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EXISTENCE AND DECAY OF SOLUTIONS OF SOME NONLINEAR PARABOLIC VARIATIONAL INEQUALITIES

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ABSTRACT. This paper discusses the existence and decay of solutions u(t) of the variational inequality of parabolic type:

$$<u'(t) + Au(t) + Bu(t) - f(t), v(t) - u(t)> \geq 0$$

for $\forall$ $v \in L^p([0,\infty);V$ $(p \geq 2)$ with $v(t) \in K$ a.e. in $[0,\infty)$, where K is a closed convex set of a separable uniformly convex Banach space V, A is a nonlinear monotone operator from V to V* and B is a nonlinear operator from Banach space W to W*. V and W are related as $V \subset W \subset H$ for a Hilbert space H. No monotonicity assumption is made on B.

KEY WORDS AND PHRASES. *Existence, Decay, Nonlinear parabolic variational inequalities.*

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Introduction

Let H be a real Hilbert space with norm $|\ |$, V be a real separable uniformly convex Banach space with norm $\|\ \|_V$ densely imbedded in H and let K be a closed convex subset of V containing 0. Moreover, let W be a Banach space with norm $\|\ \|_W$ such that $V \subset W \subset H$. We suppose that the natural injections from V into W and from W into H are compact and continuous, respectively. We identify H with its dual space H^* (i.e., $V \subset W \subset H \subset W^* \subset V^*$). Pairing between V^* and V will be denoted by $\langle v^*, v\rangle$ for $v^* \in V^*$ and $v \in V$.

Consider the following variational inequality of parabolic type :

$$\text{(1)} \qquad \langle u'(t) + Au(t) + Bu(t) - f(t),\ v(t) - u(t)\rangle \geqq 0$$

for $v(t) \in L^p([0,\infty);V)$ $(p \geqq 2)$ with $v(t) \in K$ a.e. in $(0,\infty)$.

A solution $u(t)$ of (1) should satisfy the conditions :

$$u(t) \in L^p_{loc}([0,\infty);V) \cap C([0,\infty);H),\ u'(t) \in L^2_{loc}([0,\infty);H),$$

$u(t) \in K$ for a.e. $t \in [0,\infty)$ and the initial condition

$$\text{(2)} \qquad u(0) = u_0 \in K.$$

Here A is a monotone operator from V to V^* and B is a bounded operator from W to W^*. More precisely we make the following assumptions on them.

A_1. A is the Fréchet derivative of a convex functional $F_A(u)$ on V, hemicontinuous on V and satisfies the inequalities

$$k_0 \| u \|_V^p \leqq F_A(u) \quad (\leqq \langle Au,u \rangle) \tag{3}$$

with some $k_0>0$ and $p \geqq 2$, and

$$\| Au \|_{V^*} \leqq C_0(\| u \|_V) \tag{4}$$

where $C_0(\cdot)$ is a monotone increasing function on $[0,\infty)$.

A_2. B is the Fréchet derivative of a functional $F_B(u)$ on W, continuous on W and satisfies

$$\| Bu \|_{W^*} \leqq k_1 \| u \|_W^{\alpha+1} \tag{5}$$

with some $k_1, \alpha>0$.

Regarding the forcing term $f(t)$ we assume :

A_3. $f \in L^q_{loc}([0,\infty);V^*) \cap L^2_{loc}([0,\infty);H)$ with $q=p/(p-1)$ and

$$\delta(t) \equiv \max \{ (\int_t^{t+1} \| f(s) \|_{V^*}^q \, ds)^{1/q} , (\int_t^{t+1} |f(s)|^2 \, ds)^{1/2} \}$$

$$\leqq \text{const.} < \infty.$$

Note that no monotonicity condition on B is assumed.

The problem (1) is said 'unperturbed' if $B(t) \equiv 0$, and said 'perturbed' if $B(t) \not\equiv 0$. The unperturbed problem (1) with the initial condition (2) is familiar, and the existence and uniqueness theorems are known in more general situations than ours (see Lions [5], Brezis [2], Biroli [1], Kenmochi [4], Yamada [13], etc.). However the asymptotic behaviors of solutions as $t \to \infty$ seem to be known little. In this note we first prove a decay property of solutions of the unperturbed problem (1)-(2) (with $B(t) \equiv 0$). This result is derived by combining the penalty method with the argument in our previous paper [10], where the nonlinear evolution equations (not inequalities) were treated.

Next we consider the perturbed problem (1)-(2) (i.e., $B(t) \not\equiv 0$). For the equation $u'(t)+Au(t)+Bu(t)=f(t)$ (not inequality), the existence of bounded solutions on $[0,\infty)$ in the norm $\| \ \|_V$ was proved in [8] (see also [7]). We extend this result to the variational inequality (1)-(2). Recently, similar problems were treated by Ôtani [12] and Ishii [3] in the framework of the theory of subdifferential operators. In their works it is assumed that $f(t) \equiv 0$ or $\int_0^\infty |f(s)|_H^2 \, ds$ is small, while here we require only the smallness of $M \equiv \sup_t \delta(t)$. Ishii [3] discussed the decay or blowing up properties of solutions. We also prove a decay property of solutions of the perturbed problem. Our result is much better than the corresponding result of [3].

