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ON INCLUSION RELATIONS FOR ABSOLUTE SUMMABILITY

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We obtain necessary and (different) sufficient conditions for a series summable $|\bar{N},p_n|_k$, $1 < k \le s < \infty$, to imply that the series is summable $|T|_s$, where $(\bar{N},p_n)$ is a weighted mean matrix and T is a lower triangular matrix. As corollaries of this result, we obtain several inclusion theorems.

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Let $\sum a_n$ be a given series with partial sums s_n, (C,α) the Césaro matrix of order α. If σ_n^α denotes the nth term of the (C,α)-transform of $\{s_n\}$ then, from Flett [4], $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \ge 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} \left| \sigma_n^\alpha - \sigma_{n-1}^\alpha \right|^k < \infty. \tag{1}$$

For any sequence $\{u_n\}$, the forward difference operator Δ is defined by $\Delta u_n = u_n - u_{n+1}$.

An appropriate extension of (1) to arbitrary lower triangular matrices T is

$$\sum_{n=1}^{\infty} n^{k-1} \left| \Delta t_{n-1} \right|^k < \infty, \tag{2}$$

where

$$t_n := \sum_{k=0}^{n} t_{nk} s_k. \tag{3}$$

Such an extension is used, for example, in Bor [2]. But Sarigöl, Sulaiman, and Bor and Thorpe [3] make the following extension of (1).

They define a series $\sum a_n$ to be summable $|\bar{N},p_n|_k$, $k \ge 1$ if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} \left| \Delta Z_{n-1} \right|^k < \infty, \tag{4}$$

where Z_n denotes the nth term of the weighted mean transform of $\{s_n\}$; that is,

$$Z_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k. \tag{5}$$

Apparently they have interpreted the n in (1) to represent the reciprocal of the nth diagonal term of the matrix $(\bar{N},p_n)$. (See, e.g., Sarigöl [6], where this is explicitly the case.)

Unfortunately, this interpretation cannot be correct. For if it were, then, since the nth diagonal entry of (C,α) is $O(n^{-\alpha})$, (1) would take the form

$$\sum_{n=1}^{\infty} (n^{\alpha})^{(k-1)} |\sigma_n^{\alpha} - \sigma_{n-1}^{\alpha}|^k < \infty. \tag{6}$$

However, Flett stays with (1). Thus (2) is an appropriate extension of (1) to lower triangular matrices.

Given any lower triangular matrix T, we can associate the matrices $\bar{T}$ and $\hat{T}$ with entries defined by

$$\bar{t}_{nk} = \sum_{i=k}^{n} t_{ni}, \quad \hat{t}_{nk} = \bar{t}_{nk} - \bar{t}_{n-1,k}, \tag{7}$$

respectively.

Thus, from (3),

$$\begin{aligned} t_n &= \sum_{k=0}^{n} t_{nk} s_k = \sum_{k=0}^{n} t_{nk} \sum_{i=0}^{k} a_i = \sum_{i=0}^{n} a_i \sum_{k=i}^{n} t_{nk} = \sum_{i=0}^{n} \bar{t}_{nk} a_i, \\ Y_n &:= t_n - t_{n-1} = \sum_{i=0}^{n} \bar{t}_{ni} a_i - \sum_{i=0}^{n-1} \bar{t}_{n-1,i} a_i = \sum_{i=0}^{n} \hat{t}_{ni} a_i, \quad \text{since } \bar{t}_{n-1,n} = 0. \end{aligned} \tag{8}$$

For a weighted mean matrix $A = (\bar{N}, p_n)$,

$$\bar{a}_{nk} = \sum_{i=k}^{n} \frac{p_k}{P_n} = \frac{1}{P_n}(P_n - P_{k-1}) = 1 - \frac{P_{k-1}}{P_n}. \tag{9}$$

Thus

$$\hat{a}_{nk} = \bar{a}_{nk} - \bar{a}_{n-1,k} = 1 - \frac{P_{k-1}}{P_n} - 1 + \frac{P_{k-1}}{P_{n-1}} = \frac{p_n P_{k-1}}{P_n P_{n-1}}, \tag{10}$$

so that, from (5),

$$X_n := \Delta Z_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{k=0}^{n-1} P_{k-1} a_k = \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_{\nu-1} a_\nu, \tag{11}$$

since $P_{-1} = 0$.

