$$

We can suppose, without loss of generality, that for $1 \leq i \leq n$, $p_i = m_i/2$ and for $n+1 \leq i \leq t$, $p_i = m_i$. Thus, by [\[5, Section 2.2\]](#)

$$p - 2q + 1 = \sum_{i=1}^n \left(\frac{N}{p_i}\right) = \sum_{i=1}^n \left(\frac{2N}{m_i}\right). \tag{3.4}$$

From the Riemann-Hurwitz formula $|\Gamma_1| = N|\Gamma'|$, we have

$$(q-1) + \frac{(p-2q+1)}{2} = N \left[(g'-2) + \sum_{i=1}^n \left(1 - \frac{1}{m_i}\right) + \sum_{i=n+1}^t \left(1 - \frac{1}{m_i}\right) \right]. \tag{3.5}$$

By [\(3.4\)](#) and [\(3.5\)](#),

$$\begin{aligned} (q-1) &= N \left[(g'-2) + \sum_{i=1}^n \left(1 - \frac{1}{m_i}\right) - \sum_{i=1}^n \left(\frac{1}{m_i}\right) + \sum_{i=1}^t \left(1 - \frac{1}{m_i}\right) \right] \\ &= N \left[(g'-2) + \sum_{i=1}^n \left(1 - \frac{2}{m_i}\right) + \sum_{i=n+1}^t \left(1 - \frac{1}{m_i}\right) \right], \end{aligned} \tag{3.6}$$

therefore,

$$N = \frac{|\Gamma_1|}{|\Gamma'|} = \frac{(q-1)}{(g'-2) + \sum_{i=1}^n (1 - 2/m_i) + \sum_{i=n+1}^t (1 - 1/m_i)}. \tag{3.7}$$

Since $p > 4q + 1$, then by (3.4) we have

$$\sum_{i=1}^n \left(\frac{N}{m_i} \right) > (q + 1). \tag{3.8}$$

We fix q in (3.7) and denote

$$A = g' - 2 + \sum_{i=1}^n \left(1 - \frac{2}{m_i} \right) + \sum_{i=n+1}^t \left(1 - \frac{1}{m_i} \right), \quad 2 \leq m_i \leq \infty. \tag{3.9}$$

We will consider several cases to determine the minimum positive value of A (that makes N maximal). For n and t fix

- (i) if $g' \geq 2$, then $A^{(1)} \geq \sum_{i=1}^n (1 - 2/m_i) + \sum_{i=n+1}^t (1 - 1/m_i) > 0$;
- (ii) if $g' = 1$, then

$$\begin{aligned} A^{(2)} &= \sum_{i=1}^n \left(1 - \frac{2}{m_i} \right) + \sum_{i=n+1}^t \left(1 - \frac{1}{m_i} \right) - 1 > 0, \\ A^{(1)} &> A^{(2)} > 0, \\ A^{(2)} &\geq k_2 \left(1 - \frac{2}{2} \right) + k_4 \left(1 - \frac{2}{4} \right) + k_6 \left(1 - \frac{2}{6} \right) + k'_2 \left(1 - \frac{1}{2} \right) + k'_3 \left(1 - \frac{1}{3} \right) - 1 \\ &= \frac{1}{2}(k_4 + k'_2) + \frac{2}{3}(k_6 + k'_3) - 1; \end{aligned} \tag{3.10}$$

where k_4, k_6, k'_2 , and k'_3 are nonnegative integers with $k_4 + k_6 + k'_2 + k'_3 \leq t$. The values of k_i and k'_i , for which the above expression is minimum occurs only for $k_4 + k'_2 = 1, k_6 + k'_3 = 1$, but these imply that N is even, a contradiction. Then, the suitable ones are $k_4 + k'_2 = 0$ and $k_6 + k'_3 = 2$, that is,

$$\begin{aligned} k_6 = 2, & \quad k_4 = k'_2 = k'_3 = 0; \\ k'_3 = 2, & \quad k_4 = k_6 = k'_2 = 0; \\ k'_3 = k_6 = 1, & \quad k'_2 = k_4 = 0; \end{aligned} \tag{3.11}$$