We employ the so-called penalty method introduced by Lions [5], and the argument is related to the one used in our previous paper [11], where the nonlinear wave equations in noncylindrical domains were considered.

1. Preliminaries

We prepare some lemmas concerning a penalty functioanl $\beta(u)$. Let K be a closed convex set in V and let $J:V \longrightarrow V^*$ be the duality mapping such that

$$(6) \qquad \| J(u) \|_{V^*} = \| u \|_V , \quad \langle J(u), u \rangle = \| u \|_V^2 .$$

Then the penalty functional $\beta(u)$ for K is defined by

$$(7) \qquad \beta(u) = J(u - p_K u)$$

where p_K is the projection of V to K. Recall that $p_K u$ $(\in K)$ is determined by

$$(8) \qquad \| u - p_K u \|_V = \min_{w \in K} \| u - w \|_V .$$

$p_K u$ is also characterized as the unique element of K satisfying

$$(9) \qquad \langle J(u - p_K u), w - p_K u \rangle \leqq 0 \quad \text{for} \quad w \in K.$$

For a proof see Lions [5]. The following two lemmas are well known.

Lemma 1. (Lions [5])

$\beta(u)$ is a monotone hemicontinuous mapping from V to V^*.

Lemma 2. (see, e.g., [6])

The projection p_K is continuous.

The next lemma plays an essential role in our arguments.

Lemma 3.

<u>Let</u> $u(t) \in C^1([0,\infty);V)$. <u>Then</u> $\| u(t)-p_K u(t) \|_V^2$ <u>is differentiable on</u> $[0,\infty)$ <u>and it holds that</u>

$$(10) \qquad \frac{1}{2}\frac{d}{dt} \| u(t) - p_K u(t) \|_V^2 = <\beta(u(t)), u'(t)> .$$

Proof.

The proof can be given by a variant of the way in Biroli [1, lemma 6]. By a standard argument (see Lions [5, Chap II, Prof 8.1]) we know

$$(11) \qquad \frac{1}{2} \| w - p_K w \|_V^2 - \frac{1}{2} \| v - p_K v \|_V^2 \geqq <\beta(v), w - v>$$

for $w,v \in V$. Then, if $t,t+h \geqq 0$ we have

$$\frac{1}{2} \| u(t+h) - p_K u(t+h) \|_V^2 - \frac{1}{2} \| u(t) - p_K u(t) \|_V^2$$

$$(12) \qquad \geqq < \beta(u(t)), u(t+h) - u(t)> .$$

If $h>0$, we have from (12)

$$\frac{1}{2h}\int_{t_2}^{t_2+h} \| u(s)-p_K u(s) \|_V^2 ds - \frac{1}{2h}\int_{t_1}^{t_1+h} \| u(s)-p_K u(s) \|_V^2 ds$$

$$(13) \qquad \geqq \int_{t_1}^{t_2} < \beta(u(s)), \frac{u(s+h)-u(s)}{h} > ds$$

for $t_2>t_1\geqq 0$, and hence, letting $h\downarrow 0$,

$$\frac{1}{2} \| u(t_2) - p_K u(t_2) \|_V^2 - \frac{1}{2} \| u(t_1) - p_K u(t_1) \|_V^2$$

(14) $$\geqq \int_{t_1}^{t_2} < \beta(u(s)) \, , u'(s) > ds \, .$$

Similarly, if $h<0$, we have

$$\frac{1}{2h} \int_{t_2+h}^{t_2} \| u(s)-p_K u(s) \|_V^2 \, ds - \frac{1}{2h} \int_{t_1+h}^{t_1} \| u(s)-p_K u(s) \|_V^2 \, ds$$

$$\leqq \int_{t_1}^{t_2} < \beta(u(s)), \frac{u(s+h)-u(s)}{h} > ds$$

for $t_2>t_1$ with $t_1+h\geqq 0$, and

(15) $$\frac{1}{2} \| u(t_2)-p_K u(t_2) \|_V^2 - \frac{1}{2} \| u(t_1)-p_K u(t_1) \|_V^2$$

$$\leqq \int_{t_1}^{t_2} < \beta(u(s)), u'(s) > ds$$

for $t_2>t_1\geqq 0$, where we have used the continuity of $p_K u(t)$ at $t\geqq 0$. The inequalities (14) and (15) are equiavlent to (10).

We conclude this section by stating a lemma concerning a difference inequality, which will be used for the proof of decay of solutions.