We will always assume that $\{p_n\}$ is a positive sequence with $P_n \to \infty$. Also, $\Delta_\nu \hat{t}_{n\nu} := \hat{t}_{n\nu} - \hat{t}_{n,\nu+1}$.

Theorem 1. *Let $1 < k \le s < \infty$, $\{p_n\}$ satisfying*

$$\sum_{n=\nu+1}^{\infty} n^{k-1} \left(\frac{p_n}{P_n P_{n-1}} \right)^k = O\left(\frac{1}{P_\nu^k} \right). \tag{12}$$

Let T be a lower triangular matrix. Then, the necessary conditions for $\sum a_n$ summable $|\bar{N}, p_n|_k$ to imply $\sum a_n$ is summable $|T|_s$ are

(i) $P_\nu |t_{\nu\nu}| / p_\nu = O(\nu^{1/s - 1/k})$;

(ii) $\sum_{n=\nu+1}^{\infty} n^{s-1} |\Delta_\nu \hat{t}_{n\nu}|^s = O(\nu^{s - s/k} (p_\nu / P_\nu)^s)$;

(iii) $\sum_{n=\nu+1}^{\infty} n^{s-1} |\hat{t}_{n,\nu+1}|^s = O(1)$.

Proof. We are given that

$$\sum_{n=1}^{\infty} n^{s-1}|Y_n|^s < \infty, \tag{13}$$

whenever

$$\sum_{n=1}^{\infty} n^{k-1}|X_n|^k < \infty. \tag{14}$$

Now, the space of sequences $\{a_n\}$ satisfying (14) is a Banach space if normed by

$$\|X\| = \left(|X_0|^k + \sum_{n=1}^{\infty} n^{k-1}|X_n|^k\right)^{1/k}. \tag{15}$$

We also consider the space of those sequences $\{Y_n\}$ that satisfy (13). This is also a BK-space with respect to the norm

$$\|Y\| = \left(|Y_0|^k + \sum_{n=1}^{\infty} n^{s-1}|Y_n|^s\right)^{1/s}. \tag{16}$$

Observe that (8) transforms the space of sequences satisfying (14) into the space of sequences satisfying (13). Applying the Banach-Steinhaus theorem, there exists a constant $K > 0$ such that

$$\|Y\| \le K\|X\|. \tag{17}$$

Applying (11) and (8) to $a_\nu = e_\nu - e_{\nu+1}$, where e_ν is the νth coordinate vector, we have, respectively,

$$X_n = \begin{cases} 0, & \text{if } n < \nu, \\ \dfrac{p_\nu}{P_\nu}, & \text{if } n = \nu, \\ -\dfrac{p_\nu p_n}{P_n P_{n-1}}, & \text{if } n > \nu, \end{cases} \qquad Y_n = \begin{cases} 0, & \text{if } n < \nu, \\ \hat{t}_{n\nu}, & \text{if } n = \nu, \\ \Delta_\nu \hat{t}_{n\nu}, & \text{if } n > \nu. \end{cases} \tag{18}$$

By (15) and (16), it follows that

$$\|X\| = \left\{\nu^{k-1}\left(\frac{p_\nu}{P_\nu}\right)^k + \sum_{n=\nu+1}^{\infty} n^{k-1}\left(\frac{p_\nu p_n}{P_n P_{n-1}}\right)^k\right\}^{1/k}, \tag{19}$$

$$\|Y\| = \left\{\nu^{s-1}|t_{\nu\nu}|^s + \sum_{n=\nu+1}^{\infty} n^{s-1}|\Delta_\nu \hat{t}_{n\nu}|^s\right\}^{1/s}, \tag{20}$$

recalling that $\hat{t}_{\nu\nu} = \bar{t}_{\nu\nu} = t_{\nu\nu}$.

