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SHEAR FLOW PAST A FLAT PLATE IN HYDROMAGNETICS

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ABSTRACT. The problem of simple shear flow past a flat plate has been extended to the hydromagnetic case in which a viscous, electrically conducting, incompressible fluid flows past an electrically insulated flat plate with a magnetic field parallel to the plate. For simplicity all physical parameters are assumed constant. A series solution for the velocity field has been developed for small values of a magnetic parameter. The equations governing this flow field were integrated numerically . It is found that the effect of the magnetic field is to diminish and increase respectively, the first and second order contributions for the skin friction.

KEY WORDS AND PHARASES: *Electrically conducting fluid, magnetic viscosity, external vorticity, kinematic effect, skin friction.*

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1. INTRODUCTION.

It has been recognised in recent times that an appropriate inviscid flow together with boundary layers, which adjust the slip velocity predicted by inviscid theory at solid boundaries, forms a uniformly valid leading term in the asymptotic expansion of the solution of the Navier-Stokes equations in ascending powers of viscosity for a wide class of problems, provided the solutions obtained do not break down. Attempts have been made to study higher order terms in the asymptotic expansion which take into account the second order effects related to vorticity of the main flow, the longitudinal and transverse curvature, and others. The impact of the second order effect is increasingly felt due to its considerable influence on such important fluid phenomena as skin friction and heat transfer.

The effect of external vorticity was first pointed out by Ferri and Libby [1] . Solutions of the two dimensional boundary layer equations for flow over a flat plate, including the boundary condition of finite vorticity at the edge of the boundary layer, have been presented by Li [2,3] and Ting [4] and others. These authors show that positive vortictiy increases skin friction.

The extension of this classical theory of simple shear flow past a semi-infinite flat plate to the hydromagnetic case has been attempted here. The physical phenomena investigated is the shear flow of a viscous, electrically conducting, incompressible fluid past an electrically insulated flat plate in the presenceof a uniform field parallel to the plate. All physical parameters are assumed to be constants. We wish to study, specifically, the effect of the uniform magnetic field on the second order contribution to skin friction.

2. THE EQUATIONS OF THE PROBLEM:

The classical boundary layer equations are modified so as to apply to an electrically conducting fluid in the presence of a magnetic field. The simplification of the hydromagnetic equations appropriate to the present problem, given by Greenspan and Carrier [4] yield.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} \qquad (2.1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + \frac{\bar{\mu}}{\rho}\left(H_x \frac{\partial H_x}{\partial x} + H_y \frac{\partial H_x}{\partial y}\right) \quad (2.2)$$

$$(uH_y - vH_x) + \eta' \frac{\partial H_x}{\partial y} = 0 \quad (2.3)$$

The novel feature in our problem is that the free stream has a constant vorticity. We are particularly interested in the effect of the magnetic field on this shear as the field penetrates the boundary layer and the concommitant effect produced on the skin friction.

Thus, the boundary conditions are

$$u = 0,\ v = 0,\ H_y = 0 \quad \text{at} \quad y = 0 \quad (2.4)$$

$$u = u_o + \omega y,\ H_x = H_o \quad \text{as} \quad y \to \infty \quad (2.5)$$

Here $\bar{\mu}$ is the permeability of the medium, μ is the magnetic viscosity of the liquid, η' is the magnetic viscosity and ω the constant external vorticity.

Now we rewrite the free stream velocity distribution as follows:

$$u_\infty = u_o\left[1-\left(\frac{\omega}{u_o}\sqrt{2\nu x/u_o}\right)\left(\sqrt{u_o/2\nu x}\ y\right)\right] \quad (2.6)$$

Two nondimensional parameters appear in equation (2.6):

$$\eta = \sqrt{u_o/2\nu x}\ y \quad \text{(Blasius variable)} \quad (2.7)$$

$$\xi = \frac{\omega}{u_o}\sqrt{2\nu x/u_o} \quad \text{(Vorticity number)} \quad (2.8)$$

where η is the familiar variable in the boundary layer studies; ξ is the parameter of external vorticity interpreted as the ratio of free stream vorticity and the average vortictiy in the boundary layer.