and $N_{(\max)} = 3(q - 1)$. Consequently, $|\text{Aut}(X)| \leq 6(q - 1)$ and the signature of Γ' has one of the following signatures:

$$\begin{aligned} \tau_1 &= (1, -, [2, \overset{k_2}{\cdot}, 2, 6, 6]), & q &> \frac{4}{3} \frac{1}{k_2} + 1, & p &= (3k_2 + 4)(q - 1) + 1; \\ \tau_2 &= (1, -, [2, \overset{k_2}{\cdot}, 2, 3, 3]), & q &> \left(\frac{3k_2 + 2}{3k_2 - 2} \right), & p &= (q - 1)(3k_2 + 2) + 1; \\ \tau_3 &= (1, -, [2, \overset{k_2}{\cdot}, 2, 6, 3]), & q &> \left(\frac{3k_2 + 3}{3k_2 - 1} \right), & p &= (q - 1)(3k_2 + 3) + 1. \end{aligned} \tag{3.12}$$

(II) Now, suppose that $[\Gamma' : \Gamma_1] = N$ is even. We know (see [5, 6]) that the signature of Γ' is

$$\sigma(\Gamma') = \left(g'; \pm; [m_1, \dots, m_t]; \{ (n_{i1}, \dots, n_{is_i}), i = 1, \dots, k \} \right). \tag{3.13}$$

The proper-periods of Γ_1 may proceed from elliptic elements x_i or period-cycles of Γ' . If p_i is the order of $\Gamma_1 x_i \in (\Gamma'/\Gamma_1)$, then x_i generates in Γ_1 , $\sum_1^n (N/p_i)$ proper periods, all of them being equal to 2.

Now consider the periods in Γ_1 provided by reflections of Γ' not in Γ_1 . Let q_{ij} be the smallest integer for which $(c_{ij}, c_{ij-1})^{q_{ij}} \in \Gamma_1$, since the only elements of finite order are those of order 2, we have $(n_{ij}/q_{ij}) = 2$ or 1. Let $E = \{(i, j) \mid (n_{ij}/q_{ij}) = 2\}$. Clearly, the only proper-periods in Γ_1 induced by reflections of Γ' are those provided by pairs of reflections corresponding to elements of E , and each such pair produces $(N/2q_{ij})$ periods, all of them being equal to 2. As a result, we obtain $\sum_{(i,j) \in E} (N/2q_{ij})$ proper-periods in Γ_1 . Then by (3.4)

$$p - 2q + 1 = \sum_1^n \left(\frac{2N}{m_i} \right) + \sum_{(i,j) \in E} \left(\frac{N}{2q_{ij}} \right). \tag{3.14}$$

(A) If the sign in the signature of Γ' is minus, by the Riemann-Hurwitz formula, we hold the expression

$$N^{(-)} = \frac{|\Gamma_1|}{|\Gamma'|} = \frac{(q-1)}{g' + k - 2 + \sum_{i=1}^n \left(1 - \frac{2}{m_i}\right) + \sum_{n+1}^t \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{(i,j) \in E} \left(1 - \frac{2}{n_{ij}}\right) + \frac{1}{2} \sum_{(i,j) \notin E} \left(1 - \frac{1}{n_{ij}}\right)}. \tag{3.15}$$

Since $p > 4q + 1$, then by (3.14) we have

$$\sum_1^n \left(\frac{N^{(-)}}{m_i} \right) + \sum_{(i,j) \in E} \left(\frac{N^{(-)}}{2n_{ij}} \right) > (q+1). \tag{3.16}$$

As in previous case (N odd), we are looking for the value of parameters in (3.15), that makes $N^{(-)}$ the maximum possible. Let

$$A = g' - 2 + B_1 + B_2, \tag{3.17}$$

where

$$B_1 = \sum_{i=1}^n \left(1 - \frac{2}{m_i}\right) + \sum_{i=n+1}^t \left(1 - \frac{1}{m_i}\right), \tag{3.18}$$

$$B_2 = k + \frac{1}{2} \sum_{(i,j) \in E} \left(1 - \frac{2}{n_{ij}}\right) + \frac{1}{2} \sum_{(i,j) \notin E} \left(1 - \frac{1}{n_{ij}}\right).$$