Using (19) and (20) in (17), along with (12), it follows that

$$\begin{aligned}
\nu^{s-1}\left|t_{\nu\nu}\right|^{s}+\sum_{n=\nu+1}^{\infty} n^{s-1}\left|\Delta\hat{t}_{n\nu}\right|^{s} &\leq K^{s}\left(\nu^{k-1}\left(\frac{p_\nu}{P_\nu}\right)^{k}+\sum_{n=\nu+1}^{\infty} n^{k-1}\left(\frac{p_\nu p_n}{P_n P_{n-1}}\right)^{k}\right)^{s/k} \\
&\leq K^{s}\left(\nu^{k-1}\left(\frac{p_\nu}{P_\nu}\right)^{k}+O(1)\left(\frac{p_\nu}{P_\nu}\right)^{k}\right)^{s/k} \\
&= O\left(\left(\frac{p_\nu}{P_\nu}\right)^{k}\nu^{k-1}\right)^{s/k}.
\end{aligned} \tag{21}$$

The above inequality will be true if and only if each term on the left-hand side is $O((p_\nu/P_\nu)^k\nu^{k-1})^{s/k}$.

Taking the first term,

$$\begin{aligned}
\nu^{s-1}\left|t_{\nu\nu}\right|^{s} &= O\left(\left(\frac{p_\nu}{P_\nu}\right)^{k}\nu^{k-1}\right)^{s/k}, \\
\left|t_{\nu\nu}\right|^{s} &= O\left(\left(\frac{p_\nu}{P_\nu}\right)^{s}\nu^{1-s/k}\right), \\
\left|t_{\nu\nu}\right| &= O\left(\left(\frac{p_\nu}{P_\nu}\right)^{s}\nu^{1-s/k}\right)^{1/s} \\
&= O\left(\left(\frac{p_\nu}{P_\nu}\right)\nu^{1/s-1/k}\right),
\end{aligned} \tag{22}$$

which verifies that (i) is necessary.

Using the second term, we have

$$\sum_{n=\nu+1}^{\infty} n^{s-1}\left|\Delta\hat{t}_{n\nu}\right|^{s} = O\left(\left(\frac{p_\nu}{P_\nu}\right)^{k}\nu^{k-1}\right)^{s/k} = O\left(\left(\frac{p_\nu}{P_\nu}\right)^{s}\nu^{s-s/k}\right), \tag{23}$$

which is condition (ii).

If we now apply (11) and (8) to $a_\nu = e^{\nu+1}$, we have, respectively,

$$\begin{aligned}
X_n &= \begin{cases} 0, & \text{if } n \leq \nu, \\ \dfrac{P_\nu p_n}{P_n P_{n-1}}, & \text{if } n > \nu, \end{cases} \\
Y_n &= \begin{cases} 0, & \text{if } n \leq \nu, \\ \hat{t}_{n,\nu+1}, & \text{if } n > \nu. \end{cases}
\end{aligned} \tag{24}$$

The corresponding norms are

$$\|X\| = \left\{ \sum_{n=\nu+1}^{\infty} n^{k-1} \left(\frac{P_\nu p_n}{P_n P_{n-1}} \right)^k \right\}^{1/k},$$
$$\|Y\| = \left\{ \sum_{n=\nu+1}^{\infty} n^{s-1} \left| \hat{t}_{n,\nu+1} \right|^s \right\}^{1/s}. \tag{25}$$

