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A NEW PROOF OF SOME IDENTITIES OF BRESSOUD

ROBIN CHAPMAN

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We provide a new proof of the following two identities due to Bressoud: $\sum_{m=0}^{N} q^{m^2}\left[\begin{smallmatrix}N\\m\end{smallmatrix}\right] = \sum_{m=-\infty}^{\infty}(-1)^m q^{m(5m+1)/2}\left[\begin{smallmatrix}2N\\N+2m\end{smallmatrix}\right]$, $\sum_{m=0}^{N} q^{m^2+m}\left[\begin{smallmatrix}N\\m\end{smallmatrix}\right] = (1/(1-q^{N+1}))\sum_{m=-\infty}^{\infty}(-1)^m \times q^{m(5m+3)/2}\left[\begin{smallmatrix}2N+2\\N+2m+2\end{smallmatrix}\right]$, which can be considered as finite versions of the Rogers-Ramanujan identities.

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In [1], Bressoud proves the following theorem, from which the Rogers-Ramanujan identities follow on letting $N \to \infty$.

Theorem 1. *For each integer $N \ge 0$,*

$$\begin{aligned}\sum_{m=0}^{N} q^{m^2}\begin{bmatrix}N\\m\end{bmatrix} &= \sum_{m=-\infty}^{\infty}(-1)^m q^{m(5m+1)/2}\begin{bmatrix}2N\\N+2m\end{bmatrix},\\ \sum_{m=0}^{N} q^{m^2+m}\begin{bmatrix}N\\m\end{bmatrix} &= \frac{1}{1-q^{N+1}}\sum_{m=-\infty}^{\infty}(-1)^m q^{m(5m+3)/2}\begin{bmatrix}2N+2\\N+2m+2\end{bmatrix}.\end{aligned} \tag{1}$$

Here,

$$\begin{bmatrix}N\\m\end{bmatrix} = \begin{cases}\dfrac{(q)_N}{(q)_m(q)_{N-m}} & \text{if } 0 \le m \le N;\\ 0 & \text{otherwise}\end{cases} \tag{2}$$

denotes a Gaussian binomial coefficient, where we adopt the standard q-series notation

$$(q)_n = \prod_{j=1}^{n}(1-q^j). \tag{3}$$

We give an alternative proof of Theorem 1 by showing that the left and right sides of (1) satisfy the same recurrence relations.

Define, for integers a and $N \ge 0$,

$$S_a(N) = \sum_{n=0}^{N} q^{n^2+an}\begin{bmatrix}N\\n\end{bmatrix}. \tag{4}$$

Lemma 2. *For each integer $N \ge 1$ and each a,*

$$S_a(N) = S_a(N-1) + q^{N+a}S_{a+1}(N-1), \tag{5}$$

$$S_a(N) = S_{a+1}(N-1) + q^{a+1}S_{a+2}(N-1). \tag{6}$$

Proof. Using the identity

$$\begin{bmatrix} N \\ n \end{bmatrix} = q^{N-n} \begin{bmatrix} N-1 \\ n-1 \end{bmatrix} + \begin{bmatrix} N-1 \\ n \end{bmatrix} \tag{7}$$

gives

$$\begin{aligned} S_a(N) &= q^N \sum_{n=1}^{N} q^{n^2+(a-1)n} \begin{bmatrix} N-1 \\ n-1 \end{bmatrix} + \sum_{n=0}^{N-1} q^{n^2+an} \begin{bmatrix} N-1 \\ n \end{bmatrix} \\ &= q^N \sum_{n=0}^{N-1} q^{(n+1)^2+(a-1)(n+1)} \begin{bmatrix} N-1 \\ n \end{bmatrix} + S_a(N-1) \\ &= q^{N+a} S_{a+1}(N-1) + S_a(N-1). \end{aligned} \tag{8}$$