For n, t, k , and E fix, we have the following cases:

- (i) if $g' \geq 2$: $A^{(1)} \geq B_1 + B_2$;
- (ii) if $g' = 1$: $A^{(2)} = B_1 + B_2 - 1$. (Obviously $A^{(1)} \geq A^{(2)}$.)

As we have seen, in the case where N is odd,

$$B_1 \geq B_1^* = \frac{1}{2}(k_4 + k_2') + \frac{2}{3}(k_6 + k_3') + 0k_2. \tag{3.19}$$

The minimum positive value A^* of A occurs when $k = 0$, $k_4 + k'_2 = 1$, and $k_6 + k'_3 = 1$ in B_1^* ; so $B_1^* = 7/6$ and $A^* = 7/6 - 1 = 1/6 \leq B_1(\min) - 1 \leq B_1(\min) + B_2(\min) - 1$.

Therefore, $N(\max) = 6(q - 1)$ and

$$|\text{Aut}(X)| \leq 12(q - 1). \tag{3.20}$$

The possible values of k_4 , k'_2 , k_6 , and k'_3 , lead us to conclude that Γ' has one of the following signatures:

$$\begin{aligned} \tau_4 &= (1, -, [2, \overset{k_2}{\cdot}, 2, 4, 6]), & q &> \left(\frac{6k_2 + 9}{6k_2 + 5}\right), & p &= (6k_2 + 7)(q - 1) + 1; \\ \tau_5 &= (1, -, [2, \overset{k_2}{\cdot}, 2, 2, 6]), & q &> \left(\frac{6k_2 + 2}{2k_2}\right), & p &= (6k_2 + 4)(q - 1) + 1; \\ \tau_6 &= (1, -, [2, \overset{k_2}{\cdot}, 2, 4, 3]), & q &> \left(\frac{6k_2 + 5}{6k_2 + 1}\right), & p &= (6k_2 + 5)(q - 1) + 1; \\ \tau_7 &= (1, -, [2, \overset{k_2}{\cdot}, 2, 2, 3]), & q &> \left(\frac{3k_2 + 1}{3k_2 - 1}\right), & p &= (6k_2 + 2)(q - 1) + 1. \end{aligned} \tag{3.21}$$

(B) If the sign in the signature of Γ' is positive, we have, by the Riemann-Hurwitz formula, the following expression:

$$N^{(+)} = \frac{|\Gamma_1|}{|\Gamma|} = \frac{(q - 1)}{2g' - 2 + B_1 + B_2}, \tag{3.22}$$

where B_1 and B_2 are as in (3.18).

Let $A = 2g' - 2 + B_1 + B_2$. As in the previous case we are going to minimize A . We have for n , t , k , and E fix the following cases.

CASE 1. If $g \geq 1$, then $A \geq B_1 + B_2$.

CASE 2. If $g = 0$, then $A \geq A^* = B_1 + B_2 - 2$.

Since the sign in the signature of Γ_1 is negative, then Γ' must have some period-cycle, that is, $k \neq 0$.

If $t = 0$, then $A^* = B_2 - 2$ and

$$\begin{aligned} B_2 &= 1 + \frac{1}{2} \sum_{(i,j) \in E} \left(1 - \frac{2}{n_{ij}}\right) + \frac{1}{2} \sum_{(i,j) \notin E} \left(1 - \frac{1}{n_{ij}}\right) \\ &\geq 1 + \frac{1}{2}N_2\left(1 - \frac{2}{2}\right) + \frac{1}{2}N_4\left(1 - \frac{2}{4}\right) + \frac{1}{2}N_6\left(1 - \frac{2}{6}\right) + \frac{1}{2}N'_2\left(1 - \frac{1}{2}\right) + \frac{1}{2}N'_3\left(1 - \frac{1}{3}\right) \\ &= 1 + \frac{1}{4}(N_4 + N'_2) + \frac{1}{3}(N_6 + N'_3), \end{aligned} \tag{3.23}$$