Introducing the stream function $\psi(x,y)$ and the magnetic stream function $A(x,y)$ such that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad (2.9)$$

$$H_x = \frac{\partial A}{\partial y} \quad , \quad H_y = - \frac{\partial A}{\partial x} \tag{2.10}$$

We then seek solutions of equations (2.1),(2.2), and (2,3) of the form

$$\psi\ (x,y) = \sqrt{2x/R}\left[f(\eta) + \xi\ f_1(\eta)\right] + 0(\xi^2) \tag{2.11}$$

$$A\ (x,y) = \sqrt{2x/R}\left[h(\eta) + \xi\ h_1(\eta)\right] + 0(\xi^2) \tag{2.12}$$

where R is the Reynolds number referred to a characteristic length ℓ .

Substituting equations (2.7) to (2.12) in (2.1) to (2.3), we get the equations;

$$f''' + ff'' - s\ h\ h'' = 0 \tag{2.13}$$

$$h'' - \varepsilon\ (hf'-fh') = 0 \tag{2.14}$$

$$f_1''' + ff_1'' - f'f_1' + 2f''f_1 + s(h'h_1' - 2h''h_1 - hh_1'') = 0 \tag{2.15}$$

$$h_1'' - \varepsilon(2h_1f' - 2h'f_1 + hf_1' - fh') = 0 \tag{2.16}$$

Thus we have two pairs of coupled equations. The first pair of coupled non-linear equations describe the first order flow and magnetic fields. The second pair of coupled linear equations represent the flow and magnetic fields of second order arising due to external vorticity.

Here $s = \bar{\mu}H_o^2 / \rho\ u_o^2$ is the ratio of the magnetic and kinetic energies and $\varepsilon = \nu/\eta'$ is the ratio of the kinematic viscosity to the magnetic viscosity.

The condition

$$u = 0 = v \quad \text{at} \quad y = 0 \tag{2.17}$$

gives

$$f(0) = f'(0) = 0 = f_1(0) = f_1'\ (0) \tag{2.18}$$

while the condition that

$$u \to u_o + \omega y \quad \text{as} \quad y \to \infty. \tag{2.19}$$

requires that

$$f'(\infty) = 1 \text{ and } f_1''\ (\infty) = 1 \tag{2.20}$$

Also, since

$$H_x \to H_o \qquad \text{as} \quad y \to \infty, \tag{2.21}$$

we must have

$$h'(\infty) = 1 \quad , \; h_1' \; (\infty) = 0. \tag{2.22}$$

Since on the plate,

$$- H_y = \frac{\partial A}{\partial x} = 0 \; , \tag{2.23}$$

we have

$$A(x,0) = \text{constant}. \tag{2.24}$$

Choosing the line of force at $\eta = 0$ in the external flow to be

$$A(x,0) = \text{constant}, \tag{2.25}$$

and, since this line passes through the origin, we have

$$A(x,0) = 0. \tag{2.26}$$

This leads to the conditions

$$h(0) = 0 = h_1(0) \tag{2.27}$$

Hence, the boundary conditions of the first order are

$$f(0) = f'(0) = 0 \; , \quad f'(\infty) = 1 \tag{2.28}$$

$$h(0) = 0 \; , \quad h_1' \; (\infty) = 1. \tag{2.29}$$

and that of the second order are

$$f_1(0) = f_1'(0) = 0 \; , \quad f_1'' \; (\infty) = 1 \tag{2.30}$$

$$h_1(0) = 0 \; , \quad h_1'(\infty) = 0. \tag{2.31}$$

There is mathematical difficulty involved in solving the coupled, non-linear ordinary differential equations for the velocity and the magnetic field with two parameters present. Thus, we attempt a series solution for small s and the resulting equations are solved on an IBM 1620 electronic digital computer.

3. SERIES SOLUTION FOR SMALL s :

We look for solution, of the form

$$f = f_{oo} + s f_{o1} + s^2 f_{o2} + \cdots \tag{3.1}$$

$$f_1 = f_{10} + sf_{11} + s^2 f_{12} + \cdots \tag{3.2}$$

$$h = h_{00} + s\, h_{01} + s^2 h_{02} + \cdots \tag{3.3}$$

$$h_1 = h_{10} + sh_{11} + s^2 h_{12} + \ldots . \tag{3.4}$$

for small values of the magnetic parameter s.