On the other hand, using the identity

$$\begin{bmatrix} N \\ n \end{bmatrix} = \begin{bmatrix} N-1 \\ n-1 \end{bmatrix} + q^n \begin{bmatrix} N-1 \\ n \end{bmatrix} \tag{9}$$

gives

$$\begin{aligned} S_a(N) &= \sum_{n=1}^{N} q^{n^2+an} \begin{bmatrix} N-1 \\ n-1 \end{bmatrix} + \sum_{n=0}^{N-1} q^{n^2+(a+1)n} \begin{bmatrix} N-1 \\ n \end{bmatrix} \\ &= \sum_{n=0}^{N-1} q^{(n+1)^2+a(n+1)} \begin{bmatrix} N-1 \\ n \end{bmatrix} + S_{a+1}(N-1) \\ &= q^{a+1} S_{a+2}(N-1) + S_{a+1}(N-1). \end{aligned} \tag{10}$$

□

We now equate (5) and (6).

Lemma 3. *For integers $N \geq 0$ and each a,*

$$S_a(N) + (q^{N+a+1} - 1) S_{a+1}(N) - q^{a+1} S_{a+2}(N) = 0. \tag{11}$$

Proof. Equating (5) and (6) gives

$$S_a(N-1) + (q^{N+a} - 1) S_{a+1}(N-1) - q^{a+1} S_{a+2}(N-1) = 0 \tag{12}$$

for $N \geq 1$. Replacing N by $N+1$ gives

$$S_a(N) + (q^{N+a+1} - 1) S_{a+1}(N) - q^{a+1} S_{a+2}(N) = 0. \tag{13}$$

□

We will use the $a = 0$ case of Lemma 3 which is

$$S_0(N) + (q^{N+1} - 1) S_1(N) - q S_2(N) = 0. \tag{14}$$

Clearly, $S_a(0) = 1$ for all a. Also, for $N > 0$, (5) gives

$$S_0(N) = S_0(N-1) + q^N S_1(N-1) \tag{15}$$

and, together with (14), gives

$$\begin{aligned} S_1(N) &= S_1(N-1) + q^{N+1} S_2(N-1) \\ &= S_1(N-1) + q^N [S_0(N-1) + (q^N - 1) S_1(N-1)] \\ &= q^N S_0(N-1) + (q^{2N} - q^N + 1) S_1(N-1). \end{aligned} \tag{16}$$

Together with the initial conditions $S_0(0) = S_1(0) = 1$, (15) and (16) completely define $S_0(N)$ and $S_1(N)$ for $N \geq 0$.

We now gather some consequences of these recurrences which will be used later.

Lemma 4. *For $N \geq 2$,*

$$S_0(N) = (1 + q^{2N-1}) S_0(N-1) + q^N (1 - q^N) S_1(N-2); \tag{17}$$

and for $N \geq 1$,

$$S_1(N) = q^N S_0(N) + (1 - q^N) S_1(N-1). \tag{18}$$

Proof. First of all, from (15) and (16), we have

$$S_1(N) - q^N S_0(N) = (1 - q^N) S_1(N-1) \tag{19}$$

and so, for $N \geq 2$,

$$S_1(N-1) - q^{N-1} S_0(N-1) = (1 - q^{N-1}) S_1(N-1). \tag{20}$$

Hence, by (15) again,

$$\begin{aligned} S_0(N) &= S_0(N-1) + q^N S_1(N-1) \\ &= S_0(N-1) + q^N [q^{N-1} S_0(N-1) + (1 - q^N) S_1(N-2)] \\ &= (1 + q^{2N-1}) S_0(N-1) + q^N (1 - q^N) S_1(N-2), \end{aligned} \tag{21}$$

and also by using (16),

$$\begin{aligned} S_1(N) &= q^N S_0(N-1) + (1 - q^N + q^{2N}) S_1(N-1) \\ &= q^N [S_0(N) - q^N S_1(N-1)] + (1 - q^N + q^{2N}) S_1(N-1) \\ &= q^N S_0(N) + (1 - q^N) S_1(N-1). \end{aligned} \tag{22}$$

□

The recurrences (17) and (18) with the initial conditions $S_0(0) = S_1(0) = 1$, $S_0(1) = 1 + q$ define $S_0(N)$ and $S_1(N)$ uniquely for $N \geq 0$.