then

$$A^* \geq \frac{1}{4}(N_4 + N'_2) + \frac{1}{3}(N_6 + N'_3) + 0N_2 - 1 > 0, \tag{3.24}$$

where the nonnegative integers N_i are the number of $n_{ij} = i$, $(i, j) \in E$, and N'_i is the number of $n_{ij} = i$, $(i, j) \notin E$. The value of (3.24) will be minimum for

$$N_4 + N'_2 = 3, \quad N_6 + N'_3 = 1, \tag{3.25}$$

so, $N(\max) = 12(q - 1)$ and

$$|\text{Aut}(X)| \leq 24(q - 1). \tag{3.26}$$

From conditions (3.25), we obtain the following signatures for Γ' :

$$\begin{aligned} \tau_8 &= (0, +, [-], \{(2, \overset{N_2}{\cdot}, 2, 2, 4, 4, 6)\}), \\ \tau_9 &= (0, +, [-], \{(2, \overset{N_2}{\cdot}, 2, 2, 4, 4, 3)\}), \\ \tau_{10} &= (0, +, [-], \{(2, \overset{N_2}{\cdot}, 2, 2, 2, 4, 6)\}), \\ \tau_{11} &= (0, +, [-], \{(2, \overset{N_2}{\cdot}, 2, 2, 2, 4, 3)\}), \\ \tau_{12} &= (0, +, [-], \{(2, \overset{N_2}{\cdot}, 2, 2, 2, 2, 6)\}), \\ \tau_{13} &= (0, +, [-], \{(2, \overset{N_2}{\cdot}, 2, 2, 2, 2, 3)\}), \\ \tau_{14} &= (0, +, [-], \{(2, \overset{N_2}{\cdot}, 2, 4, 4, 4, 6)\}), \\ \tau_{15} &= (0, +, [-], \{(2, \overset{N_2}{\cdot}, 2, 4, 4, 4, 3)\}). \end{aligned} \tag{3.27}$$

Now, if $t \neq 0$, then $A^* = B_1 + B_2 - 2$ and

$$A^* = \frac{1}{2}(k_4 + k'_2) + \frac{2}{3}(k_6 + k'_3) + \frac{1}{4}(N_4 + N'_2) + \frac{1}{3}(N_6 + N'_3) - 1 > 0. \tag{3.28}$$

The smallest value of A appears under the following conditions:

$$k_6 + k'_3 = 0, \quad N_6 + N'_3 = 1, \quad N_4 + N'_2 = 3, \quad k_4 + k'_2 = 0 \tag{3.29}$$

or

$$k_6 + k'_3 = 0, \quad N_6 + N'_3 = 1, \quad N_4 + N'_2 = 1, \quad k_4 + k'_2 = 1. \tag{3.30}$$

Then $N(\max) = 12(q - 1)$ and

$$|\text{Aut}(X)| \leq 24(q - 1). \tag{3.31}$$

By elementary combinatorial methods, we obtain 16 solutions for the signature of Γ'