Substituting (3.1) to (3.3) in the equation (2.15) to (2.17), we obtain for the first order flow and magnetic field, the equations

$$f_{00}''' + f_{00} f_{00}'' = 0 \tag{3.5}$$

$$f_{01}''' + f_{00} f_{01}'' + f_{00}'' f_{01} - h_{00} h_{00}'' = 0 \tag{3.6}$$

$$h_{00}'' - \varepsilon(h_{00} f_{00}' - h_{00}' f_{00}) = 0 \tag{3.7}$$

$$h_{01}'' - \varepsilon(f_{00}' h_{01} - f_{00} h_{01}' - f_{01} h_{00}' + h_{00} f_{01}') = 0 \tag{3.8}$$

etc.

and the equations for the secondary effect on the external vorticity are

$$f_{10}''' + f_{00} f_{10}'' - f_{00}' f_{10}' + 2f_{10} f_{00}'' = 0 \tag{3.9}$$

$$f_{11}''' + f_{00} f_{11}'' - f_{00}' f_{11}' + 2f_{00}'' f_{11} + f_{01} f_{10}'' - f_{01}' f_{10}' + 2f_{01}'' f_{10} + h_{10}' h_{00}' - 2h_{10} h_{00}'' - h_{00} h_{10}'' = 0. \tag{3.10}$$

etc.

$$h_{10}'' - \varepsilon(2h_{10} f_{00}' - 2h_{00}' f_{10} + h_{00} f_{10}' - f_{00} h_{10}') = 0 \tag{3.11}$$

$$h_{11}'' - \varepsilon(2h_{11} f_{\infty}' - 2h_{01}' f_{10} + h_{01} f_{10}' - f_{00} h_{11}') + (2h_{10} f_{01}' - 2h_{01}' f_{11} + h_{01} f_{11}' - f_{01} h_{11}') = 0 \tag{3.12}$$

The corresponding boundary conditions for the first order equations become

$$f_{00}(0) = f'_{00}(0) = 0, \; f'_{00}(\infty) = 1 \tag{3.13}$$

$$f_{01}(0) = f'_{01}(0) = 0, \; f'_{01}(\infty) = 0 \tag{3.14}$$

etc.

$$h_{00}(0) = 0, \; h'_{00}(\infty) = 1 \tag{3.15}$$

$$h_{01}(0) = 0 = h'_{01}(\infty) \tag{3.16}$$

etc.

and, for the second order equations, the boundary conditions become

$$f_{10}(0) = f'_{10}(0) = 0, \; f''_{10}(\infty) = 1 \tag{3.17}$$

$$f_{11}(0) = f'_{11}(0) = 0 = f''_{11}(\infty) \tag{3.18}$$

etc.

and

$$h_{10}(0) = 0 = h'_{10}(\infty) \tag{3.19}$$

$$h_{11}(0) = 0 = h'_{11}(\infty) \tag{3.20}$$

4. METHOD OF SOLUTION:

Now, the non-linear equation uncouples from others and reduces to the Prandtl-Blasius problem. The equations (3.5) and (3.9), together with their boundary conditions, constitute the vorticity interaction problem discussed by Li ((3) for the field free case. For simplicity, only equations pertaining to the first power of the magnetic parameter have been taken for investigation. All these equations have been integrated by a Runge-Kutta fourth-order process due to Gill (4). The essential results are given in the following tables.

Table 1.

$f''_{00}(0)$	$f''_{01}(0)$	$f''_{10}(0)$	$f''_{11}(0)$
0.4696	-.2727	0.7950	.3171

5. SKIN FRICTION:

The skin friction at the plate, given as the viscous force per unit area acting at the plate, is given by

$$\tau = \mu\left(\frac{\partial\mu}{\partial\mu}\right)_{y=0} \qquad (5.1)$$

$$= \mu\sqrt{u_o/2\nu x}\left\{f''(0) + \xi\, f_1''\,(0)\right\} \qquad (5.2)$$

The first factor within the brackets is the hydromagnetic skin friction for irrotational flow. The second factor is the contribution to the skin friction by interaction with external vorticity and is a measure of the kinematic effect of external vorticity at the surface. The two factors are given for various values of the magnetic parameter in the following table:

Table 2

s	f" (0)	$f_1''(0)$
0.0	0.4696	0.7950
0.1	0.4423	0.8267
0.2	0.4250	0.8584
0.3	0.3977	0.8891
0.4	0.3704	0.9208
0.5	0.3431	0.9526

6. DISCUSSION OF RESULTS:

The effect of a magnetic field on simple shear flow of a viscous, electrically conducting, incompressible liquid past an electrically insulated semi-infinite flat plate has been considered. A series solution in powers of a magnetic parameter has been attempted. The equations governing the flow and the magnetic field have been integrated numerically on a digital computer.