Lemma 4. ([9])

Let $\phi(t)$ *be a nonnegative function on* $[0,\infty)$ *such that*

$$\sup_{t\leqq s\leqq t+1} \phi(t)^{1+r} \leqq C_0(\phi(t) - \phi(t+1)) + g(t)$$

with some $C_0>0$ *and* $r\geqq 0$. *Then*

(i) *if* $r=0$ *and* $g(t)\leqq C_1 \exp(-\lambda t)$ *with some* $\lambda>0$, $C_1>0$, *then* $\phi(t)\leqq C_1' \exp(-\lambda' t)$ *for some* $C_1', \lambda'>0$,

and

(ii) *if* $r>0$ *and* $\lim_{t\to\infty} g(t)t^{1+1/r}=0$, *then*

$$\phi(t) \leqq C_1'(1+t)^{-1/r} \quad \textit{for some} \quad C_1' > 0.$$

2. Unperturbed problem

As is mentioned in the introduction we prove here a decay property of solutions of the unperturbed problem (1)-(2).

Theorem 1.

Let $u_0 \in K$ *and let* $\lim_{t\to\infty} \delta(t)t^{(p-1)/(p-2)}=0$ *if* $p>2$ *and* $\delta(t)\leqq C \exp(-\lambda t)$ $(\lambda>0)$ *if* $p=2$. *Then the problem* (1)-(2) *with* $B(t)\equiv 0$ *admits a unique solution* $u(t)$, *satisfying*

$$\| u(t) \|_V \leqq C(\| u_0 \|_V)(1+t)^{-1/(p-2)} \quad \textit{if} \quad p>2 \tag{16}$$

and

$$\| u(t) \|_V \leqq C(\| u_0 \|_V) \exp(-\lambda' t) \quad \textit{if} \quad p=2 \tag{16'}$$

with some $\lambda'>0$.

Proof.

Recall that the solution u is given by a limit function of $\{u_\varepsilon(t)\}$ as $\varepsilon \longrightarrow 0$, where $u_\varepsilon(t)$ is the solution of the modified equation

$$(17)\qquad \begin{cases} u'(t) + Au(t) + \frac{1}{\varepsilon}\beta(u) = f(t) \quad (\varepsilon>0) \\ \\ u(0) = u_0 . \end{cases}$$

Since A and β are monotone hemicontinuous operators from V to V^*, the problem (16) has a unique solution $u_\varepsilon(t)$ such that

$$u_\varepsilon(t) \in L^p_{loc}([0,\infty);V) \quad \text{and} \quad u'_\varepsilon(t) \in L^2_{loc}([0,\infty);H).$$

(Cf. Lions [5, Chap. 2, Th. 1.2., see also Biroli [1], where more general result is given.)

Let $\{w_j\}_{j=1}^{\infty}$ be a basis of V. Then, it is known that $u_\varepsilon(t)$ is given by the limit function of $\{u_{m,\varepsilon}(t)\}$ as $m \longrightarrow \infty$, where $u_{m,\varepsilon}(t) = \sum_{j=1}^{m} \alpha_{j,m}(t) w_j$ is the solution of

$$(18)\qquad \langle u'_{m,\varepsilon}(t), w_j\rangle + \langle Au_{m,\varepsilon}(t), w_j\rangle = \langle f(t), w_j\rangle$$

$$(j=1,2,\ldots,m)$$

with the initial condition

$$(19)\qquad u_{m,\varepsilon}(0) = u^0_{m,\varepsilon} \longrightarrow u_0 \text{ in } V.$$

The problem (17)-(18) is a system of ordinary differential equations with respect to $\alpha_{j,m}(t)$, $j=1,2,\dots,m$, and by the monotonicity and hemicontinuity of A and β it is easy to see that this problem admits unique solution such that

$$u_{m,\varepsilon}(t) \in C^1([0,\infty);V_m) \subset C^1([0,\infty);V)$$

where V_m is the m-dimensional subspace of V spanned by $\{w_1, \dots, w_m\}$. For the proof of Theorem 1, it suffices to show that the estimate (16) or (16)' with $u=u_{m,\varepsilon}$ holds with the constants independent of m and ε.

By Lemma 3 we have

$$E_\varepsilon(u_{m,\varepsilon}(t_2)) - E_\varepsilon(u_{m,\varepsilon}(t_1)) + \int_{t_1}^{t_2} |u'_{m,\varepsilon}(s)|^2 \, ds = \int_{t_1}^{t_2} \langle f(s), u'_{m,\varepsilon}(s) \rangle ds \tag{20}$$

for $t_2 > t_1 \geq 0$, where

$$E_\varepsilon(u(t)) \equiv F_A(u(t)) + \frac{1}{2\varepsilon} \| u(t) - p_K u(t) \|_V^2 .$$

Also we have easily by (18)

$$\int_{t_1}^{t_2} \{\langle Au_{m,\varepsilon}(s), u_{m,\varepsilon}(s) \rangle + \frac{1}{\varepsilon} \langle \beta(u_{m,\varepsilon}(s), u_{m,\varepsilon}(s) \rangle\} ds$$

$$= \int_{t_1}^{t_2} \{\langle f(s), u_{m,\varepsilon}(s) \rangle - \langle u'_{m,\varepsilon}(s), u_{m,\varepsilon}(s) \rangle\} ds. \tag{21}$$

Using the similar argument as in [10], the equalities (20)-(21)

imply the estimate (16) or (16)' with $u=u_{m,\varepsilon}$. For completeness, however, we sketch the proof briefly.