$$\begin{aligned}
 \tau_{16} &= (0, +, [2, k_2, 2], \{(2, N_2, 2, 2, 2, 2, 6)\}), \\
 \tau_{17} &= (0, +, [2, k_2, 2], \{(2, N_2, 2, 2, 2, 4, 6)\}), \\
 \tau_{18} &= (0, +, [2, k_2, 2], \{(2, N_2, 2, 2, 4, 4, 6)\}), \\
 \tau_{19} &= (0, +, [2, k_2, 2], \{(2, N_2, 2, 4, 4, 4, 6)\}), \\
 \tau_{20} &= (0, +, [2, k_2, 2, 2], \{(2, N_2, 2, 2, 6)\}), \\
 \tau_{21} &= (0, +, [2, k_2, 2, 2], \{(2, N_2, 2, 4, 6)\}), \\
 \tau_{22} &= (0, +, [2, k_2, 2, 4], \{(2, N_2, 2, 2, 6)\}), \\
 \tau_{23} &= (0, +, [2, k_2, 2, 4], \{(2, N_2, 2, 4, 6)\}), \\
 \tau_{24} &= (0, +, [2, k_2, 2], \{(2, N_2, 2, 2, 2, 2, 3)\}), \\
 \tau_{25} &= (0, +, [2, k_2, 2], \{(2, N_2, 2, 2, 2, 4, 3)\}), \\
 \tau_{26} &= (0, +, [2, k_2, 2], \{(2, N_2, 2, 2, 4, 4, 3)\}), \\
 \tau_{27} &= (0, +, [2, k_2, 2], \{(2, N_2, 2, 4, 4, 4, 3)\}), \\
 \tau_{28} &= (0, +, [2, k_2, 2, 2], \{(2, N_2, 2, 2, 3)\}), \\
 \tau_{29} &= (0, +, [2, k_2, 2, 2], \{(2, N_2, 2, 4, 3)\}), \\
 \tau_{30} &= (0, +, [2, k_2, 2, 4], \{(2, N_2, 2, 2, 3)\}), \\
 \tau_{31} &= (0, +, [2, k_2, 2, 4], \{(2, N_2, 2, 4, 3)\}).
 \end{aligned} \tag{3.32}$$

□

COROLLARY 3.2. *Let $X = D/\Gamma$ be a nonorientable q -hyperelliptic KS without boundary of algebraic genus $p > 4q + 1$ and $|\text{Aut}(X)| = \Gamma'/\Gamma$. Let Φ be the q -hyperelliptic involution such that $X/\langle\Phi\rangle$ has no boundary. Then,*

- (1) *if $|\text{Aut}(X)| = 2N$, N is odd, then $|\text{Aut}(X)| \leq (3/2)(p - 6)$;*
- (2) *if $|\text{Aut}(X)| = 2N$, N is even, and*
 - (a) *signature of Γ' has sign “-”, then $|\text{Aut}(X)| \leq 3(p - 6)$,*
 - (b) *signature of Γ' has sign “+”, then $|\text{Aut}(X)| \leq 6(p - 6)$.*

4. Examples. It is natural to ask for what values of q the bounds of [Theorem 3.1](#) are attained. We have found examples for low values of q . Here we present some examples of surfaces with maximal automorphism group for the three different bounds.

In the remainder of this section, θ denotes the canonical epimorphism from Γ' onto $G \simeq \Gamma'/\Gamma$, where Γ' has one of the signatures obtained in [Section 3](#) τ_1, \dots, τ_{31} . We denote by Φ the q -hyperelliptic involution, $\pi : G \rightarrow G/\langle\Phi\rangle$ the natural epimorphism and $\bar{\theta} = \pi\theta$.

The procedure we are going to follow in these examples is as follows:

- (a) to prove the existence of epimorphisms θ and $\bar{\theta}$ as above, such that $\ker \bar{\theta} \simeq \Gamma_1$, $\ker(\theta) \simeq \Gamma$, and $\bar{\theta} = \pi\theta$;
- (b) consequently, we obtain the realization (fulfillment) of the automorphism group $G \simeq \Gamma'/\Gamma$ which attains the bound.

EXAMPLE 4.1. If N is odd, $\sigma(\Gamma') = (1, -, [2, 2, 6, 6])$, and $q = 2$, then $|\text{Aut}(X)| \leq 6$. Let $\bar{\theta} : \Gamma' \rightarrow \bar{G} \simeq \Gamma'/\Gamma_1$ be the canonical epimorphism defined as follows:

$$\bar{\theta}(d_1) = 1, \quad \bar{\theta}(x_1) = \bar{y}, \quad \bar{\theta}(x_2) = \bar{y}^{-1}, \quad \bar{\theta}(x_3) = \bar{\theta}(x_4) = \bar{1}, \quad (4.1)$$

where d_1, x_1, x_2, x_3 , and x_4 are the generators of Γ' , and \bar{y} is an element of order three in Γ'/Γ_1 . As $|\Gamma'/\Gamma_1| = 3$ and it is a group generated by \bar{y} , with the relation $\bar{y}^3 = 1$, we have $\Gamma'/\Gamma_1 \simeq C_3$; so that $G \simeq (\Gamma'/\Gamma)$ would have order 6 and its quotient by a subgroup generated by a central element must be C_3 . Then Γ'/Γ is C_6 .