Applying (17) and (12),

$$\sum_{n=\nu+1}^{\infty} n^{s-1} \left| \hat{t}_{n,\nu+1} \right|^s \leq K^s \left\{ \sum_{n=\nu+1}^{\infty} n^{k-1} \left(\frac{P_\nu p_n}{P_n P_{n-1}} \right)^k \right\}^{s/k}, \tag{26}$$

which is condition (iii). □

Corollary 2. *Let T be a lower triangular matrix, $\{p_n\}$ satisfying (12). Then the necessary conditions for $\sum a_n$ summable $|\bar{N}, p_n|_k$ to imply $\sum a_n$ summable $|T|_k$ are*

(i) $P_\nu |t_{\nu\nu}| / p_\nu = O(1)$;
(ii) $\sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu \hat{t}_{n\nu}|^k = O(\nu^{k-1} (p_\nu / P_\nu)^k)$;
(iii) $\sum_{n=\nu+1}^{\infty} n^{k-1} |\hat{t}_{n,\nu+1}|^k = O(1)$.

To prove Corollary 2, simply set $s = k$ in Theorem 1.

A lower triangular matrix T is called a triangle if each $t_{nn} \neq 0$.

Theorem 3. *Let $1 < k \leq s < \infty$. Let T be a triangle with bounded entries such that T and $\{p_n\}$ satisfy the following:*

(i) $t_{\nu\nu} = O((p_\nu / P_\nu) \nu^{1/s - 1/k})$;
(ii) $(n |X_n|)^{s-k} = O(1)$;
(iii) $\sum_{\nu=1}^{n-1} |\Delta_\nu \hat{t}_{n\nu}| = O(|t_{nn}|)$;
(iv) $\sum_{n=\nu+1}^{\infty} (n |t_{nn}|)^{s-1} |\Delta_\nu \hat{t}_{n\nu}| = O(\nu^{s-1} |t_{\nu\nu}|^s)$;
(v) $\sum_{\nu=1}^{n-1} |t_{\nu\nu}| |\hat{t}_{n,\nu+1}| = O(|t_{nn}|)$;
(vi) $\sum_{n=\nu+1}^{\infty} (n |t_{nn}|)^{s-1} |\hat{t}_{n,\nu+1}| = O(\nu |t_{\nu\nu}|)^{s-1}$.

Then $\sum a_n$ is $\bar{N}, p_n|_k$.

Proof. Solving (11) for $\{a_n\}$ and substituting into (8) give

$$\begin{aligned} Y_n &= \sum_{\nu=1}^{n} \hat{t}_{n\nu} \left(\frac{X_\nu P_\nu}{p_\nu} - \frac{X_{\nu-1} P_{\nu-2}}{p_{\nu-1}} \right) \\ &= \sum_{\nu=1}^{n} \hat{t}_{n\nu} \frac{X_\nu P_\nu}{p_\nu} - \sum_{\nu=1}^{n} \hat{t}_{n\nu} \frac{X_{\nu-1} P_{\nu-2}}{p_{\nu-1}} \\ &= \sum_{\nu=1}^{n} \hat{t}_{n\nu} \frac{X_\nu P_\nu}{p_\nu} - \sum_{\nu=0}^{n-1} \hat{t}_{n,\nu+1} \frac{X_\nu P_{\nu-1}}{p_\nu} \\ &= \frac{\hat{t}_{nn} X_n P_n}{p_n} + \sum_{\nu=1}^{n-1} \left(\hat{t}_{n\nu} P_\nu - \hat{t}_{n,\nu+1} P_{\nu-1} \right) \frac{X_\nu}{p_\nu} \end{aligned}$$

$$\begin{aligned}
&= \frac{t_{nn}P_nX_n}{p_n} + \sum_{\nu=1}^{n-1}\left[P_\nu\left(\hat{t}_{n\nu} - \hat{t}_{n,\nu+1}\right) + \hat{t}_{n,\nu+1}\left(P_\nu - P_{\nu-1}\right)\right]\frac{X_\nu}{p_\nu} \\
&= \frac{P_nt_{nn}X_n}{p_n} + \sum_{\nu=1}^{n-1}\left(\frac{P_\nu}{p_\nu}\Delta_\nu\hat{t}_{n\nu} + \hat{t}_{n,\nu+1}\right)X_\nu \\
&= T_{n1} + T_{n2} + T_{n3}.
\end{aligned} \tag{27}$$

From Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{s-1}\left|T_{ni}\right|^s < \infty, \quad i = 1,2,3. \tag{28}$$

Using condition (i) of Theorem 3,

$$\begin{aligned}
J_1 := \sum_{n=1}^{\infty} n^{s-1}\left|T_{n1}\right|^s &= \sum_{n=1}^{\infty} n^{s-1}\left|\frac{t_{nn}P_n}{p_n}X_n\right|^s \\
&= O(1)\sum_{n=1}^{\infty} n^{s-1}\left(n^{1/s-1/k}\right)^s\left|X_n\right|^s \\
&= O(1)\sum_{n=1}^{\infty} n^{k-1}\left|X_n\right|^k\left(n^{s-s/k-k+1}\left|X_n\right|^{s-k}\right).
\end{aligned} \tag{29}$$

But

$$n^{s-s/k-k+1}\left|X_n\right|^{s-k} = \left(n^{1-1/k}\left|X_n\right|\right)^{s-k} = O\left(\left(n\left|X_n\right|\right)^{s-k}\right) = O(1), \tag{30}$$

from (ii) of Theorem 3.

Since $\sum a_n$ is summable, $|\bar{N}, p_n|_k$, $J_1 = O(1)$.

Using Hölder's inequality and conditions (i), (ii), (iii), and (iv) of Theorem 3.

$$\begin{aligned}
J_2 := \sum_{n=1}^{\infty} n^{s-1}\left|T_{n2}\right|^s &= \sum_{n=1}^{\infty} n^{s-1}\left|\sum_{\nu=1}^{n-1}\left(\frac{P_\nu}{p_\nu}\right)\left(\Delta_\nu\hat{t}_{n\nu}\right)X_\nu\right|^s \\
&= O(1)\sum_{n=1}^{\infty} n^{s-1}\left(\sum_{\nu=1}^{n-1}\nu^{1/s-1/k}\left|t_{\nu\nu}\right|^{-1}\left|\Delta_\nu\hat{t}_{n\nu}\right|\left|X_\nu\right|\right)^s \\
&= O(1)\sum_{n=1}^{\infty} n^{s-1}\left(\sum_{\nu=1}^{n-1}\nu^{1-s/k}\left|t_{\nu\nu}\right|^{-s}\left|\Delta_\nu\hat{t}_{n\nu}\right|\left|X_\nu\right|^s\right)\times\left(\sum_{\nu=1}^{n-1}\left|\Delta_\nu\hat{t}_{n\nu}\right|\right)^{s-1} \\
&= O(1)\sum_{n=1}^{\infty}\left(n\left|t_{nn}\right|\right)^{s-1}\sum_{\nu=1}^{n-1}\nu^{1-s/k}\left|t_{\nu\nu}\right|^{-s}\left|\Delta_\nu\hat{t}_{n\nu}\right|\left|X_\nu\right|^s \\
&= O(1)\sum_{\nu=1}^{\infty}\nu^{1-s/k}\left|t_{\nu\nu}\right|^{-s}\left|X_\nu\right|^s\sum_{n=\nu+1}^{\infty}\left(n\left|t_{nn}\right|\right)^{s-1}\left|\Delta_\nu\hat{t}_{n\nu}\right|
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{\nu=1}^{\infty}\nu^{1-s/k}\left|t_{\nu\nu}\right|^{-s}\left|X_\nu\right|^{s}\nu^{s-1}\left|t_{\nu\nu}\right|^{s} \\
&= O(1)\sum_{\nu=1}^{\infty}\nu^{s-s/k}\left|X_\nu\right|^{s} \\
&= O(1)\sum_{\nu=1}^{\infty}\nu^{k-1}\left|X_\nu\right|^{k}\left(\nu^{s-s/k-k+1}\left|X_\nu\right|^{s-k}\right) \\
&= O(1)\sum_{\nu=1}^{\infty}\nu^{k-1}\left|X_\nu\right|^{k} = O(1).
\end{aligned}
\tag{31}
$$