By (20) we have

$$\int_t^{t+1} |u'_{m,\varepsilon}(s)|^2\, ds \leqq 2\{E_\varepsilon(u_{m,\varepsilon}(t) - E_\varepsilon(u_{m,\varepsilon}(t+1))\} + C\delta(t)$$

(22)

$$\equiv D_\varepsilon(t)^2 . \quad (C>0 \text{ ; constant})$$

On the other hand, using the ineqaulity

$$\langle Au_{m,\varepsilon}(t), u_{m,\varepsilon}(t)\rangle + \frac{1}{\varepsilon}\langle\beta(u_{m,\varepsilon}(t)),\ u_{m,\varepsilon}(t)\rangle \geqq E_\varepsilon(u_{m,\varepsilon}(t))$$

(see (3) and (9)),

we have from (21)

$$\int_t^{t+1} E_\varepsilon(u_{m,\varepsilon}(s))ds \leqq \left(\int_t^{t+1} \|f(s)\|_{V^*}^2\, ds\right)^{1/2} \sup_{s\in[t,t+1]} \|u_{m,\varepsilon}(s)\|_V$$

$$+ \left(\int_t^{t+1} |u'_{m,\varepsilon}(s)|^2\, ds\right)^{1/2} \sup_{t\leqq s\leqq t+1} |u_{m,\varepsilon}(s)|$$

(23)

$$\leqq C(D_\varepsilon(t) + \delta(t) \sup_{t\leqq s\leqq t+1} E_\varepsilon(u_{m,\varepsilon}(s))^{1/p}$$

where hearafter C denotes various constants independent of m and ε. From (23) there exists $t^* \in [t,t+1]$ such that

$$E_\varepsilon(u_{m,\varepsilon}(t^*)) \leqq C\{D_\varepsilon(t) + \delta(t))\} \sup_{t\leqq s\leqq t+1} E_\varepsilon(u_{m,\varepsilon}(s))^{1/p}$$

and hence by (20)

$$\sup_{t \leqq s \leqq t+1} E_\varepsilon(u_{m,\varepsilon}(s)) \leqq C\{(D_\varepsilon(t) + \delta(t)) \sup_{t \leqq s \leqq t+1} E_\varepsilon(u_{m,\varepsilon}(s))^{1/p}$$
$$+ D_\varepsilon(t)^2 + D_\varepsilon(t)\delta(t)\}$$

and by Young's inequality,

$$(24) \qquad \sup_{t \leqq s \leqq t+1} E_\varepsilon(u_{m,\varepsilon}(s)) \leqq C\{(D_\varepsilon(t) + \delta(t))^{p/(p-1)} + D_\varepsilon(t)^2 + \delta(t)^2\} .$$

From (24) we can easily see that $E_\varepsilon(u_{m,\varepsilon}(t))$ is bounded on $[0,\infty)$ by a constant depending on $E_\varepsilon(u_{m,\varepsilon}(0))$. Since we may assume, without loss of generality, that $u_{m,\varepsilon}(0) \in K$ and

$$(25) \qquad E_\varepsilon(u_{m,\varepsilon}(t)) \leqq C(E_\varepsilon(u_{m,\varepsilon}(0))) \leqq C(\| u_0 \|_V)$$

where $C(\cdot)$ denotes various constants depending on the indicated quantity. By (20) and (25) we have

$$(26) \qquad \sup_{t \leqq s \leqq t+1} E_\varepsilon(u_{m,\varepsilon}(s))^{2(p-1)/p}$$
$$\leqq C(\| u_0 \|_V, M)\{E_\varepsilon(u_{m,\varepsilon}(t+1)) - E_\varepsilon(u_{m,\varepsilon}(t)) + \delta(t)^2\}$$

where we set $M \equiv \sup_t \delta(t)^2$. Applying Lemma 4 we obtain the desired result.