Finally, we need to prove that there exists a nonorientable 2-hyperelliptic KS without boundary of topological genus $p = 11$ whose automorphism group is C_6 .

Let θ be the epimorphism $\theta : \Gamma' \rightarrow C_6 \simeq \langle y : y^6 = 1 \rangle$ defined by

$$\theta(d_1) = 1, \quad \theta(x_1) = y, \quad \theta(x_2) = y^{-1}, \quad \theta(x_3) = \theta(x_4) = y^3. \quad (4.2)$$

We choose $\Phi = y^3$ as the central element, thus $\ker \theta \simeq \Gamma$. Moreover, it is straightforward to prove $\ker \bar{\theta} \simeq \theta^{-1}(\langle y^3 \rangle) \simeq \Gamma_1$.

EXAMPLE 4.2. Let N be even, $\sigma(\Gamma') = (1, -, [2, 2, 2, 3])$, and $q = 3$. Let θ and $\bar{\theta}$ be the canonical epimorphisms over

$$\begin{aligned} G &\simeq \langle x, y, z : x^2, y^3, z^2, (xy)^3, [xz], [yz] \rangle \simeq A_4 \times C_2, \\ \bar{G} &\simeq \langle x, y : x^2, y^3, (xy)^3 \rangle \simeq A_4, \end{aligned} \quad (4.3)$$

respectively,

$$\begin{aligned} \theta(d_1) &= xy, & \theta(x_1) &= \Phi, & \theta(x_2) &= \Phi, & \theta(x_3) &= x, & \theta(x_4) &= y; \\ \bar{\theta}(d_1) &= \bar{x}\bar{y}, & \bar{\theta}(x_1) &= \bar{1}, & \bar{\theta}(x_2) &= \bar{1}, & \bar{\theta}(x_3) &= \bar{x}, & \bar{\theta}(x_4) &= \bar{y}. \end{aligned} \quad (4.4)$$

It is easy to see that $\text{Ker}(\bar{\theta}) \simeq (4, -, [2, 2, 2, 2], \{-\})$ and $X = \mathcal{D}/\text{Ker}(\theta)$ is a nonorientable surface without boundary of algebraic genus $p = 29$. Now, consider $\Phi = z$. Since z is central in G we have $G/\langle \Phi \rangle = \bar{G}$ and $X/\langle \Phi \rangle$ has algebraic genus 3, therefore, X is a 3-hyperelliptic Klein surface such that $|\text{Aut}(X)| = 12(q - 1) = 24$.

EXAMPLE 4.3. Let N be even, $\sigma(\Gamma') = (0, +, [2], \{(2, 2, 2, 6)\})$, and $q = 2$. Let G, θ, \bar{G} , and $\bar{\theta}$ be defined as follows:

$$\begin{aligned} G &= \langle x, y, z : x^2, y^2, z^2, (xy)^2, (yz)^2, (zx)^6 \rangle \simeq D_6 \times C_2; \\ \theta(x_1) &= \theta(e_1) = \Phi, & \theta(c_{1,0}) &= \theta(c_{1,4}) = x, & \theta(c_{1,1}) &= \Phi xy, \\ & & \theta(c_{1,2}) &= y, & \theta(c_{1,3}) &= z; \\ \bar{G} &= \langle \bar{x}, \bar{y}, \bar{z} : \bar{x}^2, \bar{y}^2, \bar{z}^2, (\bar{x}\bar{y})^2, (\bar{y}\bar{z})^2, (\bar{z}\bar{x})^3 \rangle \simeq D_3 \times C_2, \\ \bar{\theta}(x_1) &= \bar{\theta}(e_1) = \bar{1}, & \bar{\theta}(c_{1,0}) &= \bar{\theta}(c_{1,4}) = \bar{x}, & \bar{\theta}(c_{1,1}) &= \bar{x}\bar{y}, \\ & & \bar{\theta}(c_{1,2}) &= \bar{y}, & \bar{\theta}(c_{1,2}) &= \bar{z}. \end{aligned} \quad (4.5)$$