By Hölder's inequality and conditions (v), (vi), and (iii) of Theorem 3, we have

$$
\begin{aligned}
J_3 := \sum_{n=1}^{\infty} n^{s-1}\left|T_{n3}\right|^{s} &= \sum_{n=1}^{\infty} n^{s-1}\left|\sum_{\nu=1}^{n-1}\hat{t}_{n,\nu+1}X_\nu\right|^{s} \\
&\le \sum_{n=1}^{\infty} n^{s-1}\left(\sum_{\nu=1}^{n-1}\left|\hat{t}_{n,\nu+1}\right|\left|X_\nu\right|\right)^{s} \\
&\le \sum_{n=1}^{\infty} n^{s-1}\left(\sum_{\nu=1}^{n-1}\left|t_{\nu\nu}\right|^{1-s}\left|\hat{t}_{n,\nu+1}\right|\left|X_\nu\right|^{s}\right) \\
&\qquad\times\left(\sum_{\nu=1}^{n-1}\left|t_{\nu\nu}\right|\left|\hat{t}_{n,\nu+1}\right|\right)^{s-1} \\
&= O(1)\sum_{n=1}^{\infty}\left(n\left|t_{nn}\right|\right)^{s-1}\sum_{\nu=1}^{n-1}\left|t_{\nu\nu}\right|^{1-s}\left|\hat{t}_{n,\nu+1}\right|\left|X_\nu\right|^{s} \\
&= O(1)\sum_{\nu=1}^{\infty}\left|t_{\nu\nu}\right|^{1-s}\left|X_\nu\right|^{s}\sum_{n=\nu+1}^{\infty}\left(n\left|t_{nn}\right|\right)^{s-1}\left|\hat{t}_{n,\nu+1}\right| \\
&= O(1)\sum_{\nu=1}^{\infty}\left|t_{\nu\nu}\right|^{1-s}\left|X_\nu\right|^{s}\left(\nu\left|t_{\nu\nu}\right|\right)^{s-1} \\
&= O(1)\sum_{\nu=1}^{\infty}\nu^{s-1}\left|X_\nu\right|^{s} \\
&= O(1)\sum_{\nu=1}^{\infty}\nu^{k-1}\left|X_\nu\right|^{k}\left(\nu\left|X_\nu\right|\right)^{s-k} \\
&= O(1)\sum_{\nu=1}^{\infty}\nu^{k-1}\left|X_\nu\right|^{k} = O(1).
\end{aligned}
\tag{32}
$$

□

Corollary 4 (see [5]). *Let T be a nonnegative lower triangular matrix, $\{p_n\}$ a positive sequence satisfying*

(i) $t_{ni} \ge t_{n+1,i}$, $n \ge i$, $i = 0,1,2,\ldots$;

(ii) $P_n t_{nn} = O(p_n)$;

(iii) $\bar{t}_{n0} = \bar{t}_{n-1,0}$, $n = 1,2,\ldots$;

(iv) $\sum_{i=1}^{n-1} |t_{ii}||\hat{t}_{n,i+1}| = O(t_{nn})$;
(v) $\sum_{n=i+1}^{\infty} (nt_{nn})^{k-1}|\Delta_i \hat{t}_{ni}| = O(i^{k-1}t_{ii}^k)$;
(vi) $\sum_{n=i+1}^{\infty} (nt_{nn})^{k-1}|\hat{t}_{n,i+1}| = O((it_{ii})^{k-1})$.