Let

$$\begin{aligned} B_0(N) &= \sum_m (-1)^m q^{m(5m+1)/2} \begin{bmatrix} 2N \\ N + 2m \end{bmatrix}, \\ B_1(N) &= \sum_m (-1)^m q^{m(5m+3)/2} \begin{bmatrix} 2N + 2 \\ N + 2m + 2 \end{bmatrix} \end{aligned} \tag{23}$$

denote the sums appearing on the right sides of the identities in Theorem 1. Setting $r = N+2m$ in the definition of $B_0(N)$ gives

$$B_0(N) = \sum_{r\equiv N\ (4)} q^{(5/8)(r-N)^2+(1/4)(r-N)} \begin{bmatrix} 2N \\ r \end{bmatrix} - \sum_{r\equiv N+2\ (4)} q^{(5/8)(r-N)^2+(1/4)(r-N)} \begin{bmatrix} 2N \\ r \end{bmatrix}$$
$$= q^{-1/40}\left[\sum_{r\equiv N\ (4)} q^{(5/8)(r-N+1/5)^2} \begin{bmatrix} 2N \\ r \end{bmatrix} - \sum_{r\equiv N+2\ (4)} q^{(5/8)(r-N+1/5)^2} \begin{bmatrix} 2N \\ r \end{bmatrix}\right]. \tag{24}$$

This suggests the notation

$$A(M,k,b) = \sum_{2r\equiv M+k\ (8)} q^{(5/8)(r-M/2+b)^2} \begin{bmatrix} M \\ r \end{bmatrix} \tag{25}$$

so that

$$q^{1/40}B_0(N) = A\left(2N,0,\frac{1}{5}\right) - A\left(2N,4,\frac{1}{5}\right). \tag{26}$$

Of course, $A(M,k,b) = 0$ if $M+k$ is odd, and $A(M,k,b)$ depends only on M,b and the congruence class of k modulo 8. A similar computation yields

$$q^{9/40}B_1(N) = A\left(2N+2,2,-\frac{2}{5}\right) - A\left(2N+2,-2,-\frac{2}{5}\right). \tag{27}$$

We aim at showing that $B_0(N)$ and $(1-q^{N+1})B_1(N)$ satisfy the same system of recurrences as $S_0(N)$ and $S_1(N)$.

Lemma 5. *The following holds*

$$A(M,k,b) = A(M,-k,-b) \tag{28}$$

for each M,k, and b.

Proof. Replacing r by $M-r$ in the sum for $A(M,k,b)$ yields

$$\begin{aligned} A(M,k,b) &= \sum_{2M-2r\equiv M+k\ (8)} q^{(5/8)(M/2-r+b)^2} \begin{bmatrix} M \\ M-r \end{bmatrix} \\ &= \sum_{2r\equiv M-k\ (8)} q^{(5/8)(r-M/2-b)^2} \begin{bmatrix} M \\ r \end{bmatrix} \\ &= A(M,-k,-b). \end{aligned} \tag{29}$$

□

We now wish to produce recurrences for the $A(M,k,b)$.

Lemma 6. *The following holds*

$$\begin{aligned} A(M+1,k,b) &= A\left(M,k-1,b+\frac{1}{2}\right) + q^{M/2+1/10-b}A\left(M,k+1,b+\frac{3}{10}\right), \\ A(M+1,k,b) &= A\left(M,k+1,b-\frac{1}{2}\right) + q^{M/2+1/10+b}A\left(M,k-1,b-\frac{3}{10}\right) \end{aligned} \tag{30}$$

for each M,k, and b.