We can see that $\text{Ker}(\tilde{\theta})$ has signature $\tau = (3, -, [2, \overset{14}{2}, 2], \{-\})$ and $X = \mathcal{D}/\text{Ker}(\theta)$ is a nonorientable surface without boundary and algebraic genus $p = 17$. Consider $\Phi = (zx)^3$ an order-two central element in G , we have $G/\Phi = \tilde{G}$ and $X/\langle\Phi\rangle$ has signature τ . In particular, $X/\langle\Phi\rangle$ has algebraic genus 2, then by [Theorem 2.1](#) X is a 2-hyperelliptic Klein surface. Furthermore, G is an automorphism group of X . Now, by [Theorem 3.1](#), we have $\text{Aut}(X) \leq 24(q - 1)$, then $G = \text{Aut}(X)$ attaining the upper bound.

It remains as an open problem to determine topological types of nonorientable q -hyperelliptic Klein surfaces without boundary admitting maximal automorphism group for arbitrary q .

5. Teichmüller space. In this section, we are going to study the dimension of the Teichmüller space associated to nonorientable q -hyperelliptic Klein surfaces without boundary of algebraic genus $p > 4q + 1$ such that $X/\langle\Phi\rangle$ has empty boundary. We denote this space by T_q^{-p} .

Let Γ be an NEC group. We define the *Weil space* associated to Γ , $R(\Gamma)$, as the set of monomorphisms $r : \Gamma \rightarrow \mathcal{G}$ such that $r(\Gamma)$ is discrete and $\mathcal{D}/r(\Gamma)$ is compact.

We say that two elements $r_1, r_2 \in R(\Gamma)$ are equivalents if there exists $g \in \mathcal{G}$ such that $r_1(\Gamma) = gr_2(\Gamma)g^{-1}$. The quotient space by this relation is called the Teichmüller space of Γ and denoted by $T(\Gamma)$. When Γ is a Fuchsian group (only orientation preserving elements) with signature $(g, +, [m_1, \dots, m_r])$, it is known that $T(\Gamma)$ is a cell of dimension $d(\Gamma) = 6(g - 1) + 2r$. Singerman [13] proved that if Γ is a proper NEC group, then $d(\Gamma) = (1/2)d(\Gamma^+)$ where Γ^+ is the canonical Fuchsian group associated to Γ . The Teichmüller *Modular group* of Γ , $M(\Gamma)$, is the quotient group $\text{Aut}(\Gamma)/I(\Gamma)$ where $\text{Aut}(\Gamma)$ is the full group of automorphisms of Γ and $I(\Gamma)$ the set of inner automorphisms. The Modular group $M(\Gamma)$ acts on $T(\Gamma)$ as follows: if $[\tau] \in T(\Gamma)$ and $\tilde{\alpha} \in M(\Gamma)$, then $\tilde{\alpha}[\tau] = [\tau \cdot \alpha]$.