To a first approximation, it is clear that the effect of the magnetic field on the boundary layer is to reduce skin friction. The correction to the first order skin friction for small values of the magnetic parameter s is given by $(1 - .5806\,s)$.

The magnetic field tends to increase the shear at the surface over that due to external vorticity. For the field free case, the kinematic effect of the shear associated with external vorticity at the surface is 80% of that just outside the boundary layer. There is a correction of $(1 + .3988s)$ to this part of the shear for small values of the magnetic parameter.

Thus, it appears that a uniform magnetic field parallel to the plate tends to increase the second order contribution, while reducing the first order contribution to the value of skin friction. It is easily noted that the first order term is affected more the the second order term. The above analysis is valid only for small values of s, and the results are given in tabular form.

Table 3

First order velocity and magneitc field functions

η	$f_o(\eta)$	$f_o'(\eta)$	$f_o''(\eta)$	$h_o(\eta)$	$h_o'(\eta)$
0.0	0.0000	0.0000	0.4696	0.0000	0.5120
0.3	0.2115	0.1408	0.4686	0.1537	0.5127
0.6	0.0844	0.2806	0.4617	0.3079	0.5772
0.9	0.1891	0.4167	0.4436	0.4647	0.5292
1.2	0.3337	0.5452	0.4106	0.6266	0.5517
1.5	0.5151	0.6615	0.3618	0.7969	0.5861
1.8	0.7289	0.7610	0.3004	0.9794	0.6323
2.1	0.9697	0.8411	0.2330	1.1773	0.6874
2.4	1.2315	0.9010	0.1676	1.3925	0.7471
2.7	1.5085	0.9426	0.1111	1.6255	0.8060
3.0	1.7956	0.9690	0.0677	1.8753	0.8594
3.3	2.0888	0.9845	0.0379	2.1401	0.9039
3.6	2.3856	0.9929	0.0194	2.4166	0.9381
3.9	2.6842	0.9969	0.0091	2.7019	0.9624
4.2	2.9836	0.9989	0.0039	2.9933	0.9785
4.5	3.2833	0.9996	0.0016	3.2885	0.9884
4.8	3.5832	0.9998	0.0006	3.5859	0.9941
5.1	3.8832	0.9999	0.0002	3.8837	0.9972
5.4	4.1832	0.9999	0.0001	4.1841	0.9988
5.7	4.4832	1.0000	0.0000	4.4839	0.9996
6.0	4.7832	1.0000	0.0000	4.7838	0.9999
6.3	5.0832	1.0000	0.0000	5.0838	1.0000

Table 4

First order velocity and magnetic field functions

η	$f_{o1}(\eta)$	$f'_{o1}(\eta)$	$f''_{o1}(\eta)$	$h_{o1}(\eta)$	$h'_{o1}(\eta)$
0.0	0.0000	0.0000	-.2727	0.0000	-.1455
0.3	-.0123	-.0817	-.2714	-.0427	-.1460
0.6	-.0489	-.1621	-.2624	-.0879	-.1499
0.9	-.1090	-.2375	-.2372	-.1345	-.1601
1.2	-.1905	-.3021	-.1893	-.1849	-.1785
1.5	-.2885	-.3487	-.1177	-.2422	-.2048
1.8	-.3971	-.3710	-.0294	-.3082	-.2357
2.1	-.5084	-.3661	0.0614	-.3835	-.2654
2.4	-.6142	-.3357	0.1376	-.4665	-.2861
2.7	-.7079	-.2863	0.1859	-.5535	-.2912
3.0	-.7850	-.2274	0.2014	-.6393	-.2776
3.3	-.8443	-.1682	0.1884	-.7183	-.2467
3.6	-.8867	-.1161	0.1567	-.7861	-.2039
3.9	-.9151	-.0748	0.1179	-.8402	-.1567
4.2	-.9328	-.0449	0.0811	-.8804	-.1118
4.5	-.9331	-.0253	0.0513	-.9081	-.0741
4.8	-.9488	-.0132	0.0229	-.9258	-.0541
5.1	-.9516	-.0065	0.0162	-.9298	-.0741
5.4	-.9529	-.0029	0.0082	-.9363	-.0258
5.7	-.9536	-.0011	0.0039	-.9429	-.0133
6.0	-.9538	-.0004	0.0017	-.9448	-.0059
6.3	-.9538	-.0001	0.0007	-.9459	-.0020
6.6	-.9538	-.0000	0.0002	-.9460	-.0005
6.9	-.9538	-.0000	0.0000	-.9460	-.0001

Table 5.