3. Perturbed problem

In this section we investigate the existence and decay of solutions of the problem (1)-(2) with B satisfying the assumption A_2. For this consider the approximate equations

$$(27)\quad \langle u'_{m,\varepsilon}(t) + Au_{m,\varepsilon}(t) + Bu_{m,\varepsilon}(t) + \frac{1}{\varepsilon}\beta(u_{m,\varepsilon}(t)) - f(t),\, w_j\rangle = 0,$$

$j=1,2,\ldots,m$, where we set again

$$u_{m,\varepsilon}(t) = \sum_{j=1}^{m} \alpha_{m,j}(t) w_j .$$

and we impose $u_{m,\varepsilon}(0) \in K$ and $u_{m,\varepsilon}(0) \longrightarrow u_0$ $(\in K)$ in V. Using a similar argument as in [7] we derive a priori estimates for $u_{m,\varepsilon}(t)$. We also give a rather brief discussion. First we assume $p>\alpha+2$. By (27) we have

$$(28)\quad G_{\varepsilon,0}(u_{m,\varepsilon}(t_2)) - G_{\varepsilon,0}(u'_{m,\varepsilon}(t_1)) + \int_{t_1}^{t_2} |u'_{m,\varepsilon}(s)|^2\, ds$$

$$= \int_{t_1}^{t_2} \langle f(s), u'_{m,\varepsilon}(s)\rangle ds$$

where

$$G_{\varepsilon,0}(u(t)) = F_A(u(t)) + F_B(u(t)) + \frac{1}{2\varepsilon}\|u(t) - p_K u(t)\|_V^2 ,$$

and hence, in particular,

$$(29)\quad G_{\varepsilon,0}(u_{m,\varepsilon}(t)) \leqq G_{\varepsilon,0}(u_{m,\varepsilon}(0)) + \frac{1}{4}\delta(0) \quad \text{if} \quad 0\leqq t<1$$

which together with the assumption $p>\alpha+2$ implies

$$\| u_{m,\varepsilon}(t) \|_V \leqq C(\| u_0 \|_V, \delta(0)) < \infty \tag{30}$$

if $0\leqq t<1$. Thus $u_{m,\varepsilon}(t)$ exists on an interval, say $[0,t_m]$, with $t_m>1$. If we assume $G_{\varepsilon,0}(u_{m,\varepsilon}(t))\leqq G_{\varepsilon,0}(u_{m,\varepsilon}(t+1))$ for some $t>0$, we have from (28)

$$\int_t^{t+1} |u'_{m,\varepsilon}(s)|^2 \, ds \leqq \delta(t)^2 \leqq M^2 . \tag{31}$$

Using (27) and (31) we have

$$\int_t^{t+1} G_{\varepsilon,1}(u_{m,\varepsilon}(t))ds \leqq M^2 + C \int_t^{t+1} \| u_{m,\varepsilon}(s) \|_V^2 \, ds$$

where we set

$$G_{\varepsilon,1}(u) = \langle Au + Bu + \frac{1}{\varepsilon} \beta(u), u\rangle . \tag{32}$$

Since

$$G_{\varepsilon,1}(u) \geqq k_0\| u \|_V^p - k_1\| u \|_W^{\alpha+2} + \frac{1}{\varepsilon}\| u - p_K u \|_V^2$$

and since $p>\alpha+2$, there exists a point $t^* \in [t,t+1]$ such that

$$\| u_{m,\varepsilon}(t^*) \|_V + \frac{1}{\varepsilon} \| u_{m,\varepsilon}(t^*) - p_K u_{m,\varepsilon}(t^*) \|_V^2 \leqq C(M).$$

From this and (28)

$$G_{\varepsilon,0}(u_{m,\varepsilon}(t+1)) \leqq G_{\varepsilon,0}(u_{m,\varepsilon}(t^*)) + C\delta(t)^2 \leqq C(M) .$$

Thus we conclude that

$$G_{\varepsilon,0}(u_{m,\varepsilon}(t)) \leqq \max(C(M), \max_{0\leqq s\leqq 1} G_{\varepsilon,0}(u_{m,\varepsilon}(s)))$$

$$\leqq C(M, \| u_0 \|_V) \quad \text{(by (29))}$$

and therefore $u_{m,\varepsilon}(t)$ exists on $[0,\infty)$, satisfying

$$(32)' \quad \| u_{m,\varepsilon}(t) \|_V + \frac{1}{\varepsilon} \| u_{m,\varepsilon}(t) - p_K u_{m,\varepsilon}(t) \|_V^2 \leqq C(M, \| u_0 \|_V).$$

Of course we know

$$(33) \qquad \int_t^{t+1} |u'_{m,\varepsilon}(s)|^2 \, ds \leqq C(M, \| u_0 \|_V) \quad \text{for} \quad t \geqq 0.$$