Proof. Using the formula

$$\begin{bmatrix} M+1 \\ r \end{bmatrix} = \begin{bmatrix} M \\ r-1 \end{bmatrix} + q^r \begin{bmatrix} M \\ r \end{bmatrix} \tag{31}$$

in the definition of $A(M+1,k,b)$ gives $A(M+1,k,b) = S_1 + S_2$, where

$$\begin{aligned} S_1 &= \sum_{2r \equiv M+k+1\ (8)} q^{(5/8)(r-M/2-1/2+b)^2} \begin{bmatrix} M \\ r-1 \end{bmatrix} \\ &= \sum_{2s \equiv M+k-1\ (8)} q^{(5/8)(s-M/2+1/2+b)^2} \begin{bmatrix} M \\ s \end{bmatrix} \\ &= A\left(M, k-1, b+\frac{1}{2}\right), \\ S_2 &= \sum_{2r \equiv M+k+1\ (8)} q^{r+(5/8)(r-M/2-1/2+b)^2} \begin{bmatrix} M \\ r \end{bmatrix}. \end{aligned} \tag{32}$$

But

$$r + \frac{5(r-M/2-1/2+b)^2}{8} = \frac{5(r-M/2+3/10+b)^2}{8} + \frac{M}{2} + \frac{1}{10} - b. \tag{33}$$

Hence,

$$A(M+1,k,b) = A\left(M, k-1, b+\frac{1}{2}\right) + q^{M/2+1/10-b} A\left(M, k+1, b+\frac{3}{10}\right). \tag{34}$$

Consequently, by Lemma 5 also,

$$\begin{aligned} A(M+1,k,b) &= A(M+1,-k,-b) \\ &= A\left(M, -k-1, -b+\frac{1}{2}\right) + q^{M/2+1/10+b} A\left(M, -k+1, -b+\frac{3}{10}\right) \\ &= A\left(M, k+1, b-\frac{1}{2}\right) + q^{M/2+1/10+b} A\left(M, k-1, b-\frac{3}{10}\right). \end{aligned} \tag{35}$$

□

It is convenient to note that replacing M by $M-1$ in these identities gives

$$\begin{aligned} A(M,k,b) &= A\left(M-1, k-1, b+\frac{1}{2}\right) + q^{M/2-2/5-b} A\left(M-1, k+1, b+\frac{3}{10}\right) \\ &= A\left(M-1, k+1, b-\frac{1}{2}\right) + q^{M/2-2/5+b} A\left(M-1, k-1, b-\frac{3}{10}\right). \end{aligned} \tag{36}$$

Lemma 7. *The sums $B_0(N)$ and $B_1(N)$ obey the recurrences*

$$B_0(N) = (1+q^{2N-1}) B_0(N-1) + q^N B_1(N-2) \tag{37}$$

for $N \geq 2$ and

$$B_1(N) = (1-q^{N+1}) B_1(N-1) + q^N (1-q^{N+1}) B_0(N) \tag{38}$$

for $N \geq 1$.

Proof. We compute

$$\begin{aligned}A\Big(2N,k,\frac{1}{5}\Big) &= A\Big(2N-1,k+1,-\frac{3}{10}\Big)+q^{N-1/5}A\Big(2N-1,k-1,-\frac{1}{10}\Big)\\ &= A\Big(2N-2,k,\frac{1}{5}\Big)+q^{N-3/5}A(2N-2,k+2,0)\\ &\quad +q^{N-1/5}A\Big(2N-2,k-2,\frac{2}{5}\Big)+q^{2N-1}A\Big(2N-2,k,\frac{1}{5}\Big)\\ &= (1+q^{2N-1})A\Big(2N-2,k,\frac{1}{5}\Big)+q^{N-3/5}A(2N-2,k+2,0)\\ &\quad +q^{N-1/5}A\Big(2N-2,k-2,\frac{2}{5}\Big).\end{aligned} \tag{39}$$

In particular,

$$\begin{aligned}A\Big(2N,0,\frac{1}{5}\Big) &= (1+q^{2N-1})A\Big(2N-2,0,\frac{1}{5}\Big)\\ &\quad +q^{N-3/5}A(2N-2,2,0)+q^{N-1/5}A\Big(2N-2,-2,\frac{2}{5}\Big),\\ A\Big(2N,4,\frac{1}{5}\Big) &= (1+q^{2N-1})A\Big(2N-2,4,\frac{1}{5}\Big)\\ &\quad +q^{N-3/5}A(2N-2,6,0)+q^{N-1/5}A\Big(2N-2,2,\frac{2}{5}\Big)\\ &\quad +q^{N-3/5}A(2N-2,-2,0)+q^{N-1/5}A\Big(2N-2,2,\frac{2}{5}\Big).\end{aligned} \tag{40}$$