Then $\sum a_n$ summable $|\bar{N},p_n|_k$ implies $\sum a_n$ summable $|T|_k$, $k \geq 1$.

Proof. Since $s = k$ and T is nonnegative, condition (ii) of Theorem 3 is automatically satisfied, and conditions (ii), (iv), (v), and (vi) of Corollary 4 are equivalent to conditions (i), (v), (iv), and (vi) of Theorem 3, respectively

$$\Delta_\nu \hat{t}_{n\nu} = \hat{t}_{n\nu} - \hat{t}_{n,\nu+1} = \bar{t}_{n\nu} - \bar{t}_{n-1,\nu} - \bar{t}_{n,\nu+1} + \bar{t}_{n-1,\nu+1} = t_{n\nu} - t_{n-1,\nu}. \tag{33}$$

Therefore, using conditions (i) and (iii) of Corollary 4,

$$\sum_{\nu=1}^{n-1} |\Delta_\nu \hat{t}_{n\nu}| = \sum_{\nu=1}^{n-1} (t_{n-1,\nu} - t_{n\nu}) = 1 - t_{n-1,0} - 1 + t_{nn} + t_{n0} \leq t_{nn}, \tag{34}$$

and condition (iii) of Theorem 3 is satisfied. □

Remark 5. For $1 < k \leq s < \infty$, necessary and sufficient conditions for a triangle $A : \ell^k \to \ell^s$ are known only for factorable matrices (see Bennett [1]), which include weighted mean matrices. Therefore, we should not expect to obtain a set of necessary and sufficient conditions when an arbitrary triangle is involved.

However, necessary and sufficient conditions for a matrix $A : \ell \to \ell^s$, $1 \leq s < \infty$ are known. The following result comes from Theorem 2.1 of Rhoades and Savaş [5] by setting each $\lambda_n = 1$.

Theorem 6. *Let T be a lower triangular matrix. Then $\sum a_n$ summable $|\bar{N},p_n|_k$ implies $\sum a_n$ summable $|T|_s$, $s \geq 1$ if and only if*
(i) $P_\nu |t_{\nu\nu}|/p_\nu = O(\nu^{1/s-1})$,
(ii) $\sum_{n=\nu+1}^{\infty} n^{s-1}|\Delta_\nu \hat{t}_{n\nu}|^s = O((p_\nu/P_\nu)^s)$,
(iii) $\sum_{n=\nu+1}^{\infty} n^{s-1}|\hat{t}_{n,\nu+1}|^s = O(1)$.

Remark 7. In [5], it is assumed that T has nonnegative entries and row sums one, but these restrictions are not used in the proofs.

Finally, we state necessary and sufficient conditions when $k = s = 1$.

Theorem 8. *The series $\sum a_n$ summable $|\bar{N},p_n|$ implies $\sum a_n$ summable T if and only if*
(i) $P_\nu |t_{\nu\nu}|/p_\nu = O(1)$;
(ii) $\sum_{n=\nu+1}^{\infty} |\Delta_\nu \hat{t}_{n\nu}| = O(p_\nu/P_\nu)$;
(iii) $\sum_{n=\nu+1}^{\infty} |\hat{t}_{n,\nu+1}| = O(1)$.

Proof. Note that, with $k = 1$, (12) is automatically satisfied. Therefore, the necessity of the conditions follows from Theorem 1.

To prove the conditions sufficient, use [5, Corollary 4.1] by setting each $\lambda_n = 1$. □

Corollary 9. *$\sum a_n$ summable $|C,1|$ implies $\sum a_n$ $|\bar{N},q_n|$ if and only if*
(i) $nq_n/Q_n = O(1)$.

Proof. With each $p_n = 1$, $T = (\bar{N}, q_n)$, condition (i) of Theorem 8 reduces to condition (i) of Corollary 9.