We have now derived a priori estimate for $u_{m,\varepsilon}(t)$. Using standard compactness and monotonicity arguments (see Lions [5], Biroli [1] etc.) we can suppose without loss of generality that as $m \longrightarrow \infty$,

$$u_{m,\varepsilon}(t) \longrightarrow u_\varepsilon(t) \quad \text{weakly* in} \quad L^\infty([0,\infty);V) ,$$

$$u'_{m,\varepsilon}(t) \longrightarrow u'_\varepsilon(t) \quad \text{weakly in} \quad L^2_{loc}([0,\infty);V).$$

$$(34) \quad Au_{m,\varepsilon}(t) + \frac{1}{\varepsilon}\beta(u_{m,\varepsilon}(t)) \longrightarrow \chi_\varepsilon(t) \quad \text{weakly** in} \quad L^\infty([0,\infty);V^*),$$

$$Bu_{m,\varepsilon}(t) \longrightarrow Bu_\varepsilon(t) \quad \text{strongly in} \quad L^r([0,\infty);W^*) \ (\forall r>1)$$

and

$$\chi_\varepsilon(t) = Au_\varepsilon(t) + \frac{1}{\varepsilon}\beta(u_\varepsilon(t)). \tag{35}$$

Moreover, with the aid of the inequality

$$< \beta(u) - \beta(v) , u - v > \geqq (\| u - p_K u \|_V - \| v - p_K v \|_V)^2$$

for u,v V, we know

$$\lim_{m\to\infty} \| u_{m,\varepsilon}(t) - p_K u_m(t) \|_V = \| u_\varepsilon(t) - p_K u_\varepsilon(t) \|_V \quad \text{in } L^2_{loc}([0,\infty)). \tag{36}$$

The limit function $u_\varepsilon(t)$ satisfies

$$\begin{cases} u'_\varepsilon(t) + Au_\varepsilon(t) + Bu_\varepsilon(t) + \frac{1}{\varepsilon}\beta(u(t)) = f(t) \quad \text{a.e. on } [0,\infty) \\ u_\varepsilon(0) = u_0 . \end{cases} \tag{37}$$

Furthermore, it holds from (32) and (33) that

$$\| u_\varepsilon(t) \|_V + \frac{1}{\varepsilon} \| u_\varepsilon(t) - p_K u_\varepsilon(t) \|_V^2 \leqq C(M, \| u_0 \|_V) \tag{32'}$$

and

$$\int_t^{t+1} |u'_\varepsilon(s)|^2 \, ds \leqq C(M, \| u_0 \|_V) \tag{33'}$$

for $t \geqq 0$. Then we may suppose, as $\varepsilon \longrightarrow 0$,

$$u'_\varepsilon(t) \longrightarrow u'_\varepsilon(t) \quad \text{weakly in} \quad L^2_{loc}([0,\infty);V),$$

(38) $$u_\varepsilon(t) \longrightarrow u(t) \quad \text{weakly* in} \quad L^\infty([0,\infty);V), \quad \text{and in} \quad C_{loc}([0,\infty);H),$$

$$Au_\varepsilon(t) \longrightarrow \chi(t) \quad \text{weakly** in} \quad L^\infty([0,\infty);V^*)$$

and $$Bu_\varepsilon(t) \longrightarrow Bu(t) \quad \text{strongly in} \quad L^r([0,\infty);W^*) \ (\forall r>1)$$

Moreover from (32)'

$$u_\varepsilon(t) - p_K u_\varepsilon(t) \longrightarrow 0 \quad \text{in} \quad L^\infty([0,\infty);V),$$

which implies easily

(39) $$u(t) \in K \quad \text{a.e.} \quad \text{on} \quad [0,\infty).$$

By a standard monotonicity argument (see Biroli [1]) we see $\chi(t)=Au(t)$ a.e. on $[0,\infty)$, and by (37) we have

$$< u'(t) + Au(t) + Bu(t) - f(t) \ , \ v(t) - u(t) > \geqq 0$$

for $\forall v(t) \in L^p([0,\infty);V)$ with $v(t) \in K$ a.e. on $[0,\infty)$.

We summarize above result in the following

Theorem 2.

<u>Let</u> $p>\alpha+2$. <u>Then under the assumptions</u> A_1, A_2 <u>and</u> A_3, <u>the</u>

problem (1)-(2) *admits a solution* $u(t)$ *such that*

$$\| u(t) \|_V + \int_t^{t+1} |u'(s)|^2 \, ds \leq C(M, \| u_0 \|_V) < \infty$$

for $t \geq 0$, *where we set* $M \equiv \sup_t \delta(t)$.

Next, we assume $2 \leq p < \alpha+2$. As is already seen, for the existence of solution it suffices to show the boundedness of $u_{m,\varepsilon}(t)$ by a constant independent of m and ε. For this we set further

$$\tilde{G}_{\varepsilon,0}(u) = k_0 \| u \|_V^p - k_1 S^{\alpha+2} \| u \|_V^{\alpha+2} + \frac{1}{2\varepsilon} \| u - p_K u \|_V^2$$

and

$$\tilde{G}_{\varepsilon,1}(u) = \tilde{G}_{\varepsilon,0}(u) + \frac{1}{2\varepsilon} \| u - p_K u \|_V^2 ,$$

where S is a constant such tat $\| u \|_W \leq S \| u \|_V$ for $u \in V$. Note that

$$(40) \qquad G_{\varepsilon,0}(u) \geq \tilde{G}_{\varepsilon,0}(u), \ G_{\varepsilon,1}(u) \geq \tilde{G}_{\varepsilon,1}(u) \geq \tilde{G}_{\varepsilon,0}(u),$$

and $G_{\varepsilon,1}(u) \geq G_{\varepsilon,0}(u) - 2k_1 \| u \|_W^{\alpha+2}$ for $u \in V$. Let us determine $x_0 > 0$ and $D_0 > 0$ as follows.