Noting that

$$\begin{aligned}A(2N-2,2,0) &= A(2N-2,-2,0),\\ A\Big(2N-2,2,\frac{2}{5}\Big) &= A\Big(2N-2,-2,-\frac{2}{5}\Big),\end{aligned} \tag{41}$$

subtracting gives

$$\begin{aligned}q^{1/40}B_0(N) &= A\Big(2N,0,\frac{1}{5}\Big)-A\Big(2N,4,\frac{1}{5}\Big)\\ &= (1+q^{2N-1})\Big[A\Big(2N-2,0,\frac{1}{5}\Big)-A\Big(2N-2,4,\frac{1}{5}\Big)\Big]\\ &\quad +q^{N-1/5}\Big[A\Big(2N-2,2,-\frac{2}{5}\Big)-A\Big(2N-2,-2,-\frac{2}{5}\Big)\Big]\\ &= (1+q^{2N-1})q^{1/40}B_0(N-1)+q^{N-1/5}q^{9/40}B_1(N-2)\end{aligned} \tag{42}$$

and so

$$B_0(N) = (1+q^{2N-1})B_0(N-1)+q^N B_1(N-2). \tag{43}$$

Also,

$$\begin{aligned} A\left(2N+2,k,-\frac{2}{5}\right) &= A\left(2N+1,k-1,\frac{1}{10}\right)+q^{N+1}A\left(2N+1,k+1,-\frac{1}{10}\right)\\ &= A\left(2N,k,-\frac{2}{5}\right)+q^{N+1/5}A\left(2N,k-2,-\frac{1}{5}\right)\\ &\quad +q^{N+1}A\left(2N,k,\frac{2}{5}\right)+q^{2N+6/5}A\left(2N,k+2,\frac{1}{5}\right)\\ &= A\left(2N,k,-\frac{2}{5}\right)+q^{N+1}A\left(2N,-k,-\frac{2}{5}\right)\\ &\quad +q^{N+1/5}A\left(2N,2-k,\frac{1}{5}\right)+q^{2N+6/5}A\left(2N,k+2,\frac{1}{5}\right). \end{aligned} \tag{44}$$

Consequently,

$$\begin{aligned} q^{9/40}B_1(N) &= A\left(2N+2,2,-\frac{2}{5}\right)-A\left(2N+2,-2,-\frac{2}{5}\right)\\ &= A\left(2N,2,-\frac{2}{5}\right)+q^{N+1}A\left(2N,-2-\frac{2}{5}\right)\\ &\quad -A\left(2N,-2,-\frac{2}{5}\right)-q^{N+1}A\left(2N,2,-\frac{2}{5}\right)\\ &\quad +q^{N+1/5}\left[A\left(2N,0,\frac{1}{5}\right)-A\left(2N,4,\frac{1}{5}\right)\right]\\ &\quad +q^{2N+6/5}\left[A\left(2N,4,\frac{1}{5}\right)-A\left(2N,0,\frac{1}{5}\right)\right]\\ &= (1-q^{N+1})[q^{9/40}B_1(N-1)+q^{N+1/5}q^{1/40}B_0(N)] \end{aligned} \tag{45}$$

and so

$$B_1(N) = (1-q^{N+1})B_1(N-1)+q^N(1-q^{N+1})B_0(N). \tag{46}$$

□

By Lemma 4, $S_0(N)$ and $(1-q^{N+1})S_1(N)$ satisfy the same recurrences as $B_0(N)$ and $B_1(N)$. Also, $S_0(0) = 1 = B_0(0)$, $S_0(1) = 1+q = B_0(1)$, and $(1-q)S_1(0) = 1-q = B_1(0)$. Consequently, we deduce Theorem 1: $S_0(N) = B_0(N)$ and $(1-q^{N+1})S_1(N) = B_1(N)$.

References

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Robin Chapman: School of Mathematical Sciences, University of Exeter, Exeter, EX4 4QE, UK

E-mail address: rjc@maths.ex.ac.uk