Using (33),

$$\sum_{n=\nu+1}^{\infty} |\Delta_\nu \hat{t}_{n\nu}| = \sum_{n=\nu+1}^{\infty} |t_{n\nu} - t_{n-1,\nu}| = \sum_{n=\nu+1}^{\infty} \left| \frac{p_\nu}{P_n} - \frac{p_\nu}{P_{n-1}} \right| \\ = p_\nu \sum_{n=\nu+1}^{\infty} \frac{p_n}{P_n P_{n-1}} = \frac{p_\nu}{P_\nu}, \tag{35}$$

and condition (ii) of Theorem 8 is satisfied. Since $(\bar{N}, p_n)$ has row sums one,

$$\begin{aligned} \hat{t}_{n,\nu+1} &= \bar{t}_{n,\nu+1} - \bar{t}_{n-1,\nu+1} = \sum_{i=\nu+1}^{n} t_{ni} - \sum_{i=\nu+1}^{n} t_{n-1,i} \\ &= 1 - \sum_{i=0}^{\nu} t_{ni} - 1 + \sum_{i=0}^{\nu} t_{n-1,i} \\ &= \sum_{i=0}^{\nu} (t_{n-1,i} - t_{ni}) = \sum_{i=0}^{\nu} \left(\frac{p_i}{P_{n-1}} - \frac{p_i}{P_n} \right) \\ &= \frac{p_n}{P_n P_{n-1}} \sum_{i=0}^{\nu} p_i = \frac{p_n P_\nu}{P_n P_{n-1}}. \end{aligned} \tag{36}$$

Therefore

$$\sum_{n=\nu+1}^{\infty} |\hat{t}_{n,\nu+1}| = P_\nu \sum_{n=\nu+1}^{\infty} \frac{p_n}{P_n P_{n-1}} = 1, \tag{37}$$

and condition (iii) of Theorem 8 is satisfied. □

Corollary 10. *The series $\sum a_n$ summable $|\bar{N}, p_n|_k$ implies $\sum a_n$ summable $|C,1|_k$ if and only if*

(i) $P_n/(np_n) = O(1)$.

Proof. Using $T = (C,1)$ in Theorem 8, condition (i) of Theorem 8 reduces to condition (i) of Corollary 10.

From (33) and (i) of Corollary 10,

$$\sum_{n=\nu+1}^{\infty} |\Delta_\nu \hat{t}_{n\nu}| = \sum_{n=\nu+1}^{\infty} |t_{n-1,\nu} - t_{n\nu}| = \sum_{n=\nu+1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ = \frac{1}{\nu+1} = \frac{P_\nu}{\nu p_\nu} \left(\frac{\nu}{\nu+1} \right) \left(\frac{p_\nu}{P_\nu} \right) = O\left(\frac{p_\nu}{P_\nu} \right), \tag{38}$$

and condition (ii) of Theorem 8 is satisfied.

Using (36),

$$\begin{aligned}\sum_{n=\nu+1}^{\infty}\left|\hat{t}_{n,\nu+1}\right| &= \sum_{n=\nu+1}^{\infty}\left|\sum_{i=0}^{\nu}\left(t_{n-1,i}-t_{ni}\right)\right| \\ &= \sum_{n=\nu+1}^{\infty}\left|\sum_{i=0}^{\nu}\left(\frac{1}{n}-\frac{1}{n+1}\right)\right| \\ &= \sum_{n=\nu+1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)(\nu+1) = (\nu+1)\left(\frac{1}{\nu+1}\right) = 1,\end{aligned} \tag{39}$$

and condition (iii) of Theorem 8 is satisfied. □

Combining Corollaries 9 and 10, we have the following corollary.

Corollary 11. *$|\bar{N}, p_n|$ and $|C,1|$ are equivalent if and only if*
(i) $np_n/P_n = O(1)$;
(ii) $P_n/(np_n) = O(1)$.

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