$$(41) \qquad \max_{x \geq 0} (k_0 x^p - k_3 S^{\alpha+2} x^{\alpha+2}) = k_0 x_0^p - k_3 S^{\alpha+2} x^{\alpha+2} \equiv D_0 .$$

Then 'the stable set' $\mathcal{W}$ is defined by

(42) $$\mathcal{W} = \{u \in V \mid G_{\varepsilon,1}(u) < D_0 \quad \text{and} \quad \|u\|_V < x_0\}.$$

Let us assume the initial value $u_0 \in \mathcal{W} \cap K$, and let $M < M_0' \equiv 2\sqrt{D_0 - G_{\varepsilon,0}(u_0)}$ (>0). We shall show that there exists a constant $M_0 > 0$ such tht if $M < M_0$, $u_{m,\varepsilon}(t) \in \mathcal{W}$ for $t \leqq t_m$ provided that m is sufficiently large. First, by (29),

(43) $$G_{\varepsilon,0}(u_{m,\varepsilon}(t)) \leqq G_{\varepsilon,0}(u_0) + \frac{1}{4}M + \eta < D_0$$

if $0 \leqq t \leqq \min(1, t_m)$, for sufficiently small $\eta > 0$ and large m. The inequality (43) implies $t_m > 1$. Thus, if our assertion were false, there would exist a time $\bar{t} > 1$ such that

(44) $$G_{\varepsilon,0}(u_{m,\varepsilon}(t)) < D_0 \quad \text{if} \quad 0 \leqq t < \bar{t}$$

and

(45) $$G_{\varepsilon,0}(u_{m,\varepsilon}(\bar{t})) = D_0 .$$

By (28) with $t_2 = \bar{t}$, $t_1 = \bar{t} - 1$ we have easily

(46) $$\int_{\bar{t}-1}^{\bar{t}} |u'_{m,\varepsilon}(s)|^2 \, ds \leqq M^2$$

and hence

(47) $$\int_{\bar{t}-1}^{\bar{t}} G_{\varepsilon,1}(u_{m,\varepsilon}(s))ds \leqq \int_{\bar{t}-1}^{\bar{t}} |-u'_{m,\varepsilon}(s) + f(s)| \, |u_{m,\varepsilon}(s)| ds$$
$$\leqq 2MS_1 x_0$$

where S_1 is a constant such that

$$\| u \|_H \leqq S_1 \| u \|_V \quad \text{for} \quad u \in V.$$

Therefore, if we assume $M < M_0'' \equiv D_0/2S_1x_0$, there exists a time $t^* \in [\bar{t}-1, \bar{t}]$ such that

$$G_{\varepsilon,1}(u_{m,\varepsilon}(t^*)) \leqq 2MS_1x_0 \quad \text{and} \quad \| u_{m,\varepsilon}(t^*) \|_V \leqq x(M) \tag{48}$$

where $x(M)$ $(< x_0)$ is the smaller root of the numerical equation

$$k_0 x^p - k_1 S^{\alpha+2} x^{\alpha+2} = 2MS_1 x_0 \quad (< D_0) \ . \tag{49}$$

We use again (28) to obtain

$$\begin{aligned} G_{\varepsilon,0}(u_{m,\varepsilon}(\bar{t})) &\leqq G_{\varepsilon,0}(u_{m,\varepsilon}(t^*)) + \frac{1}{4} M^2 \\ &\leqq G_{\varepsilon,1}(u_{m,\varepsilon}(t^*)) + \frac{1}{4} M^2 + 2k_1 S^{\alpha+2} \| u_{m,}\ (t^*) \|_V^{\alpha+2} \\ &\leqq 2MS_1x_0 + \frac{1}{4} M^2 + 2k_1 S^{\alpha+2} x(M)^{\alpha+2} . \end{aligned} \tag{50}$$

Now we determine $M_0''' > 0$ as the largest number such that

$$2k_1 S^{\alpha+2} x(M''') + 2M''' S_1 x_0 + \frac{1}{4} M'''^{\,2} = D_0 \ (M_0''' \leqq M_0'') \tag{51}$$

and set $M_0 \equiv \min(M_0', M_0''')$. Then, assuming $M < M_0$, we have by (51)

$$G_{\varepsilon,0}(u_{m,\varepsilon}(\bar{t})) < D_0 \tag{52}$$

which contradicts to (45). Consequently, if $M < M_0$, $u_{m,\varepsilon}(t)$ exists on $[0,\infty)$ for large m and it holds that

$$\|u_{m,\varepsilon}(t)\|_V < x_0 \ , \quad \int_t^{t+1} |u'_{m,\varepsilon}(s)|^2 \, ds \leqq \text{const.} < \infty$$

(53) and

$$G_{\varepsilon,0}(u_{m,\varepsilon}(t)) < D_0 \quad \text{for} \quad t \in [0,\infty) \ .$$

Thus, applying the monotonicity and compactness arguments, we obtain the following

Theorem 3.