Second order velocity and magnetic field functions

η	$f_{1o}(\eta)$	$f'_{1o}(\eta)$	$f''_{1o}(\eta)$	$h_{1o}(\eta)$	$h'_{1o}(\eta)$
0.0	0.0000	0.0000	0.7950	0.0000	0.2866
0.3	0.0358	0.2348	0.7933	0.0861	0.2877
0.6	0.1429	0.4750	0.7818	0.1733	0.2952
0.9	0.3203	0.7057	0.7525	0.2644	0.3148
1.2	0.5652	0.9244	0.7029	0.3637	0.3496
1.5	0.8732	1.1260	0.6399	0.4756	0.3983
1.8	1.2389	1.3086	0.5799	0.6033	0.4535
2.1	1.6569	1.4763	0.5443	0.7470	0.5027
2.4	2.1243	1.6391	0.5489	0.9028	0.5316
2.7	2.6413	1.8098	0.5965	1.0628	0.5293
3.0	3.2122	1.9998	0.6752	1.2168	0.4922
3.3	3.8439	2.2156	0.7649	1.3551	0.4259
3.6	4.5444	2.4576	0.8468	1.4706	0.3425
3.9	5.3208	2.7216	0.9101	1.5604	0.2561
4.2	6.1789	3.0014	0.9524	1.6252	0.1783
4.5	7.1226	3.2911	0.9772	1.6689	0.1157
4.8	8.1541	3.5864	0.9901	1.6963	0.0699
5.1	9.2747	3.8845	0.9961	1.7122	0.0393
5.4	10.4819	4.1838	0.9986	1.7208	0.0203
5.7	11.7849	4.4835	0.9996	1.7249	0.0093
6.0	13.1750	4.7835	0.9999	1.7267	0.0034
6.3	14.6551	5.0835	1.0000	1.7270	0.0010
6.6				1.7270	0.0005
6.9				1.7270	0.0001

Table 6.

Second order velocity and magnetic field functions.

η	$f_{11}(\eta)$	$f'_{11}(\eta)$	$f''_{11}(\eta)$	$h_{11}(\eta)$	$h'_{11}(\eta)$
0.0	0.0000	0.0000	0.3171	0.0000	0.3623
0.3	0.0136	0.0886	0.2736	0.1087	0.3630
0.6	0.0519	0.1646	0.2345	0.2183	0.3681
0.9	0.1113	0.2305	0.2062	0.3305	0.3818
1.2	0.1844	0.2897	0.1919	0.4870	0.4092
1.5	0.2849	0.3461	0.1841	0.5781	0.4573
1.8	0.3968	0.3985	0.1606	0.7260	0.5339
2.1	0.5228	0.4379	0.0929	0.9019	0.6446
2.4	0.6566	0.4479	-.0371	1.1159	0.7868
2.7	0.7866	0.4103	-.2208	1.3756	0.9445
3.0	0.8967	0.3141	-.4211	1.6812	1.0879
3.3	0.9897	-.0303	-.6786	2.0233	1.1813
3.6	0.9693	-.1395	-.6780	2.3819	1.1957
3.9	0.9498	-.2359	-.6010	2.7317	1.1212
4.2	0.8496	-.4295	-.5989	3.0472	0.9708
4.5	0.6957	-.5917	-.4735	3.3097	0.7742
4.8	0.4990	-.7143	-.3372	3.4097	0.5670
5.1	0.2715	-.7981	-.2166	3.5108	0.3791
5.4	0.0386	-.8503	-.1243	3.6521	0.2279
5.7	-.2358	-.8796	-.0616	3.7422	0.1181
6.0	-.5017	-.8942	-.0226	3.7931	0.0451
6.3	-.7706	-.9002	-.0062	3.8167	0.0092
6.6	- 1.0412	-.9004	-.0009	3.8172	0.0010
6.9	- 1.3124	-.9004	-.0001	3.8172	0.0000

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