Let $X = \mathcal{D}/\Gamma$ be a nonorientable q -hyperelliptic Klein surfaces without boundary of algebraic genus $p > 4q + 1$ such that $X/\langle\Phi\rangle$ has empty boundary. Following Harvey [8],

$$T_q^{-p} = \bigcup_{\tilde{\alpha} \in M(\Gamma)} \tilde{\alpha} \left(\bigcup_{\Phi \in \Phi(\Gamma, \Gamma_1, \Gamma_1/\Gamma)} \text{Im } \hat{i}_\Phi \right), \tag{5.1}$$

where Γ_1 is the group of the q -hyperellipticity of X with signature

$$(q + 1, -, [2, \overset{p+1}{\dots}, \overset{2q}{2}], \{-\}), \quad \Phi(\Gamma, \Gamma_1, \Gamma_1/\Gamma) \tag{5.2}$$

is the set of all equivalence classes of surjections $\phi : \Gamma_1 \rightarrow \mathbb{Z}_2$ with $\text{Ker}(\phi) = \Gamma$ modulo the action of $\text{Aut}(\Gamma_1)$ and $\text{Aut}(\mathbb{Z}_2)$, and $\text{Im } \hat{i}_\phi$ is the image set $\hat{i}_\phi(T(\Gamma_1))$ under the isometry \hat{i}_ϕ induced by the inclusion $i_\phi : \text{Ker } \phi \rightarrow \Gamma_1$.

Here the family $\Phi(\Gamma, \Gamma_1, \Gamma_1/\Gamma)$ consists of $q + 2$ classes each of them characterized by the number of glide reflections between the canonical generators of Γ_1 in $\text{Ker } \phi$ (to know d_1, \dots, d_{q+1}). If we call these classes $[\phi_i], i = 0, \dots, q + 1$; a representative of each

class is defined by

$$\begin{aligned}
 \phi_i(d_j) &= \bar{0} && \text{for } j = 1, \dots, i; \\
 \phi_i(d_j) &= \bar{1} && \text{for } j = i + 1, \dots, q + 1; \\
 \phi_i(x_j) &= \bar{1} && \text{for } j = 1, \dots, p + 1 - 2q.
 \end{aligned}
 \tag{5.3}$$

Then,

$$T_q^{-p} = \bigcup_{\bar{\alpha} \in M(\Gamma)} \bar{\alpha}(\text{Im } \hat{i}_0 \cup \dots \cup \text{Im } \hat{i}_{q+1}),
 \tag{5.4}$$

where $\hat{i}_j : T(\Gamma_1) \rightarrow T(\Gamma)$, $\tau \in R(\Gamma_1)$ $[\tau] \mapsto [\tau \cdot i_j]$ is the real analytic homeomorphism (see [8]) induced by the inclusion $i_j : \text{Ker } \phi_j \rightarrow \Gamma_1$.

The situation allows us to use Maclachlan’s method (see [10, Lemma 3]), to prove that (5.4) is a disjoint union.

Suppose that there exist $l, m \in \{0, \dots, q + 1\}$, such that

$$\bar{\alpha} \hat{i}_l(T(\Gamma_1)) \cap \hat{i}_m(T(\Gamma_1)) \neq \emptyset.
 \tag{5.5}$$

Therefore, there exist $\tau_1, \tau_2 \in R(\Gamma_1)$ satisfying

$$[\tau_1 \cdot i_l \cdot \alpha] = [\tau_2 \cdot i_m]
 \tag{5.6}$$

if and only if there exists $t \in \mathcal{G}$ such that

$$\tau_1 \cdot i_l \cdot \alpha(f) = t(\tau_2 \cdot i_m)(f)t^{-1}, \quad \forall f \in \Gamma_1.
 \tag{5.7}$$

Consider $\tau_3 : \tau_3(f) = t\tau_2(f)t^{-1}$, for all $f \in \Gamma_1$; then $\tau_1 \cdot i_l \cdot \alpha = \tau_3 \cdot i_m$. Now, as Γ_1 is unique we can consider the inverse image of τ_3 to obtain

$$(\tau_3^{-1} \cdot \tau_1) \cdot i_l \cdot \alpha = i_m
 \tag{5.8}$$

being $\tau_3^{-1} \cdot \tau_1$ an automorphism of Γ_1 . This leads us to conclude that $l = m$. Thus, T_q^{-p} is a disjoint union of copies $\hat{i}_l(T(\Gamma_1))$, that is, T_q^{-p} is a submanifold of $T(\Gamma)$ of dimension

$$d(\Gamma_1) = 2p - q - 1.
 \tag{5.9}$$

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