Let $2 \leqq p < \alpha+2$ *and* $M < M_0$. *Then the problem* (1)-(2) *admits a solution* u *satisfying*

$$\| u(t) \|_V \leqq x_0 \quad \underline{\text{and}} \quad \int_t^{t+1} |u'_{m,\varepsilon}(s)|^2 \, ds \leqq \text{const.} < \infty \ .$$

Moreover, we note that the approximate solutions $u_{m,\varepsilon}(t)$ (m: large) satisfy

$$(54) \qquad G_{\varepsilon,0}(u_{m,\varepsilon}(t)) \geqq \tilde{G}_{\varepsilon,0}(u_{m,\varepsilon}(t))$$

$$\geqq (k_0 - k_1 S^{\alpha+2} x_0^{(\alpha+2)-p}) \| u_{m,\varepsilon}(t) \|_V^p$$

$$+ \frac{1}{2\varepsilon} \| u - p_K u \|_V^2$$

with $(k_0 - k_1 S^{\alpha+2} x_0^{(\alpha+2)-p}) > 0$. Therefore the same argument as in the section 2 yields the following

Theorem 4.

Let $2 \leqq p < \alpha+2$ *and* $M < M_0$. *Then the solution in* Theorem 3 *satisfies the decay property* :

(i) *If* $p>2$ *and* $\lim_{t\to\infty} \delta(t)t^{(p-1)/(p-2)}=0$, *then*

$$\| u(t) \|_V \leqq C(\| u_0 \|_V)(1+t)^{-1/(p-2)}$$

or

(ii) *If* $p=2$ *and* $\delta(t) \leqq C \exp\{-\lambda t\}$ $(C,\lambda>0)$, *then*

$$\| u(t) \|_V \leqq C' \exp\{-\lambda' t\}$$

for some $C',\lambda'>0$.

Remark. In [3], Ishii proved that $|u(t)| \leqq C(1+t)^{-1/(p-2)}$ if $p>2$ and $|u(t)| \leqq C \exp\{-\lambda t\}$ $(C,\lambda>0)$ if $p=2$ for the case $f\equiv 0$. It is clear that our result is much better, because the norm $\|\cdot\|_V$ is essentially stronger than the norm $|\cdot|$.

4. An example

Here we give an typical example. Let Ω be a bounded domain in R^n and set

$$V \equiv W_0^{1,p}(\Omega),\ H = L^2(\Omega) \quad \text{and} \quad W = L^{\alpha+2}(\Omega)$$

with $0<\alpha<pn/(n-1)+2$ if $n \geqq p+1$ and $0<\alpha<\infty$ if $n \leqq p$. We define $A;V \longrightarrow V^*$ by

$$<Au,v> = \int_\Omega \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i}\, dx \qquad (p \geqq 2)$$

for $u, v \in W_0^{1,p}(\Omega)$, and $B: W \longrightarrow W^*$ by

$$Bu = d(x)|u|^\alpha u \quad \text{for} \quad u \in L^{\alpha+2}(\Omega)$$

where $d(x)$ is a bounded measurable function on Ω. Moreover we set

$$K = \{u \in W_0^{1,p}(\Omega) \mid b(x) \leqq u(x) \leqq a(x) \quad \text{a.e. on} \quad \Omega\}$$

where a, b are measurable function on Ω with $a(x) \geqq 0 \geqq b(x)$. Then all the assumptions A_1-A_2 are satisfied. The problem (1) -(2) is equivalent in this case to the problem

$$\begin{cases} Lu(x,t) = f(x,t) & \text{a.e. on } \Omega \times [0,\infty) \text{ where } b(x)<u(x,t)<a(x), \\ Lu(x,t) \leqq f(x,t) & \text{a.e. on } \Omega \times [0,\infty) \text{ where } u(x,t)=a(x) \\ Lu(x,t) \geqq f(x,t) & \text{a.e. on } \Omega \times [0,\infty) \text{ where } u(x,t)=b(x) \\ \text{with the conditions} & \\ u|_{\partial\Omega}=0 \text{ a.e. on } \partial\Omega\ [0,\infty) & \text{and } u(x,0)=u_0(x)\ (\in K) \text{ a.e. on } \Omega, \end{cases}$$

where

$$Lu = \frac{\partial u}{\partial t} - \sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial}{\partial x_i} u\right) + d(x)|u|^\alpha u .$$

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